Topological and Geometrical Aspects of the Study of Projective Plane Curves

Enrique ARTAL BARTOLO

Departmento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

LIB*60*BER. Topology of Algebraic Varieties
A Conference in Honor of the 60th Birthday
of Anatoly Libgober
Jaca (Aragón) June 23rd 2009

Contents

Contents

Contents

Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

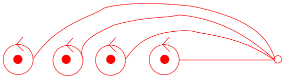
Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- Goal: Compute $\pi_1(\mathbb{P}^2 \setminus C)$, $C = F(x,y,z) = 0$ of degree d.

Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- Goal: Compute $\pi_1(\mathbb{P}^2 \setminus C)$, $C = F(x,y,z) = 0$ of degree d.
- Choose a generic line L_0, $L_0 \pitchfork C$. By Lefschetz Hyperplane Section Theorem:

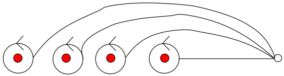
$$\mathbb{F} := \langle a_1, \dots, a_{d-1} \mid \_ \rangle = \langle a_1, \dots, a_d \mid a_d \cdot \ldots \cdot a_2 \cdot a_1 = 1 \rangle = \pi_1(L_0 \setminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C)$$



Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- Goal: Compute $\pi_1(\mathbb{P}^2 \setminus C)$, $C = F(x,y,z) = 0$ of degree d.
- Choose a generic line L_0, $L_0 \pitchfork C$. By Lefschetz Hyperplane Section Theorem:

$$\mathbb{F} := \langle a_1, \ldots, a_{d-1} \mid \_ \rangle = \langle a_1, \ldots, a_d \mid a_d \cdot \ldots \cdot a_2 \cdot a_1 = 1 \rangle = \pi_1(L_0 \setminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C)$$

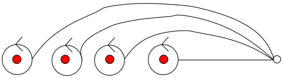


- $P \in L_0 \setminus C$ *generic*, $P \in L_\infty$, $L_\infty \pitchfork C$. Choose equations: $P := [0:1:0]$, $L_0 : x = 0$, $L_\infty : z = 0$.

Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- Goal: Compute $\pi_1(\mathbb{P}^2 \setminus C)$, $C = F(x,y,z) = 0$ of degree d.
- Choose a generic line L_0, $L_0 \pitchfork C$. By Lefschetz Hyperplane Section Theorem:

$$\mathbb{F} := \langle a_1, \ldots, a_{d-1} \mid \_ \rangle = \langle a_1, \ldots, a_d \mid a_d \cdot \ldots \cdot a_2 \cdot a_1 = 1 \rangle = \pi_1(L_0 \setminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C)$$



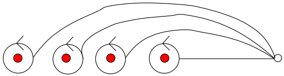
- $f(x,y) := F(x,y,1)$, f monic in y.

$$R := \{t \in \mathbb{C} \mid f(t,y) \text{ has no multiple root}\}$$

Zariski foundational paper

- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- Goal: Compute $\pi_1(\mathbb{P}^2 \setminus C)$, $C = F(x,y,z) = 0$ of degree d.
- Choose a generic line L_0, $L_0 \pitchfork C$. By Lefschetz Hyperplane Section Theorem:

$$\mathbb{F} := \langle a_1, \ldots, a_{d-1} \mid \_ \rangle = \langle a_1, \ldots, a_d \mid a_d \cdot \ldots \cdot a_2 \cdot a_1 = 1 \rangle = \pi_1(L_0 \setminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C)$$



- $f(x,y) := F(x,y,1), f$ monic in y.

$$R := \{t \in \mathbb{C} \mid f(t,y) \text{ has no multiple root}\}$$

- Take a loop α in $\mathbb{C} \setminus R$ based at 0. The motion $\{L_t\}$ defined by α deform $a_1, \ldots, a_d$ to $a_1^\alpha, \ldots, a_d^\alpha$. In $\pi_1(\mathbb{P}^2 \setminus C)$:

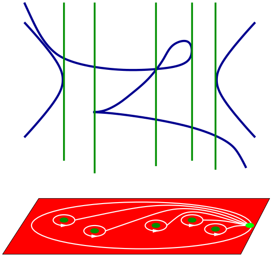
$$a_j = a_j^\alpha$$

Van Kampen proves these relations are enough.

Braid monodromy

Modern language

There is a morphism $\nabla : \pi_1(\mathbb{C} \setminus R) \to \mathbb{B}_d$. The group $\mathbb{B}_d$ acts geometrically on the free group in $a_1, \ldots, a_d$

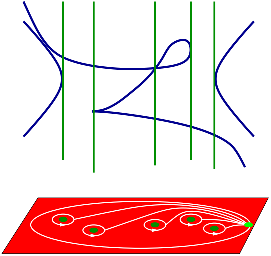


Braid monodromy

Modern language

There is a morphism $\nabla : \pi_1(\mathbb{C} \setminus R) \to \mathbb{B}_d$. The group $\mathbb{B}_d$ acts geometrically on the free group in $a_1, \ldots, a_d$

$$\pi_1(\mathbb{C}^2 \setminus C) = \left\langle a_1, \ldots, a_d \mid a_j = a_j^{\nabla(\beta)}, \beta \in \pi_1(\mathbb{C} \setminus R) \right\rangle$$

