LIB60BER
Topology of Algebraic Varieties
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On the topology of real Calabi-Yau varieties

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Plan of the talk

- Calabi-Yau varieties and Mirror Symmetry
- Anatoly and me
- Topology of Calabi-Yau varieties over $\mathbb{C}$:
  review of some known results and methods
- Topology of Calabi-Yau varieties over $\mathbb{R}$:
  recent research progress
  (joint work with Florian Schwerteck)

Calabi-Yau n-folds

Definition

A smooth complex projective algebraic variety X of dimension n is called Calabi-Yau n-fold if

- $\Omega_X^n \cong \mathcal{O}_X$ ($\Leftrightarrow K_X = 0$);
- $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$.

If $n = 1$ then X is an elliptic curve $X \cong \mathbb{C}/\Gamma$.
If $n = 2$ then X is called $K3$ surface.

Topology of Calabi-Yau varieties in dimensions 1 and 2

Every elliptic curve E over $\mathbb{C}$ is homeomorphic to the 2-dimensional torus $(S^1)^2$.

Every $K3$ surface S over $\mathbb{C}$ is homeomorphic to the surface obtained from the quotient $E^2/(\pm id)$ by blowing up 16 singular points coming from fixed points of the involution $x \to -x$ (E is an elliptic curve).

Open questions about the topology of Calabi-Yau n-folds ($n \geq 3$)

The topology of a Calabi-Yau n-fold X ($n \geq 3$) is not uniquely determined, because e.g. the Euler number $e(X)$ is not uniquely determined by n.

Open questions:

- Are there finitely many topological types of Calabi-Yau 3-folds X?
- Is the Euler number $e(X)$ of a Calabi-Yau n-fold X ($n \geq 3$) bounded by a constant depending only on n ?

Hodge numbers of complex manifolds

The Hodge numbers $h^{p,q}(X)$ of a smooth projective algebraic n-fold X is defined as

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^q(X, \Omega_X^p), \;\; 0 \leq p, q \leq n.$$

The Hodge numbers $h^{p,q}(X)$ allow to compute the Betti numbers

$$b^i(X) = \dim_C H^i(X, \mathbb{C}) = \sum_{p+q=i} h^{p,q}(X)$$

and the Euler number

$$e(X) = \sum_{i=0}^{2n} (-1)^i b_i = \sum_{p=0}^{n} \sum_{q=0}^{n} (-1)^{p+q} h^{p,q}(X).$$

Examples

If E is an elliptic curve then

$$h^{0,0}(E) = h^{1,0}(E) = h^{0,1}(E) = h^{1,1}(E) = 1.$$

if S is a $K3$ surface then

$$h^{0,0}(S) = h^{2,0}(S) = h^{0,2}(S) = h^{2,2}(S) = 1,$$

$$h^{1,0}(S) = h^{0,1}(S) = h^{2,1}(S) = h^{1,2}(S) = 0,$$

$$h^{1,1}(S) = 20.$$

Hodge numbers of Calabi-Yau 3-folds

If X is a Calabi-Yau 3-fold then

$$h^{0,0}(X) = h^{3,0}(S) = h^{0,3}(X) = h^{3,3}(X) = 1,$$
$$h^{1,0}(X) = h^{0,1}(X) = h^{2,0}(X) = h^{0,2}(X) = 0,$$
$$h^{1,3}(X) = h^{3,1}(X) = h^{2,3}(X) = h^{3,2}(X) = 0,$$

$$h^{1,1}(X) = h^{2,2}(X) = a(X), \;\; h^{2,1}(X) = h^{1,2}(X) = b(X),$$
$$b_0(X) = b_6(X) = 1, \;\; b_1(X) = b_5(X) = 0,$$
$$b_2(X) = a(X), \;\; b_3(X) = 2 + b(X),$$
$$e(X) = 2(a(X) - b(X)).$$

Examples of Calabi-Yau 3-folds:

- If $X \subset \mathbb{CP}_4$ is a smooth hypersurface of degree 5 then

$$a(X) = 1, \quad b(X) = 101, \quad e(X) = -200.$$

- If $X \subset \mathbb{CP}_5$ is a smooth complete intersection of two hypersurfaces of degree 3 then

$$a(X) = 1, \quad b(X) = 73, \quad e(X) = -144.$$

- If $X \subset \mathbb{CP}_2 \times \mathbb{CP}_2$ is a smooth hypersurface of bidegree $(3,3)$ then

$$a(X) = 2, \quad b(X) = 83, \quad e(X) = -162.$$

Mirror Symmetry

In the middle of 80's physicists working in String Theory became interested in Calabi-Yau 3-folds and at the end of 80's come to a conclusion that for any Calabi-Yau 3-fold X there must exist another Calabi-Yau 3-fold Y such that

$$a(X) = b(Y), \;\; b(X) = a(Y).$$

In particular, for the Euler numbers of X and Y one obtains

$$e(X) = 2(a(X) - b(X)) = 2(b(Y) - a(Y)) = -e(Y).$$

This conclusion of physicists was motivated by a phenomenon which they call Mirror Symmetry.

The first opinion of mathematicians

The first opinion of mathematicians about conclusion of physicists concering pairs of Calabi-Yau 3-folds X and Y satisfying the conditions

$$h^{p,q}(X) = h^{3-p,q}(Y), \ \ \forall p, q$$

was very skeptic.

The point was that there exist so called rigid Calabi-Yau 3-folds X having $b(X) = h^{2,1}(X) = 0$ which would imply that there exist also Calabi-Yau 3-folds Y having $a(Y) = h^{1,1}(Y) = 0$. The latter is impossible for any projective algebraic variety Y.

The calculation of Candelas et. al.

The opinion of mathematicians has changed completely after the pioneer paper of 4 physicists:

Ph. Candelas, X. de la Ossa, Green, Parkes, *A A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21-74.

This paper used Mirror Symmetry in order to predict the number n_d of rational curves of degree d on a generic Calabi-Yau quitic 3-fold in $\mathbb{C}P_4$ for any $d \geq 1$! Such a calculation was far beyond of abilities of mathematicians working in algebraic geometry.

Anatoly and me

The result of Candelas et. al. became a subject of very active discussions between mathematicians after the first Workshop on Mirror Symmetry in MSRI at the begining of 1991.
I was very impressed by the paper of Anatoly with Jeremy Teiltelbaum (1992) where the result of Candelas et.al. was generalized for complete intersections.
Mirror Symmetry was the reason why we met in Chicago in January 1993. I was very impressed by the paper of Anatoly and Jeremy, because it was a clear evidence that the mirror construction for hypersurfaces in toric varieties must have a generalisation to the case of complete intersections, but such a generalization was not clear to me in that time.

Reflexive polytopes

Let $M := \mathbb{Z}^n$ and $N := Hom(M, \mathbb{Z})$. We consider the real extensions $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ together with the canonical pairing

$$\langle *, * \rangle \ : \ M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}.$$

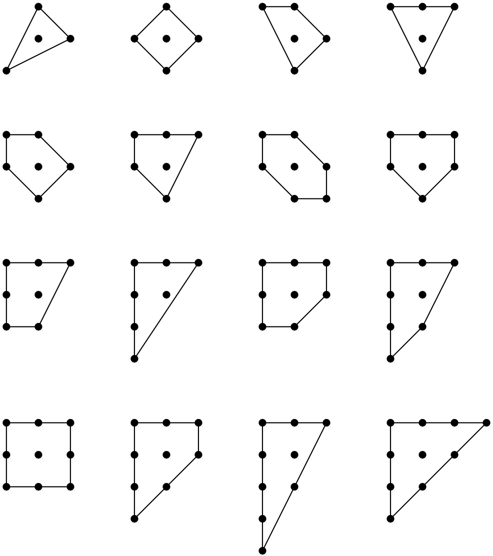
Definition

A convex n-dimensional polytope $\Delta \subset M_{\mathbb{R}}$ is called reflexive if it contains $0 \in M$ in its interior, all vertices of Δ belong to M and all vertices of the dual polytope

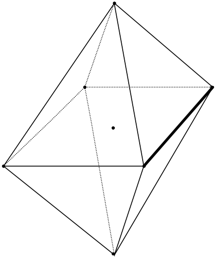
$$\Delta^* := \{y \in N_{\mathbb{R}} \ : \ \langle x, y \rangle \geq -1 \forall x \in \Delta\}$$

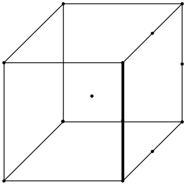
belong to N.

All 16 reflexive polygons up to isomorphism



A pair of dual to each other 3-dimesional reflexive polytopes





Mirror duality for Calabi-Yau 3-folds

Theorem

For any Calabi-Yau 3-fold $\hat{Z}_f$ *defined by a* Δ*-regular Laurent polynomial* f *whose Newton polyhedron is a reflexive 4-dimensional polyhedron* Δ*, one has the following formulas for the Hodge numbers*

$$h^{1,1}(\hat{Z}_f)$$

and

$$h^{2,1}(\hat{Z}_f)$$

are permuted by the duality $\Delta \leftrightarrow \Delta^*$*.*

Real Calabi-Yau varieties

By a real Calabi-Yau *n*-fold $X(\mathbb{R})$ we mean a smooth irreducible projective algebraic variety X defined over $\mathbb{R}$ such that it complex extension $X_{\mathbb{C}}$ is a complex projective Calabi-Yau *n*-fold. We remark that $X(\mathbb{R})$ is always an oriented real manifold.

Motivations:

- The set of real points $X(\mathbb{R})$ is a special Lagrangian *n*-cycle in $X_{\mathbb{C}}$ which play important role in Homological Mirror Symmetry;
- It would be interesting to develope methods for counting rational curves on real Calabi-Yau 3-folds.

Real elliptic curves

There exist only 3 topological possibilities for real elliptic curves $E(\mathbb{R})$:

$$\emptyset, S^1, 2S^1.$$

If E is defined by an equation $y^2 = P_4(x)$ where P_4 is a polynomial of degree 4 then

- $E(\mathbb{R}) = \emptyset$ if $P_4 = -x^4 - 1$;
- $E(\mathbb{R}) \cong_{\mathrm{top}} S^1$ if $P_4 = x^4 - 1$;
- $E(\mathbb{R}) \cong_{\mathrm{top}} 2S^1$ if $P_4 = (x^2 - 1)(x^2 - 2)$.

Real $K3$ surfaces

Topological types of real $K3$ surfaces $S(\mathbb{R})$ have been completely classified in works of Kharlamov and Nikulin about 30 years ago. They have shown that there exist exactly 66 topologically different types for $S(\mathbb{R})$. The number of connected components of $S(\mathbb{R})$ can be at most 10 and apart from one exception $(2(S^1 \times S^1))$ only one connected component of $S(\mathbb{R})$ may have positive genus.

Our aim

Many examples of topologically different Calabi-Yau 3-folds can be constructed from hypersurfaces in toric varieties. On the other hand there exists a powerful method due to Oleg Viro which allows to control the topology of real hypersurfaces in toric varieties. The aim ist to use this combinatorial patchworking method of Viro in order to construct examples of real Calabi-Yau 3-folds and to compute their topological invariants from the combinatorial data.

