Topology at infinity of polynomials and Danielewski surfaces

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NEWTON TREES: 1) Newton trees for $\mathbb{C}[x^{-1}][[x,y]]$, 2) Newton trees for $\mathbb{C}[x,y]$.

CLASSIFICATION OF RATIONAL POLYNOMIALS 1) Results of Miyanishi, Sugie and Neuman, Norbury, 2) Results of Sasao.

STUDY OF $\mathbb{C}[y, xy, x^2y + a_1x, \cdots, x^{k-1}y + \cdots + a_{k-2}x, x\phi(x^{k-1}y + \cdots + a_{k-2}x)]$

NEWTON TREES FOR $\mathbb{C}[x^{-1}][[x,y]]$

Newton algorithm

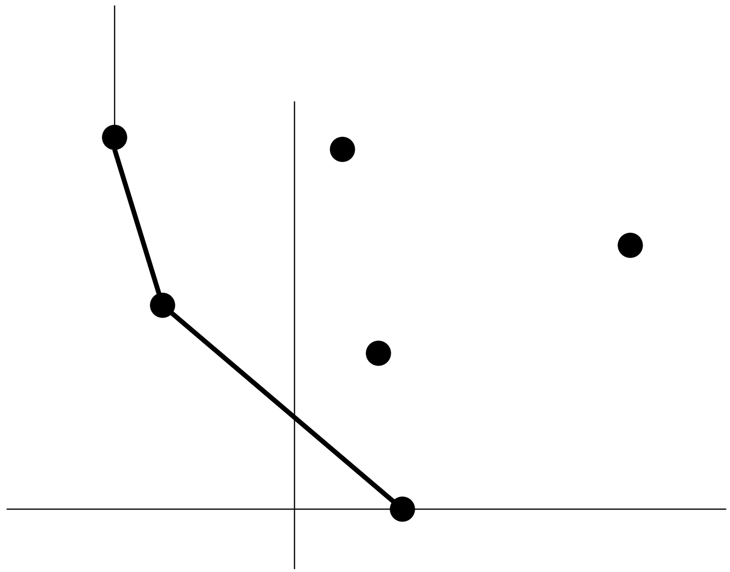
Let

$$f(x,y) = \sum_{(\alpha,\beta)\in\mathbb{Z}\times\mathbb{N}} c_{\alpha,\beta}x^{\alpha}y^{\beta} \in \mathbb{C}[x^{-1}][[x,y]]$$

We define

$$\mathrm{Supp}f = \{(\alpha,\beta)\in\mathbb{Z}\times\mathbb{N} | c_{\alpha,\beta} \neq 0\}.$$

Let $\Delta(f) = \Delta(\mathrm{Supp}f)$ and $\mathcal{N}(f) = \mathcal{N}(\Delta(f))$.



For all $S \in \mathcal{N}(f)$ let in$(f, S) = \sum_{(\alpha,\beta)\in S} c_{\alpha,\beta} x^{\alpha} y^{\beta}$, be the *face polynomial*.

If l_S has equation $p\alpha + q\beta = N$, with gcd$(p, q) = 1$, then in$(f, S) = x^{a_S} y^{b_S} F_S(x^q, y^p)$ where $F_S(x, y) = \prod_i (x - \mu_i y)^{\nu_i}$.

We say that $f \in \mathbb{C}[x^{-1}][[x, y]]$ is in *good coordinates* if

1. $\mathcal{N}(f)$ doesn't hit the x-axis or

2. if $\mathcal{N}(f)$ hits the x-axis and if S_m is the corresponding face

   (a) either l_{S_m} has equation $p\alpha + q\beta = N$ with $p \neq 1$ or

   (b) if $p = 1$,then F_{S_m} has at least two factors.

Lemma 1 *If* $f \in \mathbb{C}[x^{-1}][[x,y]]$ *is not in good coordinates, there exist changes of variables in* $\mathbb{C}[[x,y]]$ *in which it is in good coordinates.*

Idea: If $\mathrm{in}(f, S_m) = x^{a_{S_m}}(x^q - \mu y)^{\nu}$ the face disappears by a change of coordinates.

Let $(p,q) \in \mathbb{N}^2$, gcd$(p,q) = 1$. Let $(p',q') \in \mathbb{N}^2$ such that $qq' - pp' = 1$. Let $\mu \in \mathbb{C}^*$. We define

$$\begin{array}{rcccc} \sigma_{(p,q,\mu)} & : & \mathbb{C}[[x,y]] & \longrightarrow & \mathbb{C}[[x_1,y_1]] \\ & & f(x,y) & \mapsto & f(\mu^{q'}x_1^p, x_1^q(y_1+\mu^{p'})) \end{array}$$

We say that $\sigma_{(p,q,\mu)}$ is a *Newton map*.

The "interesting" Newton maps correspond to (p,q) such that $px+qy = N$ is the equation of a face S of $\mathcal{N}(f)$ and μ a root of F_S.

Let $f \in \mathbb{C}[x^{-1}][[x,y]]$ in good coordinates. Let $\sigma = \sigma_{(p,q,\mu)}$ be a Newton map. We denote by f_σ the result of $\sigma(f)$ after a change of variables so that f_σ is in good coordinates. Let $\Sigma_n = (\sigma_1, \cdots, \sigma_n)$ where σ_i is a Newton map for all i, we define f_{Σ_n} by induction: $f_{\Sigma_1} = f_{\sigma_1}$, $f_{\Sigma_i} = (f_{\Sigma_{i-1}})_{\sigma_i}$.