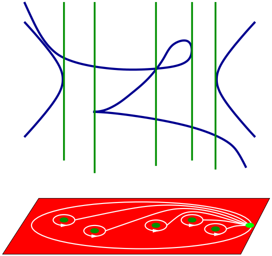


Braid monodromy

Modern language

There is a morphism $\nabla : \pi_1(\mathbb{C} \setminus R) \to \mathbb{B}_d$. The group $\mathbb{B}_d$ acts geometrically on the free group in $a_1, \ldots, a_d$

$$\pi_1(\mathbb{P}^2 \setminus C) = \left\langle a_1, \ldots, a_d \ \middle|\ a_j = a_j^{\nabla(\beta)}, \beta \in \pi_1(\mathbb{C} \setminus R), a_d \cdot \ldots \cdot a_1 = 1 \right\rangle.$$

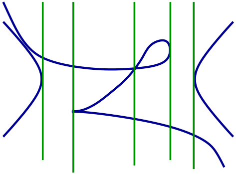


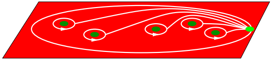
Braid monodromy

Modern language

There is a morphism $\nabla:\pi_1(\mathbb{C}\setminus R)\to\mathbb{B}_d$. The group $\mathbb{B}_d$ acts geometrically on the free group in $a_1,\dots,a_d$

$$\pi_1(\mathbb{P}^2\setminus C)=\left\langle a_1,\dots,a_d \mid a_j=a_j^{\nabla(\beta)},\beta\in\pi_1(\mathbb{C}\setminus R),a_d\cdot\ldots\cdot a_1=1\right\rangle.$$

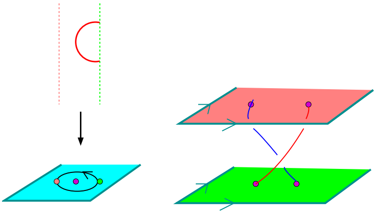




The group $\pi_1(\mathbb{C}\setminus R)$ is generated by elements $\alpha\gamma\alpha^{-1}$, where γ runs a small circle around a point of R.

Puiseux relations and Zariski results

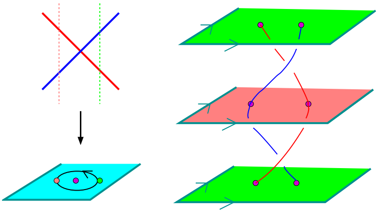
Turn around $x = y^2$



$a_1 = a_2$

Puiseux relations and Zariski results

Turn around $x^2 = y^2$



$a_1 a_2 = a_2 a_1$

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\dots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\ldots$, where a'_i is conjugated to a_i. **Difficult** Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\ldots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\ldots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\dots$, where a'_i is conjugated to a_i. **Difficult** Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\ldots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- The proof depends on a statement of Severi with a wrong proof: families of nodal curves with fixed degree and number of nodes are irreducible. It becomes Zariski conjecture.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\dots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- Zariski conjecture is proved by Fulton and Deligne.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\ldots$, where a'_i is conjugated to a_i. **Difficult** Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- Harris proves Severi's statement $\Rightarrow$ Zariski's proof is correct.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, $\dots$, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- C tricuspidal quartic, $\pi_1(\mathbb{P}^2 \setminus C)$ finite non-abelian.
- C hexacuspidal sextic, cusps in a conic, $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a'_i = a'_j$, $[a'_i, a'_j] = 1$, $a'_i a'_j a'_i = a'_j a'_i a'_j$, …, where a'_i is conjugated to a_i. Difficult Find which conjugated elements a'_i.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- C tricuspidal quartic, $\pi_1(\mathbb{P}^2 \setminus C)$ finite non-abelian.
- C hexacuspidal sextic, cusps in a conic, $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.
- If there exists C hexacuspidal sextic, cusps not in a conic, $\pi_1(\mathbb{P}^2 \setminus C) \ncong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. He proves later abelianity for non explicit examples coming from deformation theory

Puiseux relations and Zariski results

Braids define a finite set of relations of the type $a_i' = a_j'$, $[a_i', a_j'] = 1$, $a_i' a_j' a_i' = a_j' a_i' a_j'$, $\ldots$, where a_i' is conjugated to a_i. Difficult Find which conjugated elements a_i'.

Zariski results

- If $\{C_t\}_{t\in[0,1]}$ is a continuous equisingular family $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are isotopic.
- If $\{C_t\}_{t\in[0,1]}$ is a continuous family which is equisingular in $(0,1]$ and C_0 is reduced, then $\pi_1(\mathbb{P}^2 \setminus C_0) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C_1)$.
- If C is smooth of degree d, then $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ (choose a curve with a high order flex).
- If C is nodal, then $\pi_1(\mathbb{P}^2 \setminus C)$ is abelian (start from a generic line arrangement).
- C tricuspidal quartic, $\pi_1(\mathbb{P}^2 \setminus C)$ finite non-abelian.
- C hexacuspidal sextic, cusps in a conic, $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.
- If there exists C hexacuspidal sextic, cusps not in a conic, $\pi_1(\mathbb{P}^2 \setminus C) \not\cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. He proves later abelianity for non explicit examples coming from deformation theory
- Oka and A. find explicit examples. Degtyarev proves the space of hexacuspidal sextics with cusps not in a conic is irreducible.

