Euler-Poincaré Characteristic, Todd genus and signature of singular varieties

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Preamble: 1. Euler - Poincaré Characteristic

Definition (Poincaré)

Let \( X \) be a triangulated compact (smooth or singular) variety, the **Euler - Poincaré characteristic** of \( X \) is defined as

\[
e(X) = \sum_{i=0}^{m} (-1)^i k_i
\]

where \( m = \dim_{\mathbb{R}} X \) and \( k_i \) is the number of \( i \)-dimensional simplexes.
Example 1 (Lhuilier)

Let $X$ be a complex algebraic curve, i.e. a compact Riemann surface. $X$ is homeomorphic to a sphere with $g$ handles. The Euler–Poincaré characteristic of $X$ is

$$e(X) = 2 - 2g.$$
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- 2-dimensional sphere: $e(S^2) = 2$,
- 2-dimensional torus: $e(T) = 0$,

Example 2

The Euler - Poincaré characteristic of the pinched torus is $e(P) = 1$. 
Theorem (Poincaré-Hopf)

Let $X$ be a compact manifold and let $v$ be a (continuous) vector field with (finitely many) isolated singularities $(a_j)_{j \in J}$ of index $I(v, a_j)$, then

$$e(X) = \sum_{j \in J} I(v, a_j).$$
Preamble : 2. The arithmetic genus

Let $X$ be a complex algebraic manifold, $n = \dim_{\mathbb{C}} X$. Let $g_i$ be the number of $\mathbb{C}$-linearly independent holomorphic differential $i$-forms on $X$.

**Definition (Arithmetic Genus)**

The **arithmetic genus** of $X$ is defined as :

$$\chi(X) := \sum_{i=0}^{n} (-1)^i g_i$$
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- $g_n$ is called geometric genus of $X$,
- $g_1$ is called irregularity of $X$,

**Definition (Arithmetic Genus)**

*The arithmetic genus of $X$ is defined as :

$$\chi(X) := \sum_{i=0}^{n} (-1)^i g_i$$*
Example

Let $X$ be a complex algebraic curve, i.e. a compact Riemann surface. $X$ is homeomorphic to a sphere with $g$ handles. Then $g_0 = 1$ and $g_1 = g_n = g$.

The arithmetic genus of $X$ is:

$$\chi(X) = 1 - g$$
The Todd genus $T(X)$ has been defined (by Todd) in terms of Eger-Todd fundamental classes (polar varieties), using Severi results. The Eger-Todd classes are homological Chern classes of $X$.

Todd “proved” that

$$T(X) = \chi(X).$$

In fact, the Todd proof uses a Severi Lemma which has never been completely proved. The Todd result has been proved by Hirzebruch.
Preamble : 5. The signature

Definition (Thom-Hirzebruch)

Let $M$ be a (real) compact oriented $4k$-dimensional manifold. Let $x$ and $y$ two elements of $H_{2k}(M; \mathbb{R})$, then

$$\langle x \cup y, [M] \rangle \in \mathbb{R}$$

defines a bilinear form on the vector space $H_{2k}(M; \mathbb{R})$. The index (or signature) of $M$, denoted by $\text{sign}(M)$, is defined as the index of this form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.
What shall we do?

<table>
<thead>
<tr>
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Hirzebruch Theory
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Hirzebruch Theory  
Motivic Theory (BSY)
Hirzebruch Series

\[ Q_y(\alpha) := \frac{\alpha(1 + y)}{1 - e^{-\alpha(1 + y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]] \]
Hirzebruch Series

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- \[ Q_1(\alpha) = \frac{\alpha}{\tanh \alpha} \quad y = 1 \]
Characteristic Classes of Manifolds.

Let $X$ be a complex manifold with dimension $\dim_{\mathbb{C}} X = n$, let us denote by

$$c^*(TX) = \sum_{j=0}^{n} c^j(TX), \quad c^j(TX) \in H^{2j}(X; \mathbb{Z})$$

the total Chern class of the (complex) tangent bundle $TX$.