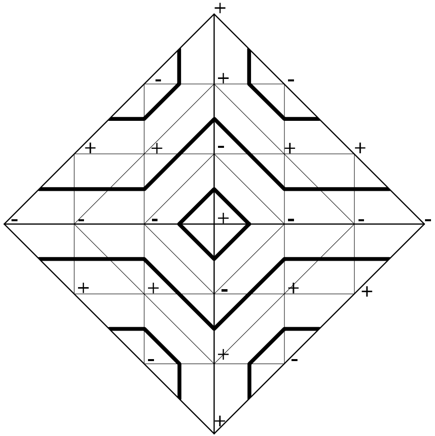
The patchworking method of Viro

The patchworking method of Viro turned out to be very helpful for constructing algebraic curves in $\mathbb{P}_2(\mathbb{R})$ with a prescribed isotopy class.
However this method works for real hypersurfaces (and complete intersections) in toric varieties of arbitrary dimension.
The main idea of this method is to use a moment map

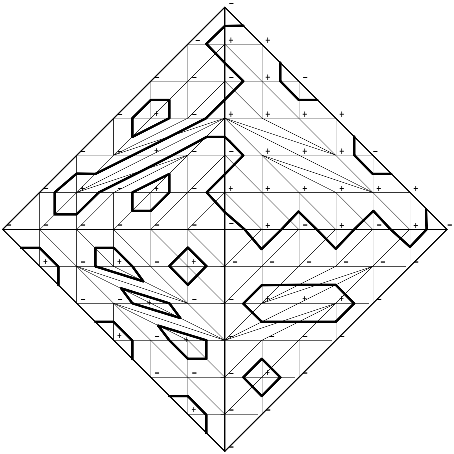
$$\mathbb{P}_\Delta(\mathbb{R}) \to \Delta$$

and a triangulation of Δ in order to degenerate $\mathbb{P}_\Delta(\mathbb{R})$ into a union of very simple toric varieties $\mathbb{P}_{\Delta_i}(\mathbb{R})$ (real weighted projective spaces). Such a degeneration induces a degeneration of a real hypersurface $X \subset \mathbb{P}_\Delta(\mathbb{R})$ into a union of very simple pieces X_i. This allows to reconstruct the topology of X via glueing of X_i.

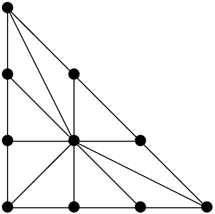
Topological model of a real cubic curve



Topological model of a degree 6 curve in $\mathbb{P}_2(\mathbb{R})$



An unimodular triangulation of a reflexive polygon



Theorem

Let Δ be a 3-dimensional reflexive polytope together with a unimodular triangulation and let $\widehat{Z}(\mathbb{R}) \subset \widehat{\mathbb{P}_{\Delta}}(\mathbb{R})$ be a generic real $K3$ surface assigned to some sign fuction. Then $\widehat{Z}(\mathbb{R})$ is homeomorphic to the disjoint union of a 2-dimensional sphere and an oriented surface of genus 10.

Idea of proof: One computes the Euler number of $\widehat{Z}(\mathbb{R})$ and shows that it equals -16. Moreover, one can compute the Betti numbers with the coefficients in $\mathbb{Z}/2\mathbb{Z}$:

$$b_0 = 2, b_1 = 20, b_2 = 2.$$

This shows that $\widehat{Z}(\mathbb{R})$ consists of 2 connected components. Together with the classification of Nikulin and Kharlamov this implies a unique possibility for $\widehat{Z}(\mathbb{R})$.

Further questions

- It is not difficult to show that for any unimodular triangulation of a 4-dimensional reflexive poltope Δ the corresponding Calabi-Yau 3-fold $\widehat{Z}(\mathbb{R}) \subset \widehat{\mathbb{P}_\Delta}(\mathbb{R})$ consists of 2 connected components. What oriented 3-dimensional manifolds could appear in this way?
- Compute the orbifold elliptic genus of the real Calabi-Yau variety $\widehat{Z}(\mathbb{R}) \subset \widehat{\mathbb{P}_\Delta}(\mathbb{R})$ via combinatorial invariants of the n-dimensional reflexive polytope Δ in any dimension $n \geq 4$.

A. Libgober, *Elliptic genera, real algebraic varieties and quasi-Jacobi forms*, arXiv: 0904.1026

Happy Birthday, Anatoly!