Braid monodromy, surfaces and topology of curves

Definition

Chisini realizes that Zariski-van Kampen method gives not only the fundamental group but a stronger invariant for curves

Braid monodromy, surfaces and topology of curves

Moishezon

Papers of B. Moishezon (alone and with M. Teicher) show that braid monodromy is a extremely powerful tool in the study of complex surfaces (via projection).

Braid monodromy, surfaces and topology of curves

Moishezon

Papers of B. Moishezon (alone and with M. Teicher) show that braid monodromy is a extremely powerful tool in the study of complex surfaces (via projection).

Libgober I

Libgober proves that Puiseux presentation determine the homotopy type of $\mathbb{C}^2 \setminus C$ (line at infinity can be chosen non generic as long as there is no vertical asymptote: braid monodromies with vertical asymptotes are more subtle, see Carmona and Lönne).

Braid monodromy, surfaces and topology of curves

Moishezon

Papers of B. Moishezon (alone and with M. Teicher) show that braid monodromy is a extremely powerful tool in the study of complex surfaces (via projection).

Libgober I

Libgober proves that Puiseux presentation determine the homotopy type of $\mathbb{C}^2 \setminus C$ (line at infinity can be chosen non generic as long as there is no vertical asymptote: braid monodromies with vertical asymptotes are more subtle, see Carmona and Lönne).

Libgober II

Libgober proves that braid monodromy and braid representations can provide new invariants of irreducible components of equisingular families of curves (unexplored! up to Burau representation).

Braid monodromy, surfaces and topology of curves

Moishezon

Papers of B. Moishezon (alone and with M. Teicher) show that braid monodromy is a extremely powerful tool in the study of complex surfaces (via projection).

Libgober I

Libgober proves that Puiseux presentation determine the homotopy type of $\mathbb{C}^2 \setminus C$ (line at infinity can be chosen non generic as long as there is no vertical asymptote: braid monodromies with vertical asymptotes are more subtle, see Carmona and Lönne).

Libgober II

Libgober proves that braid monodromy and braid representations can provide new invariants of irreducible components of equisingular families of curves (unexplored! up to Burau representation).

Topology

V. Kulikov and M. Teicher prove that braid monodromy determines the topology for nodal and cuspidal curves (using explict description of centralizer of braids). Carmona proves it in general (using Neumann plumbing calculus on Waldhausen graph manifolds).

Definition and first results

Definition by Libgober

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- $\operatorname{Deck}(\rho)$ induces a finitely generated $\mathbb{C}[t, t^{-1}]$-module structure M_C on $H_1(X; \mathbb{C})$

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- Deck(ρ) induces a finitely generated $\mathbb{C}[t, t^{-1}]$-module structure M_C on $H_1(X; \mathbb{C})$
- $\mathbb{C}[t, t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- $\mathrm{Deck}(\rho)$ induces a finitely generated $\mathbb{C}[t, t^{-1}]$-module structure M_C on $H_1(X; \mathbb{C})$
- $\mathbb{C}[t, t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

First results

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- $\mathrm{Deck}(\rho)$ induces a finitely generated $\mathbb{C}[t, t^{-1}]$-module structure M_C on $H_1(X; \mathbb{C})$
- $\mathbb{C}[t, t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

First results

- Zariski: Alexander polynomial is trivial if C is irreducible with degree a prime power.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- Deck(ρ) induces a finitely generated $\mathbb{C}[t,t^{-1}]$-module structure M_C on $H_1(X;\mathbb{C})$
- $\mathbb{C}[t,t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

First results

- Zariski: Alexander polynomial is trivial if C is irreducible with degree a prime power.
- Zariski, for nodal and cuspidal curves: nodal points do not contribute to Alexander polynomial. Position of cusps is important. For hexacuspidal sextics, Δ_C equals either 1 or $t^2 - t + 1$.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- $\mathrm{Deck}(\rho)$ induces a finitely generated $\mathbb{C}[t,t^{-1}]$-module structure M_C on $H_1(X;\mathbb{C})$
- $\mathbb{C}[t,t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

First results

- Zariski: Alexander polynomial is trivial if C is irreducible with degree a prime power.
- Zariski, for nodal and cuspidal curves: nodal points do not contribute to Alexander polynomial. Position of cusps is important. For hexacuspidal sextics, Δ_C equals either 1 or $t^2 - t + 1$.
- Formulæ to compute Δ_C in general by Libgober, Esnault, Loeser-Vaquié, A.