**Definition**

The Chern roots $\alpha_i$ of $TX$ are defined by:

$$\sum_{j=0}^{n} c^j(TX) t^j = \prod_{i=1}^{n} (1 + \alpha_i t)$$

$$\alpha_i \in H^2(X; \mathbb{Z}).$$
One defines the Todd-Hirzebruch class: 

$$ \widetilde{td}_y(TX) := \prod_{i=1}^{n} Q_y(\alpha_i) $$ 

$$ c^*(TX) = \prod_{i=1}^{n} (1 + \alpha_i) y = -1 $$

Todd class,

$$ L^*(TX) = \prod_{i=1}^{n} (\alpha_i \tanh \alpha_i) y = 1 $$

Thom-Hirzebruch L-class.
One defines the Todd-Hirzebruch class: $\widetilde{td}_{(y)}(TX) := \prod_{i=1}^{n} Q_{y}(\alpha_{i})$
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\begin{align*}
\overset{\sim}{td}(y)(TX) &= \left\{ \begin{array}{l}
c^*(TX) = \prod_{i=1}^{n} (1 + \alpha_i) \\
y = -1 \\
\text{Chern class,}
\end{array} \right.
\end{align*}
\]
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\tilde{td}_{(y)}(TX) &= \begin{cases} 
    c^*(TX) &= \prod_{i=1}^{n} (1 + \alpha_i) & y = -1 \\
    td^*(TX) &= \prod_{i=1}^{n} \left( \frac{\alpha_i}{1 - e^{-\alpha_i}} \right) & y = 0
\end{cases}
\end{align*}
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Chern class,

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\end{array} \right.$$
The $\chi_y$-characteristic

Let $X$ be a complex projective manifold.

**Definition**

One defines the $\chi_y$-characteristic of $X$ by

$$
\chi_y(X) := \sum_{p=0}^{<\infty} \left( \sum_{i=0}^{<\infty} (-1)^i \dim_{\mathbb{C}} H^i(X, \wedge^p T^*X) \right) \cdot y^p
$$

$\chi_0(X) = \chi(X)$, arithmetic genus of $X$ (definition)

$\chi_1(X) = \sign(X)$, signature of $X$ (Hodge)
The $\chi_y$-characteristic

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- $y = -1$ \quad $\chi_{-1}(X) = e(X)$, Euler - Poincaré characteristic of $X$ (Hodge)
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\[ \quad c^*(TX) \]
\[ \quad L^*(TX) \]

**Hirzebruch Riemann-Roch Theorem**

One has:

\[ \chi_y(X) = \int_X \tilde{td}_y(TX) \cap [X] \in \mathbb{Q}[y]. \]
The three particular cases

\[ \chi(X) = \int_X c^* (\mathcal{T}X) \cap [X] = -1 \]

Euler-Poincaré characteristic of \( X \)

\[ \chi(X) = \int_X td^* (\mathcal{T}X) \cap [X] = 0 \]

Poincaré-Hopf Theorem

\[ \text{sign}(X) = \int_X L^* (\mathcal{T}X) \cap [X] = 1 \]

Arithmetic genus of \( X \)

\[ \text{Hirzebruch-Riemann-Roch Theorem} \]

\[ \text{signature of } X \]

\[ \text{Hirzebruch signature Theorem} \]
The three particular cases

\[ e(X) = \int_X c^*(TX) \cap [X] \]  
Euler - Poincaré characteristic of \( X \)

*Poincaré-Hopf Theorem*
The three particular cases

- $e(X) = \int_X c^*(TX) \cap [X]$  
  Euler - Poincaré characteristic of $X$
  \textit{Poincaré-Hopf Theorem}

- $\chi(X) = \int_X td^*(TX) \cap [X]$  
  arithmetic genus of $X$
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The three particular cases

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  arithmetic genus of \( X \) 
  \textit{Hirzebruch-Riemann-Roch Theorem}

- \( \text{sign}(X) = \int_X L^*(TX) \cap [X] \) 
  signature of \( X \) 
  \textit{Hirzebruch signature Theorem}
Question

What can we do for singular varieties?
Three generalisations in the case of singular varieties.
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Chern Transformation (MacPherson)

\[ F(X) : \text{Group of constructible functions (ex. } 1_X) \]

\[ c_* : F(X) \to H_*(X) \]

One defines \( c_*(X) := c_*(1_X) : \text{Schwartz-MacPherson class of } X. \)
Three generalisations in the case of singular varieties.

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<td>$\mathbb{F}(X)$: Group of constructible functions (ex. $1_X$)</td>
<td>$G_0(X)$: Grothendieck Group of coherent sheaves (ex. $\mathcal{O}_X$)</td>
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<td>$c_* : \mathbb{F}(X) \rightarrow H_*(X)$</td>
<td>$td_* : G_0(X) \rightarrow H_*(X) \otimes \mathbb{Q}$</td>
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<td>One defines $c_<em>(X) := c_</em>(1_X)$: Schwartz-MacPherson class of $X$.</td>
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Three generalisations in the case of singular varieties.