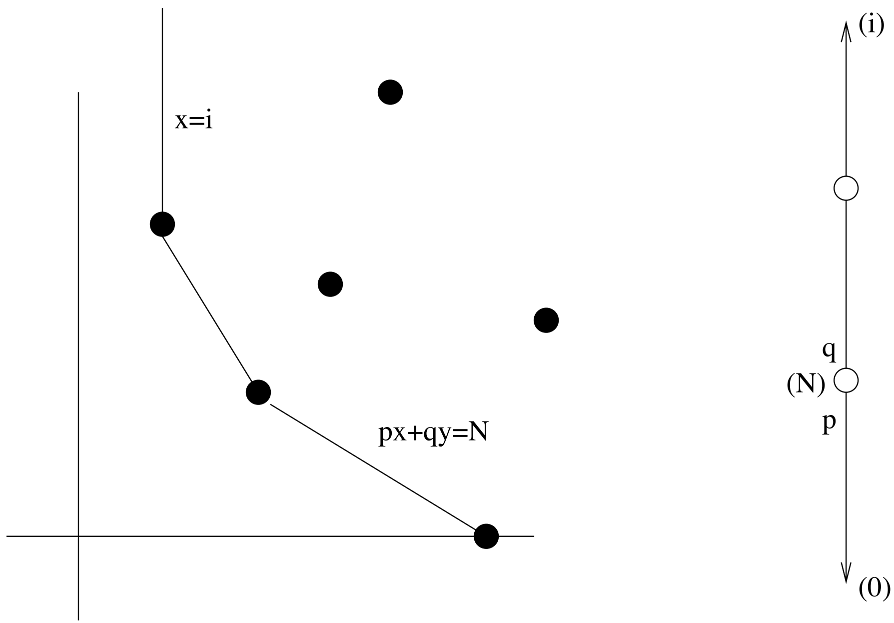
Theorem 2 *For all $f(x,y) \in \mathbb{C}[x^{-1}][[x,y]]$ the set of $n \in \mathbb{N}$ such that there exists $\Sigma_n = (\sigma_1, \cdots, \sigma_n)$ where σ_i is a Newton map for all i, and f_{Σ_n} is a monomial up to a unit in $\mathbb{C}[x^{-1}][[x,y]]$ is bounded in $\mathbb{N}$.*

If f is in good coordinates, we define the *depth* of f, $d(f)$ by induction. If f is a monomial up to a unit, then $d(f) = 0$. Otherwise $d(f) = \max d(f_\sigma) + 1$ where the maximum is taken over all faces S of the Newton polygon and all roots of F_S. Note that the definition of the depth depends on the choice of good coordinates at each step of the Newton algorithm.

Newton trees

Given $f \in \mathbb{C}[x^{-1}][[x, y]]$ in good coordinates, the *Newton process* consists in applying successive Newton maps attached to successive Newton polygons until the result is a monomial times a unit.

Newton trees are trees that encode the Newton process. They are build by induction. First of all, we need to build a graph associated to a Newton diagram.



We can recover the whole Newton polygon from the graph since we can read the equations of the supporting lines of the faces on the graph.

Newton tree of $f \in \mathbb{C}[x^{-1}][[x,y]]$

We assume that f is in good coordinates. We build the Newton tree of f by induction on the depth.

Assume that f has depth 0, i.e. $f = x^a y^b u$. Its Newton tree is the graph of its Newton polygon, i.e. an edge with two arrows decorated with (a) and (b).

Assume that we have constructed the Newton tree for all f of depth less or equal to $n-1$.

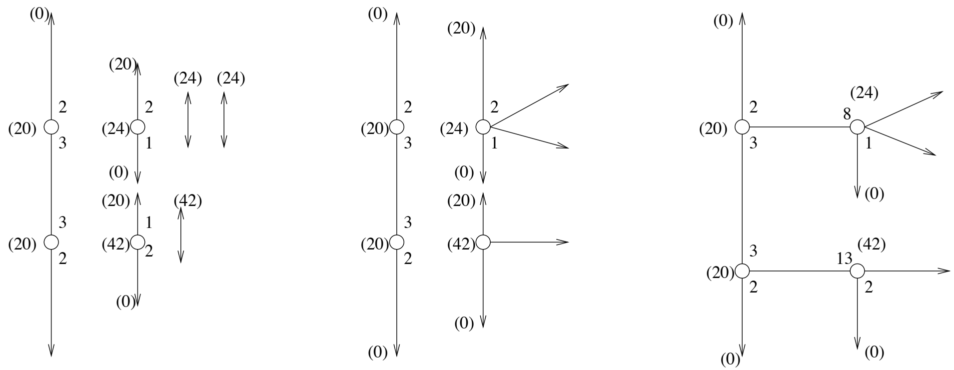
Let $f \in \mathbb{C}[x^{-1}][[x,y]]$ in good coordinates of depth n. On one hand we can construct the graph associated to its Newton polygon. On the other hand, for each face of the Newton polygon and each root of the face polynomials the resulting functions are of depth at most $n-1$. Then by the hypothesis, we can construct their Newton trees.

To obtain the Newton tree of f, we will glue the Newton trees of f_σ to the graph of the Newton polygon of f, removing the top arrow and gluing the edge to the corresponding vertex. The decorations on the edges change during the glueing, but not the decorations of vertices.

In the process, the edges that are glued become *horizontal edges*. All others are *vertical edges*. The vertices on a vertical line correspond to faces on a Newton polygon, the edges on an horizontal line are produced with successive Newton maps. To each vertex corresponds a face polynomial. The number of roots of this face polynomial is the number of horizontal edges arising from the vertex.

If we forget about horizontal and vertical edges, the Newton trees are the same as Eisenbud and Neumann diagrams.

As an example the following is the Newton tree of $f(x,y) = (x^2 - y^3)^2(x^3 - y^2)^2 + x^6y^3 + x^5y^5 + x^4y^7$