Definition and first results

Definition by Libgober

- $C \subset \mathbb{C}^2$ defined by $f = 0$, f reduced. The mapping $f : \mathbb{C}^2 \setminus C \to \mathbb{C}^*$ provides an epimorphism $f_* : \pi_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}$.
- Any such morphism produces an Alexander polynomial Δ_C.
- Let $\rho : X \to \mathbb{C}^2 \setminus C$ be the unramified covering associated to ρ.
- Deck(ρ) induces a finitely generated $\mathbb{C}[t, t^{-1}]$-module structure M_C on $H_1(X; \mathbb{C})$
- $\mathbb{C}[t, t^{-1}]$ principal and the torsion part of $M_C \Rightarrow$ Alexander polynomial.

First results

- Zariski: Alexander polynomial is trivial if C is irreducible with degree a prime power.
- Zariski, for nodal and cuspidal curves: nodal points do not contribute to Alexander polynomial. Position of cusps is important. For hexacuspidal sextics, Δ_C equals either 1 or $t^2 - t + 1$.
- Formulæ to compute Δ_C in general by Libgober, Esnault, Loeser-Vaquié, A.
- Libgober: Δ_C determined by Burau representation and braid monodromy.

Local and global Alexander polynomials

Links associated to affine curves

Local and global Alexander polynomials

Links associated to affine curves

- Local links: Given $P \in C$ singular point, $0 < \varepsilon \ll 1$: $(\mathbb{S}^3_\varepsilon, K_\varepsilon)$, $K_\varepsilon := C \cap \mathbb{S}^3_\varepsilon$ is the local link of C at P. We denote it by K_C^P.

Local and global Alexander polynomials

Links associated to affine curves

- Local links: Given $P \in C$ singular point, $0 < \varepsilon \ll 1$: $(\mathbb{S}^3_\varepsilon, K_\varepsilon)$, $K_\varepsilon := C \cap \mathbb{S}^3_\varepsilon$ is the local link of C at P. We denote it by K_C^P.
- Global link: $(\mathbb{S}^3_M, K_M)$, $K_M := C \cap \mathbb{S}^3_N$ is the link at infinity of C. We denote it by K_C^∞.

Local and global Alexander polynomials

Links associated to affine curves

- Local links: Given $P \in C$ singular point, $0 < \varepsilon \ll 1$: $(\mathbb{S}^3_\varepsilon, K_\varepsilon)$, $K_\varepsilon := C \cap \mathbb{S}^3_\varepsilon$ is the local link of C at P. We denote it by K_C^P.
- Global link: $(\mathbb{S}^3_M, K_M)$, $K_M := C \cap \mathbb{S}^3_N$ is the link at infinity of C. We denote it by K_C^∞.

Theorem (Libgober)

Δ_C divides both $\prod_{P \in \mathrm{Sing}(C)} \Delta_{K_C^P}$ and $\Delta_{K_C^\infty}$.

Local and global Alexander polynomials

Links associated to affine curves

- Local links: Given $P \in C$ singular point, $0 < \varepsilon \ll 1$: $(\mathbb{S}^3_\varepsilon, K_\varepsilon)$, $K_\varepsilon := C \cap \mathbb{S}^3_\varepsilon$ is the local link of C at P. We denote it by K_C^P.
- Global link: $(\mathbb{S}^3_M, K_M)$, $K_M := C \cap \mathbb{S}^3_N$ is the link at infinity of C. We denote it by K_C^∞.

Theorem (Libgober)

Δ_C *divides both* $\prod_{P \in \mathrm{Sing}(C)} \Delta_{K_C^P}$ *and* $\Delta_{K_C^\infty}$.

Remark

Cogolludo and Florens compute the extra factor $(\prod_{P \in \mathrm{Sing}(C)} \Delta_{K_C^P})/\Delta_C$.

Local and global Alexander polynomials

Links associated to affine curves

- Local links: Given $P \in C$ singular point, $0 < \varepsilon \ll 1$: $(\mathbb{S}^3_\varepsilon, K_\varepsilon)$, $K_\varepsilon := C \cap \mathbb{S}^3_\varepsilon$ is the local link of C at P. We denote it by K_C^P.
- Global link: $(\mathbb{S}^3_M, K_M)$, $K_M := C \cap \mathbb{S}^3_N$ is the link at infinity of C. We denote it by K_C^∞.

Theorem (Libgober)

Δ_C divides both $\prod_{P \in \mathrm{Sing}(C)} \Delta_{K_C^P}$ and $\Delta_{K_C^\infty}$.

Remark

Cogolludo and Florens compute the extra factor $(\prod_{P \in \mathrm{Sing}(C)} \Delta_{K_C^P})/\Delta_C$.

Remark

If $C = f^{-1}(0)$ is smooth and K_C^∞ is isotopic to $K^\infty_{f^{-1}(t)}$, t small, then K_C^∞ determines the topology of $(\mathbb{C}^2, C)$ (Neumann) and $\pi_1(\mathbb{C}^2 \setminus C) \cong \mathbb{Z}$ (Kaliman).

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

- Y be a space (with the homotopy type of a cellular complex) such that $\pi_1(Y) = G$.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

- Y be a space (with the homotopy type of a cellular complex) such that $\pi_1(Y) = G$.
- $\rho : X \to Y$ the universal abelian unramified covering associated to $G \to H$

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

- Y be a space (with the homotopy type of a cellular complex) such that $\pi_1(Y) = G$.
- $\rho : X \to Y$ the universal abelian unramified covering associated to $G \to H$
- $M_G := H_1(X;\mathbb{C}) = G'/G'' \otimes \mathbb{C}$ is a finitely generated Λ-module.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

- Y be a space (with the homotopy type of a cellular complex) such that $\pi_1(Y) = G$.
- $\rho : X \to Y$ the universal abelian unramified covering associated to $G \to H$
- $M_G := H_1(X;\mathbb{C}) = G'/G'' \otimes \mathbb{C}$ is a finitely generated Λ-module.
- $J_k(G) \subset \Lambda$ be the Fitting ideals of M_G.

Definitions

- Characteristic varieties are a generalization of Alexander polynomial for reducible curves.
- Invariant of $G := \pi_1(\mathbb{C}^2 \setminus C)$ and depends only on G/G''.
- The invariant is related with the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant.
- Fix G a finitely presented group, and let $H := G/G'$.
- $k := \operatorname{rk} H$ and h number of torsion elements.
- $\Lambda := \mathbb{C}[H]$ the group algebra (if H is free it is a ring of Laurent polynomials).
- T_H algebraic variety associated to Λ; it is a complex abelian Lie group diffeomorphic to h disjoint copies of tori $(\mathbb{C}^*)^k$.

Alexander Invariant

- Y be a space (with the homotopy type of a cellular complex) such that $\pi_1(Y) = G$.
- $\rho : X \to Y$ the universal abelian unramified covering associated to $G \to H$
- $M_G := H_1(X;\mathbb{C}) = G'/G'' \otimes \mathbb{C}$ is a finitely generated Λ-module.
- $J_k(G) \subset \Lambda$ be the Fitting ideals of M_G.

Definition

The characteristic variety $V'_k(G) \subset T_H$ is the zero locus of $J_k(G)$.

Definitions II

Twisted cohomology

Definitions II

Twisted cohomology

- Identify $T_H \equiv H^1(Y;\mathbb{C}^*) = \operatorname{Hom}(G, \operatorname{GL}(\mathbb{C};1))$.

Definitions II

Twisted cohomology

- Identify $T_H \equiv H^1(Y;\mathbb{C}^*) = \mathrm{Hom}(G, \mathrm{GL}(\mathbb{C};1))$.
- For $\sigma \in T_H$, define a local system of coefficients $\mathbb{C}_\sigma$

Definitions II

Twisted cohomology

- Identify $T_H \equiv H^1(Y;\mathbb{C}^*) = \mathrm{Hom}(G,\mathrm{GL}(\mathbb{C};1))$.
- For $\sigma \in T_H$, define a local system of coefficients $\mathbb{C}_\sigma$

Definition

$$V_k''(G) := \{\sigma \in T_H \mid \dim H^1(Y;\mathbb{C}_\sigma) \geq k\}$$

Definitions II

Twisted cohomology

- Identify $T_H \equiv H^1(Y;\mathbb{C}^*) = \mathrm{Hom}(G, \mathrm{GL}(\mathbb{C};1))$.
- For $\sigma \in T_H$, define a local system of coefficients $\mathbb{C}_\sigma$

Definition

$$V_k''(G) := \{\sigma \in T_H \mid \dim H^1(Y;\mathbb{C}_\sigma) \geq k\}$$

Remark

These two definitions may differ in $\mathbf{1} \in T_H$.

Definitions II

Twisted cohomology

- Identify $T_H \equiv H^1(Y;\mathbb{C}^*) = \operatorname{Hom}(G, \operatorname{GL}(\mathbb{C};1))$.
- For $\sigma \in T_H$, define a local system of coefficients $\mathbb{C}_\sigma$

Definition

$$V_k''(G) := \{\sigma \in T_H \mid \dim H^1(Y;\mathbb{C}_\sigma) \geq k\}$$

Remark

These two definitions may differ in $\mathbf{1} \in T_H$.

Remark

More general characteristic varieties can be defined using $\operatorname{Hom}(G, \operatorname{GL}(\mathbb{C};m))$, $m > 1$.

Properties

- Characteristic varieties are defined over $\mathbb{Z}$.

Properties

- Characteristic varieties are defined over $\mathbb{Z}$.
- Let $\boldsymbol{\sigma} \neq \mathbf{1}$ a torsion element of T_H of order $\ell > 1$

Properties

- Characteristic varieties are defined over $\mathbb{Z}$.
- Let $\sigma \neq \mathbf{1}$ a torsion element of T_H of order $\ell > 1$
- ρ is associated to a cyclic ℓ-fold covering $\rho_\sigma : X_\sigma \to Y$

Properties

- Characteristic varieties are defined over $\mathbb{Z}$.
- Let $\sigma \neq \mathbf{1}$ a torsion element of T_H of order $\ell > 1$
- ρ is associated to a cyclic ℓ-fold covering $\rho_\sigma : X_\sigma \to Y$
- There is a natural decomposition of $H^1(X_\sigma; \mathbb{C}) = \oplus_{j=0}^{\ell-1} H_{\sigma^j}$

Properties

- Characteristic varieties are defined over $\mathbb{Z}$.
- Let $\sigma \neq \mathbf{1}$ a torsion element of T_H of order $\ell > 1$
- ρ is associated to a cyclic ℓ-fold covering $\rho_\sigma : X_\sigma \to Y$
- There is a natural decomposition of $H^1(X_\sigma; \mathbb{C}) = \oplus_{j=0}^{\ell-1} H_{\sigma^j}$
- $\sigma \in V_k(G) \Leftrightarrow \dim H_\sigma \geq k$.