**Chern Transformation (MacPherson)**

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\[ c_* : \mathbb{F}(X) \to H_*(X) \]

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**Todd Transformation (Baum-Fulton-MacPherson)**

\[ G_0(X) : \text{Grothendieck Group of coherent sheaves (ex. } \mathcal{O}_X) \]

\[ td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q} \]

One defines \( td_*(X) := td_*([\mathcal{O}_X]) \).

**L-Transformation (Cappell-Shaneson)**

\[ \Omega(X) : \text{Group of constructible self-dual sheaves (ex. } \mathcal{IC}_X) \]

\[ L_* : \Omega(X) \to H_{2*}(X; \mathbb{Q}) \]

One defines \( L_*(X) := L_*([\mathcal{IC}_X]) \).
The three transformations are defined on different spaces:

\[ \mathcal{F}(X), \quad G_0(X) \quad \text{and} \quad \Omega(X) \]
Where the “motivic” arrives...

**Definition**

The Grothendieck relative group of algebraic varieties over $X$ $K_0(\text{var}/X)$ is the quotient of the free abelian group of isomorphism classes of algebraic maps $Y \to X$, modulo the "additivity relation":

$$[Y \to X] = [Z \to Y \to X] + [Y \to Z \to X]$$

for closed algebraic sub-spaces $Z$ in $Y$. 
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**Definition**

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$$K_0(\text{var}/X)$$

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$$[Y \rightarrow X] = [Z \rightarrow Y \rightarrow X] + [Y \setminus Z \rightarrow Y \rightarrow X]$$

for closed algebraic sub-spaces $Z$ in $Y$. 
Three results…
Theorem

The map $e : K_0(var/X) \rightarrow \mathcal{F}(X)$ defined by $e([f : Y \rightarrow X]) := f!1_Y$ is the unique group morphism which commutes with direct images for proper maps and such that $e([id_X]) = 1_X$ for $X$ smooth and pure dimensional.
Theorem

The map $e : K_0(var/X) \longrightarrow \mathbb{F}(X)$ defined by $e([f : Y \to X]) := \tilde{f}_! 1_Y$ is the unique group morphism which commutes with direct images for proper maps and such that $e([id_X]) = 1_X$ for $X$ smooth and pure dimensional.

Theorem

There is an unique group morphism $mC : K_0(var/X) \longrightarrow G_0(X)$ which commutes with direct images for proper maps and such that $mC([id_X]) = [\mathcal{O}_X]$ for $X$ smooth and pure dimensional.
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Theorem

The morphism \( sd : K_0(\text{var}/X) \longrightarrow \Omega(X) \) defined by

\[
sd([f : Y \to X]) := [Rf_* \mathbb{Q}_Y [\dim_{\mathbb{C}}(Y) + \dim_{\mathbb{C}}(X)]]
\]

is the unique group morphism which commutes with direct images for proper maps and such that \( sd([\text{id}_X]) = [\mathcal{O}_X [2 \dim_{\mathbb{C}}(X)]] = [\mathcal{I} \mathcal{C}_X] \) for \( X \) smooth and pure dimensional.
Theorem

There is an unique group morphism

\[ T_y : K_0(\text{var}/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y] \]

which commutes with direct images for proper maps and such that

\[ T_y([\text{id}_X]) = \tilde{td}_y(TX) \cap [X] \quad \text{for } X \text{ smooth and pure dimensional.} \]
Theorem

There is an unique group morphism

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which commutes with direct images for proper maps and such that
\[ T_y([id_X]) = \tilde{td}(y)(TX) \cap [X] \]
for \( X \) smooth and pure dimensional.

In particular, one has: \( T_{-1}([id_X]) = c_*(X) \)
Theorem

There is an unique group morphism

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which commutes with direct images for proper maps and such that

\[ T_y([id_X]) = \tilde{\text{td}}(y)(TX) \cap [X] \]

for \( X \) smooth and pure dimensional.

In particular, one has: \( T_{-1}([id_X]) = c_\ast(X) \)

Remark

If a complex algebraic variety \( X \) has only rational singularities (for example if \( X \) is a toric variety), then:

\[ mC([id_X]) = [\mathcal{O}_X] \in G_0(X) \]

and in this case \( T_0([id_X]) = td_\ast(X) \).