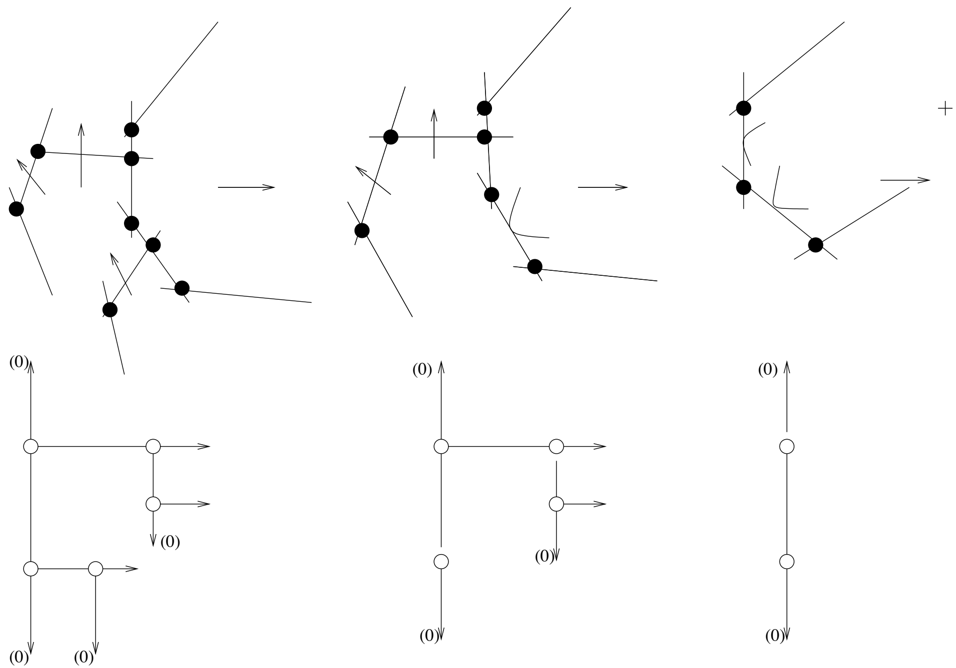


We have the following interpretation of Newton trees:

Theorem 3 *(C.-N., Libgober) There exist a normal variety* $\overline{X}$ *with at most quotient singularities and a proper morphism* $\overline{\pi} : \overline{X} \to \mathbb{C}^2$ *such that*

$$\overline{\pi}^* f = \sum \tilde{C}_n + \sum N_m E_m$$

such that the strict transforms of f *are smooth, this divisor has normal crossing and the Newton tree of* f *is the dual graph of this divisor and the* N_m *are the decorations of the corresponding vertices. The singularities of* $\overline{X}$ *are on the intersections of the divisors* E_m *and they can be computed from the Newton tree.*



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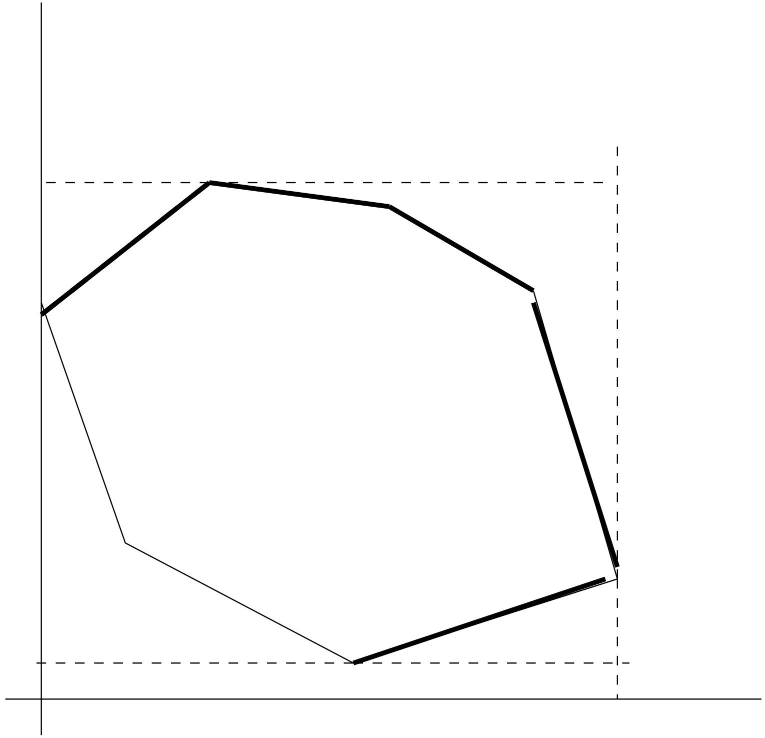
NEWTON TREES FOR $\mathbb{C}[x, y]$

Let

$$f(x, y) = \sum_{(\alpha,\beta)\in\mathbb{N}^2} c_{\alpha,\beta} x^{\alpha} y^{\beta} \in \mathbb{C}[x, y]$$

We consider the convex hull of Supp$f$. The Newton polygon at infinity is the bolded part.



We write equations of faces:

1. For $S \in \mathcal{N}_{0,\infty}$: $px + qy = N, p \leq 0, q > 0$

2. For $S \in \mathcal{N}_{\infty,\infty}$: $px + qy = N, p > 0, q > 0$

3. For $S \in \mathcal{N}_{\infty,0}$: $px + qy = N, p > 0, q \leq 0$

Let $(p,q) \in \mathbb{N}^2$, gcd$(p,q) = 1$. Let $(p',q') \in \mathbb{N}^2$ such that $qq' - pp' = 1$. Let $\mu \in \mathbb{C}^*$. We define

$$\begin{array}{rcccc} \sigma_{(p,q,\mu)} & : & \mathbb{C}[x,y] & \longrightarrow & \mathbb{C}[x_1^{-1}][x_1, y_1] \\ & & f(x,y) & \mapsto & f(\mu^{q'} x_1^{-p}, x_1^{-q}(y_1 + \mu^{p'})) \end{array}$$

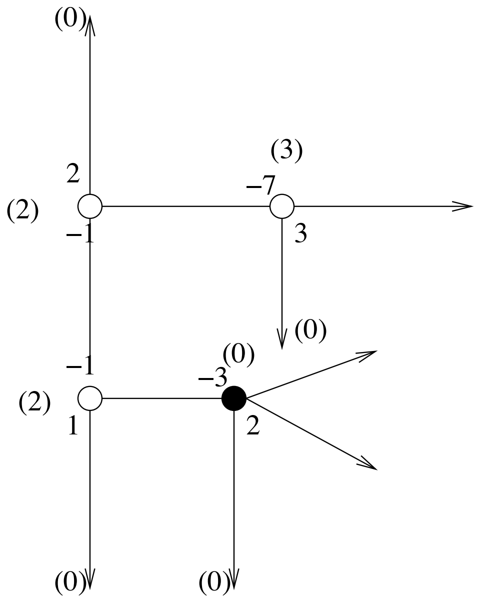
Again, the "interesting" Newton maps come from faces of the Newton polygon at infinity and roots of the face polynomials.

The Newton tree at infinity of the curve $f = t$ is obtained by glueing to the graph of the Newton polygon at infinity of $f-t$ the Newton tree of all $(f-t)_\sigma \in \mathbb{C}[x_1^{-1}][x_1, y_1]$. The decorations of the edges are changed and the decorations of the vertices change sign. The Newton tree depends on t. As an example, we consider

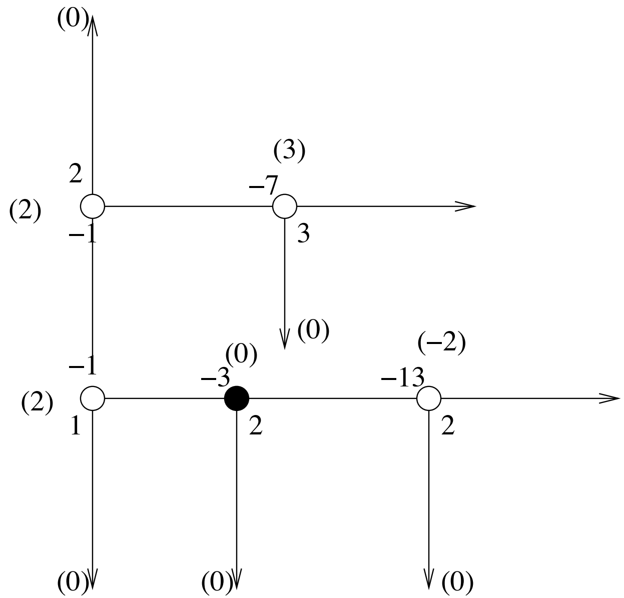
$$f(x,y) = x^6y^4+(4x^5+3x^4)y^3+(6x^2+11x^3+2x^2)y^2+$$

$$(4x^3+13x^2+8x+1)y+x^2+5x+5$$

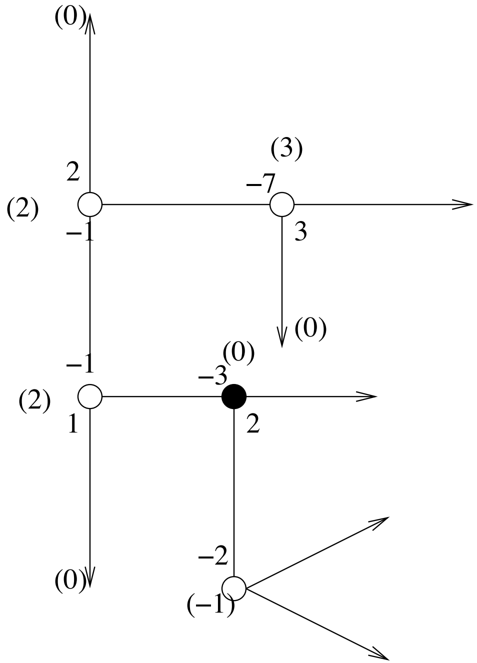
The Newton tree of $f-t$ for t generic is



The Newton tree of $f + 1$ is



The Newton tree of f is



The vertices decorated with (0) are called *dicritical vertices*. The corresponding face polynomials are what Abhyankar calls the "univariate polynomials strategically located in the belly of a bivariate polynomial" The number of roots of the face polynomial attached to a dicritical vertex is called the *degree of the dicritical.* If the degree is one the dicritical is also called a *section*. If all the dicriticals of a polynomial are sections, the polynomial is said *simple*.

CLASSIFICATION OF RATIONAL POLYNOMIALS

A *rational polynomial* $f \in \mathbb{C}[x,y]$ is a polynomial whose generic (not singular, not special) fiber $\{f = t\}$ is a rational curve. It is equivalent to say that f is a *field generator*, which means that there exists $g \in \mathbb{C}(x,y)$ such that $\mathbb{C}(f,g) = \mathbb{C}(x,y)$. We say that f is a *good field generator* if there exists $g \in \mathbb{C}[x,y]$ such that $\mathbb{C}(f,g) = \mathbb{C}(x,y)$, otherwise we say that f is a *bad field generator*.

Russell proved that a rational polynomial is a bad field generator if and only if it has no sections.

The classification of rational polynomials is a very difficult problem.

The classification of simple rational polynomials was begun by Miyanishi and Sugie and completed by Neumann and Norbury.

Up to the birational morphisms,

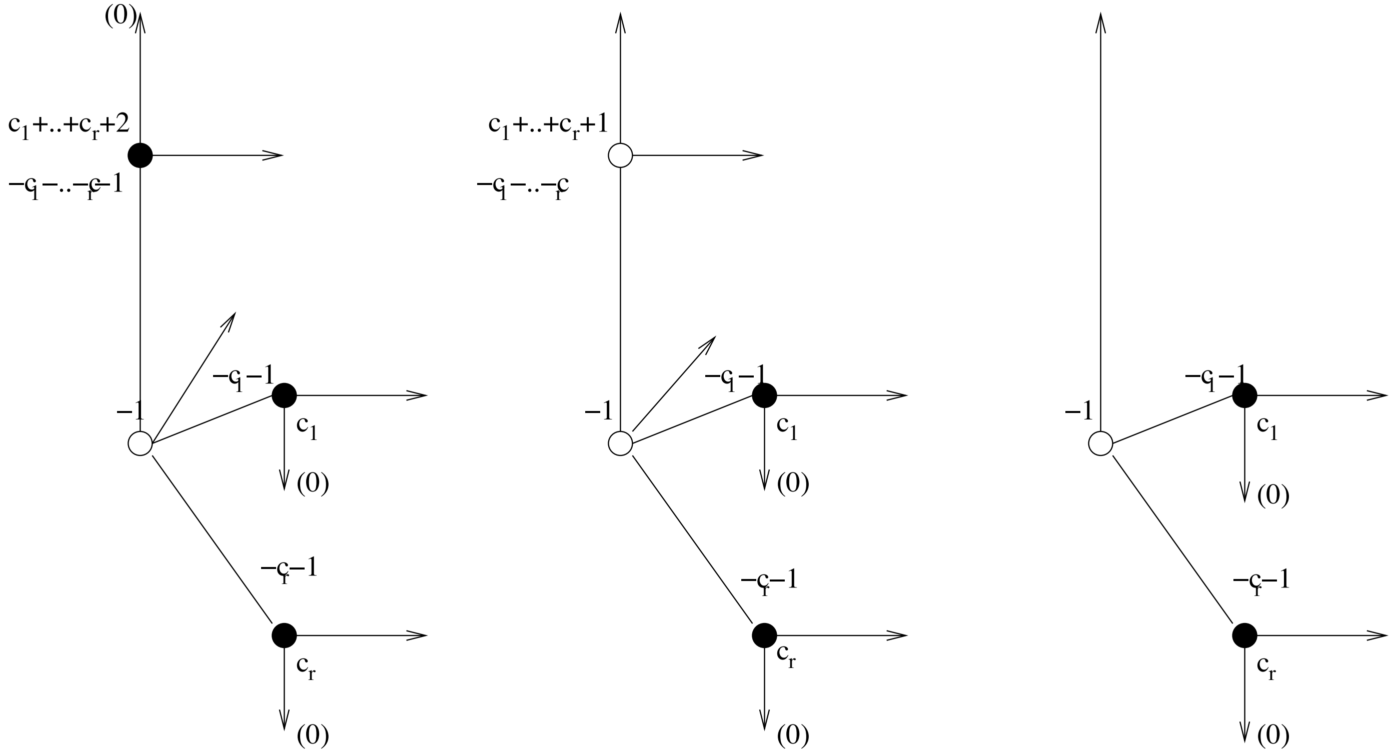
$$x = x^{a_1}y^{a_2}, y = x^{b_1}y^{b_2}, a_1b_2 - a_2b_1 = \pm 1$$

$$x = x, y = x^k y + p(x), \deg p < k$$

one gets the following polynomials:

$$xy + x^2 y \prod (\beta_i - xy)^{a_i}, x^2 y \prod (\beta_i - xy)^{a_i}, y \prod (\beta_i - x)^{a_i} + h(x)$$

with the following Newton trees:



Sasao has classified quasi-simple rational polynomials, which means that the dicriticals have at most degree 2. Part of the classification is, up to the same birationnal maps than before:

$$(1)x\prod(xy+\beta_i)^{c_l}+\gamma(xy+\beta_i)(xy+\beta_j)+$$

$$\gamma'(xy+\beta_j)+t$$

$$(3)(xy+\eta)(x^{1+c}y^c(xy+\gamma)^{c'}+\beta_1)(x^{1+c}y^c(xy+\gamma)^{c'}+\beta_2)+$$

$$\gamma'(x^{1+c}y^c(xy+\gamma)^{c'}+\beta_1)+t$$

$$(3')(xy+\gamma_1)$$

$$(x(xy+\gamma_1)^c(xy+\gamma_2)^{c'}+\beta_1)(x(xy+\gamma_1)^c(xy+\gamma_2)^{c'}+\beta_2)+$$

$$\gamma'(x(xy+\gamma_1)^c(xy+\gamma_2)^{c'}+\beta_1)+t$$

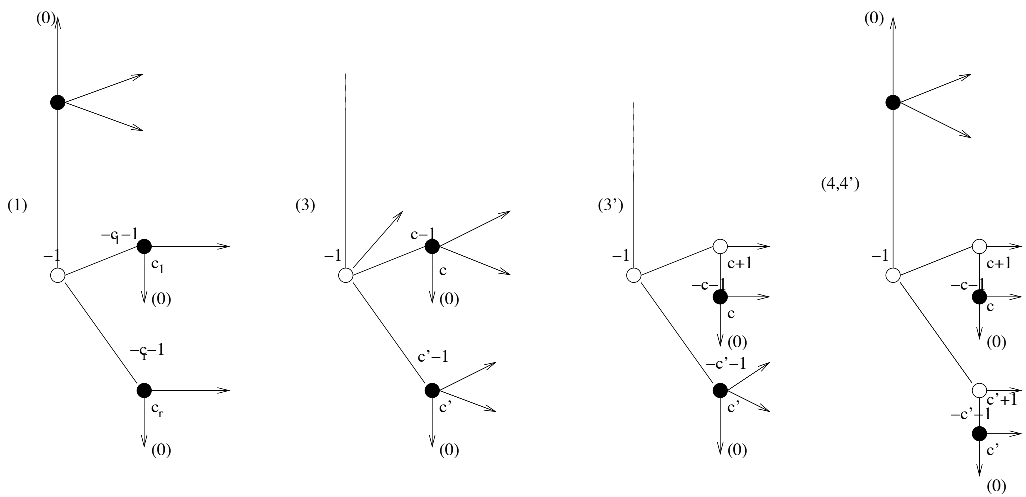
$$(4)(x^{1+c+1}y^{c+1}(xy+\gamma)^{c'}+\beta_1)$$

$$(x^{1+c}y^{c}(xy+\gamma)^{c'+1}+\beta_2)+t$$

$$(4')(x(xy+\gamma_1)^{c+1}(xy+\gamma_2)^{c'}+\beta_1)$$

$$(x(xy+\gamma_1)^{c+1}(xy+\gamma_2)^{c'}+\beta_2)+t$$

with the following Newton trees:



First remark: We notice that the two first cases of Neumann Norbury as well as those cases of Sasao all belong to a ring $\mathbb{C}[y, xy, x\phi(xy)] \subset \mathbb{C}[x, y]$ with $\phi \in \mathbb{C}[X]$.

$$\begin{array}{ccccc} \mathbb{C}^2 & \rightarrow & S = \{XZ = \phi(Y)\} & \rightarrow & \mathbb{C} \\ (x, y) & \mapsto & (y, xy, x\phi(xy)) & \mapsto & f(y, xy, x\phi(xy)) \end{array}$$

Other cases in Sasao and up to the same birational maps, give:

$$(2) x(xy^2+\gamma)^c[(xy^2+\gamma)^{c+1}+\gamma' y]+t$$

$$(2') x^{-1}[(xy+1)(x^2y+x+\gamma)^c+\beta]$$

$$[(x^2y+x+\gamma)^{c+1}+\gamma' x]+t$$

$$(5) x[(xy^2+\gamma)^c+\beta_1 y][(xy^2+\gamma)^c+\beta_2 y]+t$$

$$(5') x^{-1}[(xy+1)\{(x^2y+x+\gamma)^c+\beta x\}+\gamma']$$

$$[(x^2y+x+\gamma)^c+\beta x]+t$$

$$(6)[(xy^2+\gamma)+\beta_1 y][x(xy^2+\gamma)+(\beta_2+\beta_3)xy+\beta_2\beta_3]+t$$

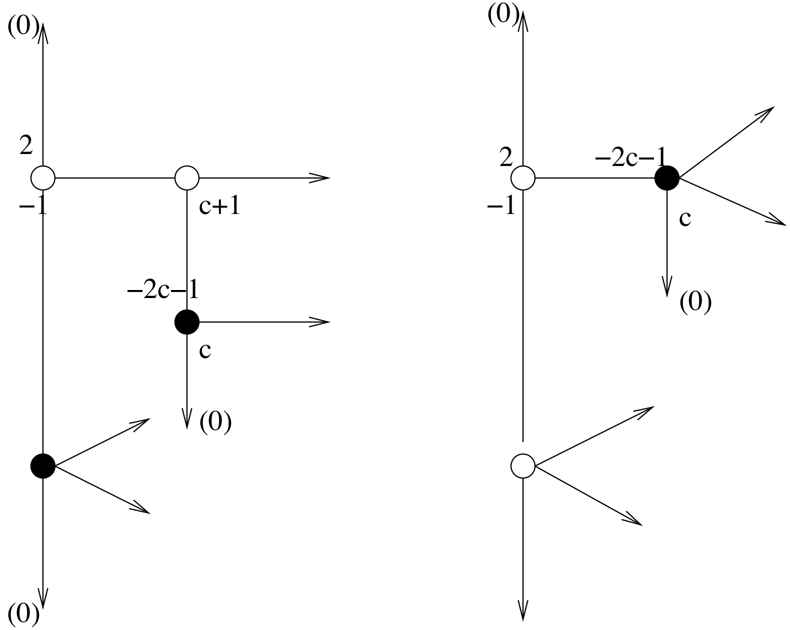
There are two more cases of the form $p(x)y+q(x)$ that we will not study here.

For (2) and (5) and (6) the polynomials belong to $\mathbb{C}[x, x^2y, xy(x^2y+\gamma)^c, y(x^2y+\gamma)^{2c}]$, for (2') and (5') they belong to

$$\mathbb{C}[x, x^2y+x+\gamma, (xy+1)(x^2y+x+\gamma)^c,$$

$$x^{-1}((xy+1)(x^2+x+\gamma)^c+\beta))(x^2y+x+\gamma)^c]$$

and the Newton trees are



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With Peter Russell we began the classification of birational maps whose missing curves are lines and one curve. We obtained rational polynomials in

$$\mathbb{C}[y, xy, x^2y+a_1x, \cdots, x^ky+a_1x^{k-1}+\cdots+a_{k-1}x]$$

a ring which was also studied by David Wright.

Concerning bad field generators, very little is known.

The first field generator was given by Jan and unpublished; it has degree 25:

$$J = y(x^8y^4-1)^2+3x^3y^2(x^8y^4-1)+3x^6y^3+x$$

and belongs to

$$\mathbb{C}[x, x^2y, xy(x^8y^4-1), y(x^8y^4-1)^2]$$

Then Peter Russell proved that there is no bad field generators of degree less than 21, and for degree 22,23,24, and gave an example of bad field generator of degree 21:

$$R = (y^2(xy+1)^4 + y(2xy+1)(xy+1)+1)$$

$$(y(xy+1)^5 + 2xy(xy+1)^2 + x)$$

which is in $\mathbb{C}[x, xy, y(xy+1)^3]$.

All rational polynomial that I mentioned, good or bad can be seen in the same frame.

STUDY OF

$$A = \mathbb{C}[y, xy, x^2y + a_1x, \cdots, x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x,$$
$$x\phi(x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x)] \subset \mathbb{C}[x, y]$$

If we summarize, we got field generators in

$$(a)\mathbb{C}[y, xy, x^2y+a_1x, \cdots, x^ky+a_1x^{k-1}+\cdots+a_{k-1}x]$$

$$(b)\mathbb{C}[y, xy, x\phi(xy)]$$

$$(c)\mathbb{C}[x, x^2y, xy(x^2y+\gamma)^c, y(x^2y+\gamma)^{2c}]$$

$$(d)\mathbb{C}[x, x^2y+x+\gamma, (xy+1)(x^2y+x+\gamma)^c,$$

$$x^{-1}((xy+1)(x^2+x+\gamma)^c+\beta))(x^2y+x+\gamma)^c]$$

$$(e)\mathbb{C}[x, x^2y, xy(x^8y^4-1), y(x^8y^4-1)^2].$$

Actually all of our polynomials come from

$$\mathbb{C}[y, xy, x^2y + a_1x, \cdots, x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x,$$

$$x\phi(x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x)]$$

where $\phi \in \mathbb{C}[X]$. It is clear for case (a) and (b).

In case (c) let $x = Y(XY^2 + \gamma)^c, y = X/(XY^2 + \gamma)^{2c}$, then

$$\mathbb{C}[x, x^2y, xy(x^2y + \gamma)^c, y(x^2y + \gamma)^{2c}] =$$

$$\mathbb{C}[Y(XY^2 + \gamma)^c, XY^2, XY, X]$$

Case (e) is similar using $x = X(X^8Y^4-1), y = Y/(X^8Y^4-1)^2$, we have

$$\mathbb{C}[x, x^2y, xy(x^8y^4-1), y(x^8y^4-1)^2] =$$

$$\mathbb{C}[Y, XY, X^2Y, X(X^8Y^4-1)]$$

In case (d), if $x = X(X^2Y - \beta X + \gamma)^c, y = X^{-1}((XY-\beta) - (X^2Y - \beta X + \gamma)^c)/(X^2Y - \beta X + \gamma)^{2c}$, one has

$$\mathbb{C}[x, x^2y + x + \gamma, (xy+1)(x^2y + x + \gamma)^c,$$

$$x^{-1}((xy+1)(x^2+x+\gamma)^c+\beta))(x^2y+x+\gamma)^c] =$$

$$\mathbb{C}[Y, XY, X^2Y - \beta X, X(X^2Y - \beta X + \gamma)^c]$$

Lemma 4 *Let* $p(x,y) = x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x$ *and* $\tilde{p}(v,w) = v^{k-1}w + b_1v^{k-2} + \cdots + b_{k-2}v$ *There exist* $b_1, b_2, \cdots, b_{k-2}$ *such that if* $x = v/\phi(\tilde{p}(v,w))$ *and* y *satisfies* $p(x,y) = \tilde{p}(v,w)$, *then* $y = Q_0(v,w) \in \mathbb{C}[v,w]$.

Then we also have

$$xy = Q_1(v,w) \in \mathbb{C}[v,w]$$

$$\cdots$$

$$x^{k-2}y + a_1x^{k-3} + \cdots a_{k-3}x = Q_{k-2}(v,w) \in \mathbb{C}[v,w]$$

If $B = \mathbb{C}[v, \tilde{p}(v,w), Q_{k-2}(v,w), \cdots, Q_0(v,w)] \subset \mathbb{C}[v,w]$, we say that B is the *mirror* of A. One goes from A to B by the birational map $\tau : x = v/\phi(\tilde{p}(v,w)), y = Q_0(v,w)$

In particular,

if $A = \mathbb{C}[y, xy, x\phi(xy)]$, then A coincides with its mirror. The corresponding surface is $XZ = Y\phi(Y)$ and the birational map τ from A to $B = A$ is the exchange of X and Z.

if $a_1 = 0, \cdots, a_{k-2} = 0$, that means for $A = \mathbb{C}[y, xy, x^2y, \cdots, x^{k-1}y, x\phi(x^{k-1}y)]$, then $B = \mathbb{C}[v, v^{k-1}w, v^{k-2}w\phi(v^{k-1}w),$
$\cdots, vw(\phi(v^{k-1}w))^{k-2}, w(\phi(v^{k-1}w))^{k-1}]$.

The surface $S := \{XZ = \phi(Y)\}$ has been studied by affine geometers. In particular one knows the group of automorphisms of S, given by Makar-Limanov. In terms of the automorphisms of $A = \mathbb{C}[y, xy, x\phi(xy)] \subset \mathbb{C}[x, y]$, they are generated by τ and Jonquières automorphisms of $\mathbb{C}^2$, (this subgroup is called the tame automorphism group), and $(x, y) \to (cx, c^{-1}y)$ (and some special automorphisms depending of some special ϕ).

For the moment we don't know in general the group of automorphisms of A. It is interesting for us because this group acts on the set of rational polynomials belonging to A.

One other interesting result for $S := \{XZ = \phi(Y)\}$, was given by Daigle. It says that the tame automorphism group of $S := \{XZ = \phi(Y)\}$, acts transitively on the kernels of locally nilpotent derivations of S. One can show that any generator of the kernel of a locally nilpotent derivation of S is a rational polynonial in A, and there exits $g \in A$ such that $\mathbb{C}(f, g) = \mathbb{C}(x, y)$.

We will say that a rational polynomial in A is *A-good* if there exits $g \in A$ such that $\mathbb{C}(f, g) = \mathbb{C}(x, y)$.

Question 5 *1. What is the automorphism group of A?*

2. What are the orbits of the kernels of the locally nilpotent derivations in A?

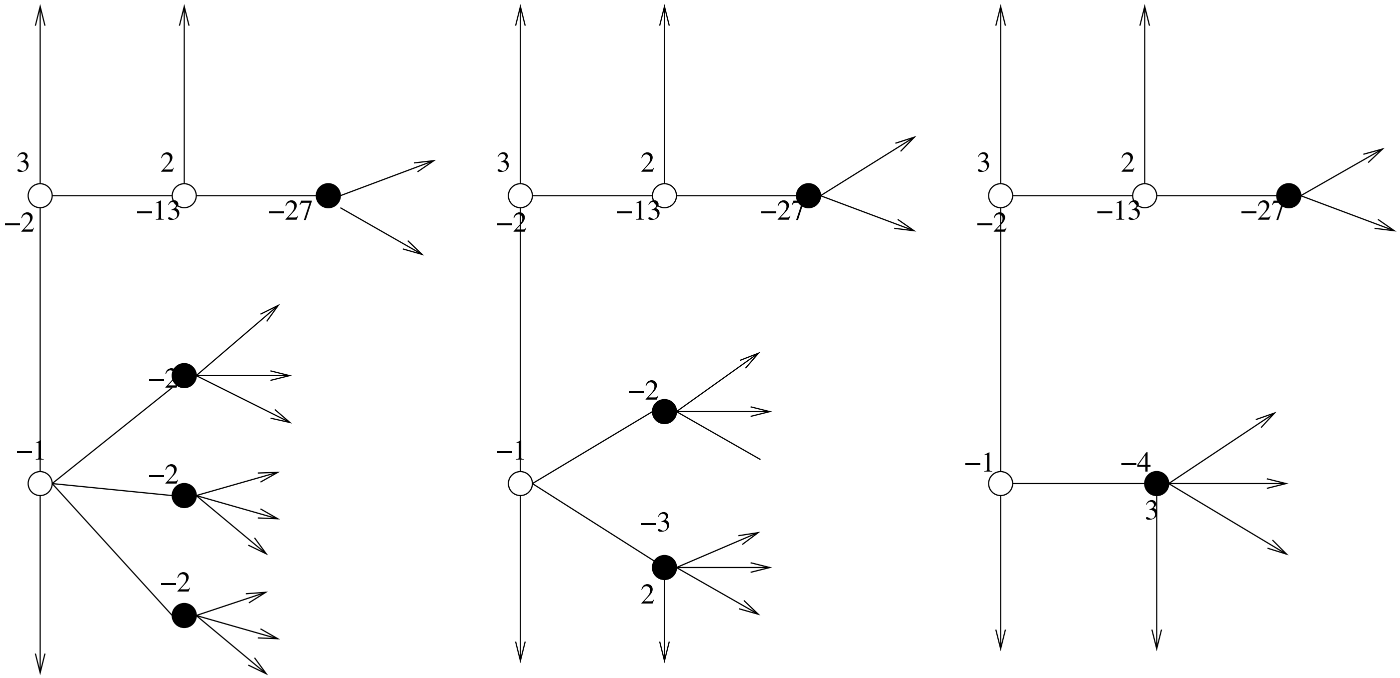
3. What are the orbits of the A-good rational polynomials in A?

Coming back to the classification of bad field generators in A and B.

In $A = B = \mathbb{C}[y, xy, x\phi(xy)]$, let $\phi(X) = X^3 + c_3X^2 + c_1X + c_0$.

$$f_0(x, y) = y + x\phi(xy) + 2x^2y^2 + c_2xy \in A$$

We notice that f_0 is invariant by τ. Let ρ : $x = x + 1, y = y$, then $\tau(\rho(f_0))$ is a bad field generator of degree 21. There are 3 types depending on the number of roots of ϕ. Russell's one corresponds to $\phi(X) = (X - c)^3$.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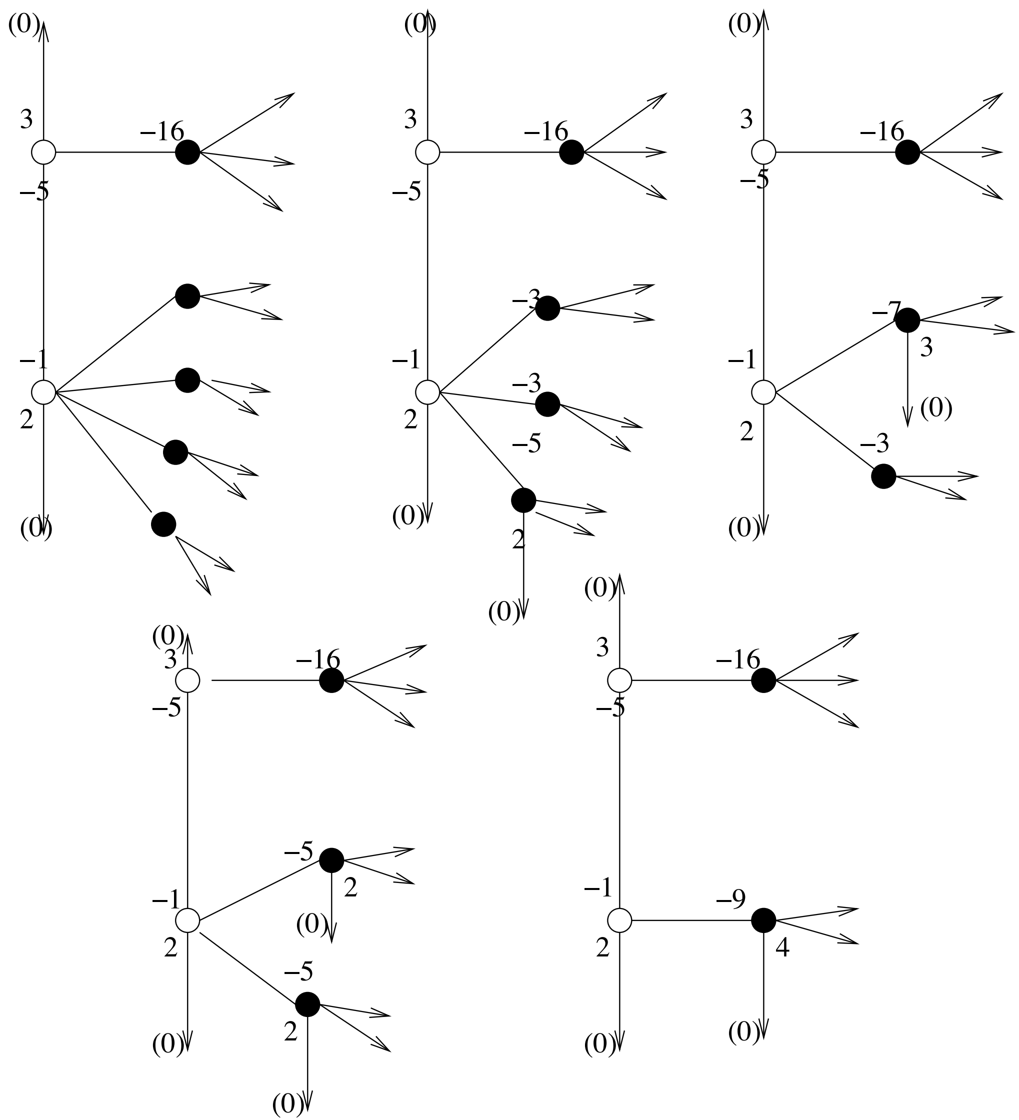
3

In $A = \mathbb{C}[y, xy, x^2y + ax, x\phi(x^2y + ax)]$, let $\phi(X) = X^4 + c_3X^3 + c_2X^2 + c_1X + c_0$

$$f_0(x, y) = y + 3xy(x^2y + ax) + 3(x^2y + ax)^3 +$$

$$x\phi(x^2y+ax)+A_1xy+A_2(x^2y+ax)^2+A_3(x^2y+ax)$$

with $A_1 = c_3, A_2 = 2c_3, A_3 = c_2 + 2a$. The polynomial f_0 is in A, it is A-simple. Its image by τ in B is a bad field generator of degree 25. Jan's polynomial corresponds to $a = 0$ and $\phi(X) = (x^4 - 1)$. We get 5 types of badfield generators of degree 25, depending on the multiplicities of the roots of ϕ.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In the mirror of $A = \mathbb{C}[y, xy, x^2y + a_1, \cdots, x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x, x\phi(x^{k-1}y + a_1x^{k-2} + \cdots + a_{k-2}x)] \in \mathbb{C}[x, y]$, with $\phi(X) = X^{k+1} + c_kX^k + \cdots + c_0$ we have bad field generators of degree $k^3 - k + 1$. Then we can ask:

Question 6 *Do we get all bad field generators this way and composing by τ and Jonquières automorphismes of A and B?*