Computation of characteristic varieties

In general, it is very expensive to compute the characteristic varieties of a group. In case of $\pi_1(\mathbb{C}^2 \setminus C)$ or $\pi_1(\mathbb{P}^2 \setminus C)$ it is also expensive to compute it in terms of C.
Libgober's theory of quasiadjunction polytopes provide a way to compute the characteristic varieties of $\pi_1(\mathbb{P}^2 \setminus C)$ in terms of superabondance of some linear systems associated to the singularities of C.

Computation of characteristic varieties

In general, it is very expensive to compute the characteristic varieties of a group. In case of $\pi_1(\mathbb{C}^2\setminus C)$ or $\pi_1(\mathbb{P}^2\setminus C)$ it is also expensive to compute it in terms of C.
Libgober's theory of quasiadjunction polytopes provide a way to compute the characteristic varieties of $\pi_1(\mathbb{P}^2\setminus C)$ in terms of superabondance of some linear systems associated to the singularities of C.

Theorem

Let $C=C_1\cup\cdots\cup C_r$ be the irreducible decomposition of a curve, $G:=\pi_1(\mathbb{P}^2\setminus\mathbb{C})$. Then, the non-coordinate irreducible components of $V_k(G)$ can be computed using quasiadjunction polytopes of the singular points of C.

Computation of characteristic varieties

In general, it is very expensive to compute the characteristic varieties of a group. In case of $\pi_1(\mathbb{C}^2 \setminus C)$ or $\pi_1(\mathbb{P}^2 \setminus C)$ it is also expensive to compute it in terms of C.
Libgober's theory of quasiadjunction polytopes provide a way to compute the characteristic varieties of $\pi_1(\mathbb{P}^2 \setminus C)$ in terms of superabondance of some linear systems associated to the singularities of C.

Theorem

Let $C = C_1 \cup \cdots \cup C_r$ be the irreducible decomposition of a curve, $G := \pi_1(\mathbb{P}^2 \setminus \mathbb{C})$. Then, the non-coordinate irreducible components of $V_k(G)$ can be computed using quasiadjunction polytopes of the singular points of C.

Remark

As in the case of Alexander polynomial, nodal points are irrelevant.

Computation of characteristic varieties

In general, it is very expensive to compute the characteristic varieties of a group. In case of $\pi_1(\mathbb{C}^2\setminus C)$ or $\pi_1(\mathbb{P}^2\setminus C)$ it is also expensive to compute it in terms of C.
Libgober's theory of quasiadjunction polytopes provide a way to compute the characteristic varieties of $\pi_1(\mathbb{P}^2\setminus C)$ in terms of superabondance of some linear systems associated to the singularities of C.

Theorem

Let $C = C_1\cup\cdots\cup C_r$ be the irreducible decomposition of a curve, $G := \pi_1(\mathbb{P}^2\setminus \mathbb{C})$. Then, the non-coordinate irreducible components of $V_k(G)$ can be computed using quasiadjunction polytopes of the singular points of C.

Remark

As in the case of Alexander polynomial, nodal points are irrelevant.

Example (A., Cogolludo, Tokunaga)

Let C_1 be a smooth conic and C_2 be an irreducible quartic, such that C_1 and C_2 intersect tangentially at four points. Let $C := C_1\cup C_2$; then $V_1(\pi_1(\mathbb{P}^2\setminus C))$ is trivial if C_2 is smooth but it has a non trivial point if C_2 has 3 nodes.

Orbifold groups

Orbifold

An orbifold X_φ is a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points.

Orbifold groups

Orbifold

An orbifold X_φ is a quasiprojective Riemann surface X with a function $\varphi: X \to \mathbb{N}$ with value 1 outside a finite number of points.

Orbifold group

For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $\varphi(p_j) := m_j > 1$. Then

$$\pi_1^{\text{orb}} := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1,\ldots,m_n}$.

Orbifold groups

Orbifold

An orbifold X_φ is a quasiprojective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points.

Orbifold group

For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $\varphi(p_j) := m_j > 1$. Then

$$\pi_1^{\text{orb}} := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j. We denote X_φ by $X_{m_1,\ldots,m_n}$.

Definition

A dominant algebraic morphism $\rho : Y \to X$ defines an orbifold morphism $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p)$ is an n-multiple.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Example

$G := \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/6\mathbb{Z}$, $T_H = \{\zeta \in \mathbb{C}^* \mid \zeta^6 = 1\}$ and $V_1(G)$ consists of $\{\exp(\pm\frac{2i\pi}{6})\}$.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Example

$G := \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/6\mathbb{Z}$, $T_H = \{\zeta \in \mathbb{C}^* \mid \zeta^6 = 1\}$ and $V_1(G)$ consists of $\{\exp(\pm\frac{2i\pi}{6})\}$.