That is not true in general!
Verdier Riemann-Roch Formula

Let \( f : X' \rightarrow X \) be a smooth map (or a map with constant relative dimension), then one has

\[
\tilde{td}(y)(T_f) \cap f^* T_y([Z \rightarrow X]) = T_yf^*([Z \rightarrow X]).
\]

Here \( T_f \) is the bundle over \( X' \) of tangent spaces to fibres of \( f \).
Still for specialists...

Let us define $td_{1+y}([F]) := \sum_{i=0}^{\infty} \widetilde{td}_i([F]) \cdot (1 + y)^{-i}$. 
Let us define \( td_{(1+y)}([\mathcal{F}]) := \sum_{i=0}^{\infty} \tilde{td}_i([\mathcal{F}]) \cdot (1 + y)^{-i} \).

Then one has:

**Factorisation of \( T_y \)**

\[ T_y = td_{(1+y)} \circ mC : K_0(var/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]. \]
The main result

The following diagrams commute:

\[
\begin{array}{ccc}
F(X) & \xleftarrow{e} & K_0(\text{var}/X) \\
& & \xrightarrow{mC} \\
& & G_0(X) \\
& & \downarrow \text{sd} \\
c_* & \downarrow & T_y & \downarrow & \Omega(X) & \downarrow & \text{td}_* \\
& & & & & & \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] \\
& & & & & & \\
& & & & & & \downarrow \text{L}_* & \xrightarrow{y=0} \\
& & & & & & & \downarrow y=1 & \xrightarrow{y=1} \\
& & & & & & & H_*(X) \otimes \mathbb{Q}
\end{array}
\]
The main result

Theorem

The following diagrams commute:

\[
\begin{align*}
\mathbb{F}(X) & \xleftarrow{e} K_0(\text{var}/X) & mC & \rightarrow G_0(X) \\
c_* \downarrow & & & sd \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} H_*(X) \otimes \mathbb{Q}[y] & L_* \downarrow & y=0 \\
& & H_*(X) \otimes \mathbb{Q} & y=1 \\
& & H_*(X) \otimes \mathbb{Q}
\end{align*}
\]
The main result

Theorem

The following diagrams commute:

\[ F(X) \xleftarrow{e} K_0(\text{var}/X) \xrightarrow{mC} G_0(X) \]

\[ \begin{array}{ccccccc}
F(X) & \xrightarrow{e} & K_0(\text{var}/X) & \xrightarrow{mC} & G_0(X) \\
\downarrow{c_*} & & \downarrow{T_y} & & \downarrow{\Omega(X)} & & \downarrow{td_*} \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] & \xrightarrow{y=0} & H_*(X) \otimes \mathbb{Q} \\
\downarrow{y=1} & & \downarrow{y=0} & & \downarrow{y=1} \\
H_*(X) \otimes \mathbb{Q} & & & & \end{array} \]
The main result

**Theorem**

The following diagrams commute:

\[ \begin{array}{ccc}
F(X) & \xleftarrow{e} & K_0(\text{var}/X) \\
\downarrow{c_*} & & \downarrow{T_y} \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] \\
\downarrow{L_*} & & \downarrow{y=0} \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=1} & H_*(X) \otimes \mathbb{Q} \\
\end{array} \rightarrow \begin{array}{ccc}
& & G_0(X) \\
& \downarrow{sd} & \\
& \downarrow{\Omega(X)} & \\
& \downarrow{td_*} & \\
& H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=0} \\
\end{array} \]

Jean-Paul Brasselet (CNRS - Marseille)

Euler-Poincaré, Todd and signature

Jaca, 22 June 2009
The main result

\[ \mathbb{F}(X) \leftrightarrow e \quad K_0(\text{var}/X) \quad mC \quad G_0(X) \]

\[ c_* \downarrow \quad T_y \downarrow \quad \Omega(X) \quad \downarrow td_* \]

\[ H_*(X) \otimes \mathbb{Q} \quad y=-1 \quad H_*(X) \otimes \mathbb{Q}[y] \quad L_* \downarrow \quad y=0 \quad H_*(X) \otimes \mathbb{Q} \]

\[ y=1 \quad H_*(X) \otimes \mathbb{Q} \]

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Thanks for your attention

Happy birthday Anatoly