Milnor Fibers of Line Arrangements

Alexandru Dimca

Université de Nice, France

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Outline

Anatoly, one of my guiding stars...

- **Alternative approaches to a question**, e.g.
  A. Libgober: Eigenvalues for the monodromy of the Milnor fibers of arrangements. In: Libgober, A., Tibăr, M. (eds) Trends in Mathematics: Trends in Singularities. Birkhäuser, Basel (2002).
  D. