Morse theory for plane algebraic curves

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Maciej Borodzik

Institute of Mathematics, University of Warsaw

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Setup

Our setup is the following

- $C \subset \mathbb{C}^2$ algebraic curve
- Intersect C with a sphere S_r of radius r.
- Links for small r are understood.
- Links at infinity are understood.
- Study properties of C by these links.

What happens with L_r if we change r?

Motto

C introduces a "cobordism" between links of singular points and the link at infinity.

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Classical arguments from Morse theory

Lemma

If for all $r \in [r_1, r_2]$, C is transverse to S_r, then L_{r_1} is isotopic to L_{r_2}.

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Crossing a non transversality point is either 0, or 1 or 2 – handle addition to the link.

Classical Morse theory

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Handles

Adding handles mean

- 0-handles: adding an unknot to L_r.
- 2-handles: deleting an unknot to L_r.
- 1-handles: adding a band.

Lemma

If C is a complex curve, there are no 2-handles.

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What about singularities

Crossing a singular point of multiplicity m can be viewed as follows

- Take a disconnected sum of L_r with a link of singularity...
- And then join them with precisely m one handles.

Example

Passing through a double point corresponds to changing an undercrossing to an overcrossing on some planar diagram of the link.

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Examples

Now, please, hold Your breath, I will try to show some real pictures.

Knot invariants

Take your favourite link invariant such that

- It is computable for many algebraic knots
- You can control its changes when adding a handle
- It is not too good. It is not equal to genus for positive links.

And this invariant yields obstruction for the existence of a plane curve with given singularities.

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My favourite invariant up to now is is

Tristram–Levine signature

Definition

If S is Seifert matrix of the link L and $|\zeta| = 1$, then $\sigma_L(\zeta)$ is the signature of the form

$$(1-\zeta)S + (1-\bar{\zeta})S^T.$$

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Theorem

If $L_1, \ldots, L_n$ are links of singular points of C, L_∞ is a link at infinity, then for almost all ζ

$$|\sigma_{L_\infty}(\zeta) - \sum_{k=1}^{n} \sigma_{L_k}(\zeta)| \leq b_1(C).$$

In the proof we use the absence of 2–handles, but this can be done in general, i.e. non-complex case, too.

Corollaries

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Applications

- A polynomial curve of bidegree (m, n), having an A_{2k} singularity at the origin, has $k \lesssim \frac{1}{4}mn$.
- BMY-like inequality for polynomial curves.
- Possible proof of Zajdenberg–Lin theorem using the fact that $b_1(C) = 0$ and relations among signatures of torus knots.
- Studying deformations of singular points: we get new relations.
- Find maximal number of cusps on a curve in $\mathbb{C}P^2$ of degree d.
- Possible ways to improve everything if we apply better invariants.

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Thank You!