Example

$G := \mathbb{F}_k =, k > 1$, $T_H = V_1(G) = (\mathbb{C}^*)^k$.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Example

$G := \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/6\mathbb{Z}$, $T_H = \{\zeta \in \mathbb{C}^* \mid \zeta^6 = 1\}$ and $V_1(G)$ consists of $\{\exp(\pm\frac{2i\pi}{6})\}$.

Example

$G := \pi_1^{\text{orb}}((\mathbb{C} \setminus \{1, \ldots, r\})_n) = \mathbb{F}_r * \mathbb{Z}/n\mathbb{Z}$, $r > 1$, $T_H = V_1(G) = (\mathbb{C}^*)^k \times \mathbb{Z}/n\mathbb{Z}$.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Example

$G := \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/6\mathbb{Z}$, $T_H = \{\zeta \in \mathbb{C}^* \mid \zeta^6 = 1\}$ and $V_1(G)$ consists of $\{\exp(\pm\frac{2i\pi}{6})\}$.

Example

$G := \pi_1^{\text{orb}}((\mathbb{C} \setminus \{1, \ldots, r\})_n) = \mathbb{F}_r * \mathbb{Z}/n\mathbb{Z}$, $r > 1$, $T_H = V_1(G) = (\mathbb{C}^*)^k \times \mathbb{Z}/n\mathbb{Z}$.

Example

$G := \pi_1^{\text{orb}}((\mathbb{C}^*)_n) = \mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$, $T_H = (\mathbb{C}^*) \times \mathbb{Z}/n\mathbb{Z}$ and $V_1(G)$ contains $\mathbf{1}$ and the components not passing through $\mathbf{1}$.

Orbifold groups II

Example

If C is a reduced curve with equation $f_2^3 - f_3^2 = 0$, f_j homogeneous of degree j, then the mapping $\mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{1\}$, given by $x \mapsto \frac{f_2^3(x)}{f_3^2}$, defines an orbifold morphism for $\varphi(0) = 3$ and $\varphi(\infty) = 2$.

Example

$G := \pi_1^{\text{orb}}(\mathbb{C}_{2,3}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/6\mathbb{Z}$, $T_H = \{\zeta \in \mathbb{C}^* \mid \zeta^6 = 1\}$ and $V_1(G)$ consists of $\{\exp(\pm \frac{2i\pi}{6})\}$.

Example

$G := \pi_1^{\text{orb}}((\mathbb{C} \setminus \{1, \ldots, r\})_n) = \mathbb{F}_r * \mathbb{Z}/n\mathbb{Z}$, $r > 1$, $T_H = V_1(G) = (\mathbb{C}^*)^k \times \mathbb{Z}/n\mathbb{Z}$.

Example

$G := \pi_1^{\text{orb}}(\text{Torus}_{2,2})$, $T_H = (\mathbb{C}^*)^2 \times \mathbb{Z}/2\mathbb{Z}$ and $V_1(G)$ contains $\mathbf{1}$ and the component not passing through $\mathbf{1}$.

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.

Theorem (Arapura)

Let Σ be an irreducible component of $V_1(G)$. Then,

- *If* $\dim\Sigma > 0$ *then there exists a surjective morphism* $\rho : X \to C$, *C algebraic curve, and a torsion element* σ *such that* $\Sigma = \sigma\rho^*(H^1(C;\mathbb{C}^*))$.
- *If* $\dim\Sigma = 0$ *then* Σ *is unitary.*

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.
Following ideas coming from works of Delzant, Simpson or Dimca, we prove

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.
Following ideas coming from works of Delzant, Simpson or Dimca, we prove

Theorem (A., Cogolludo)

Let Σ be an irreducible component of $V_1(G)$. Then one of the two following statements holds:

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $V_1(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ consists of a torsion point.*

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.
Following ideas coming from works of Delzant, Simpson or Dimca, we prove

Theorem (A., Cogolludo)

Let Σ be an irreducible component of $V_1(G)$. Then one of the two following statements holds:

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $V_1(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ consists of a torsion point.*

Remark

Finite cyclic coverings determine $V_1(G)$

Quasiprojective groups

Let G be the fundamental group of a quasiprojective manifold X.
Following ideas coming from works of Delzant, Simpson or Dimca, we prove

Theorem (A., Cogolludo)

Let Σ be an irreducible component of $V_1(G)$. Then one of the two following statements holds:

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $V_1(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ consists of a torsion point.*

Remark

Finite cyclic coverings determine $V_1(G)$

Example

Let C_1 be a quintic curve having three $\mathbb{A}_4$ singular points. Degtyarev proved that its fundamental group is finite and non abelian of order 320. Let C be the union of C_1 and a tangent line to a singular point. Then, the Alexander polynomial of C is $t^4 - t^3 + t^2 - t + 1$ but the points in the characteristic variety cannot come from a morphism onto an orbifold.

Corollaries

Let G be a quasiprojective group.

Corollary

The irreducible components of $V_1(G)$ are subtori translated by torsion elements.

Corollaries

Let G be a quasiprojective group.

Corollary

The irreducible components of $V_1(G)$ are subtori translated by torsion elements.

Corollary

Σ_1, Σ_2 *irreducible components of* $V_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$.

Corollaries

Let G be a quasiprojective group.

Corollary

The irreducible components of $V_1(G)$ are subtori translated by torsion elements.

Corollary

Σ_1, Σ_2 *irreducible components of* $V_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$.

Corollary

Σ_1, Σ_2 *irreducible components of* $V_1(G)$ *of positive dimension* $\Rightarrow \Sigma_1 \cap \Sigma_2 \subset V_2(G)$ *and consists of torsion points.*

Corollaries

Let G be a quasiprojective group.

Corollary

The irreducible components of $V_1(G)$ are subtori translated by torsion elements.

Corollary

Σ_1, Σ_2 *irreducible components of* $V_1(G)$, $\dim(\Sigma_1 \cap \Sigma_2) > 0 \Rightarrow \Sigma_1 = \Sigma_2$.

Corollary

Σ_1, Σ_2 *irreducible components of* $V_1(G)$ *of positive dimension* $\Rightarrow \Sigma_1 \cap \Sigma_2 \subset V_2(G)$ *and consists of torsion points.*

Corollary

Σ *irreducible component of* $V_1(G)$, $\dim \Sigma > 2 \Rightarrow$ *the subgroup parallel to* Σ *is also contained in* $V_1(G)$. *Also true if* $\dim \Sigma = 2$ *and* G *rational.*

Corollaries

Let G be a quasiprojective group.

Corollary

The irreducible components of $V_1(G)$ are subtori translated by torsion elements.

Corollary

Σ_1,Σ_2 *irreducible components of* $V_1(G)$, $\dim(\Sigma_1\cap\Sigma_2)>0 \Rightarrow \Sigma_1=\Sigma_2$.

Corollary

Σ_1,Σ_2 *irreducible components of* $V_1(G)$ *of positive dimension* $\Rightarrow \Sigma_1\cap\Sigma_2\subset V_2(G)$ *and consists of torsion points.*

Corollary

Σ *irreducible component of* $V_1(G)$, $\dim\Sigma>2 \Rightarrow$ *the subgroup parallel to* Σ *is also contained in* $V_1(G)$. *Also true if* $\dim\Sigma=2$ *and* G *rational.*

Corollary

Σ *irreducible component of* $V_1(G)$, $\dim\Sigma=1\Rightarrow \mathbf{1}\notin\Sigma$.

Applications

Γ finite graph with edges weighted by elements of $\mathbb{N}_{>1}$ (no loops or multiple edges). The Artin group G_Γ has the following presentation:

- A generator g_v for each vertex v of Γ

Applications

Γ finite graph with edges weighted by elements of $\mathbb{N}_{>1}$ (no loops or multiple edges). The Artin group G_Γ has the following presentation:

- A generator g_v for each vertex v of Γ
- For each edge e with weight n and extremities v, w a relation

$$\underbrace{g_v g_w g_v \cdots}_{n \text{ times}} = \underbrace{g_w g_v g_w \cdots}_{n \text{ times}}$$

Applications

Γ finite graph with edges weighted by elements of $\mathbb{N}_{>1}$ (no loops or multiple edges). The Artin group G_Γ has the following presentation:

- A generator g_v for each vertex v of Γ
- For each edge e with weight n and extremities v, w a relation

$$\underbrace{g_v g_w g_v \dots}_{n \text{ times}} = \underbrace{g_w g_v g_w \dots}_{n \text{ times}}$$

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides p, q, r

Applications

Γ finite graph with edges weighted by elements of $\mathbb{N}_{>1}$ (no loops or multiple edges). The Artin group G_Γ has the following presentation:

- A generator g_v for each vertex v of Γ
- For each edge e with weight n and extremities v, w a relation

$$\underbrace{g_v g_w g_v \cdots}_{n \text{ times}} = \underbrace{g_w g_v g_w \cdots}_{n \text{ times}}$$

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides p, q, r

- *If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ then there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r} = \pi_1(\mathbb{C}^2 \setminus C_{p,q,r})$*

Applications

Γ finite graph with edges weighted by elements of $\mathbb{N}_{>1}$ (no loops or multiple edges). The Artin group G_Γ has the following presentation:

- A generator g_v for each vertex v of Γ
- For each edge e with weight n and extremities v, w a relation

$$\underbrace{g_v g_w g_v \cdots}_{n \text{ times}} = \underbrace{g_w g_v g_w \cdots}_{n \text{ times}}$$

Theorem

Let $G_{p,q,r}$ the Artin group associated to a triangle with sides p, q, r

- *If $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1$ then there exists an affine curve $C_{p,q,r}$ such that $G_{p,q,r} = \pi_1(\mathbb{C}^2 \setminus C_{p,q,r})$*
- *The groups $G_{2,2n,2m}$, $2 \leq n \leq m$, $m > 2$, are not quasiprojective.*

Happy Birthday, Anatoly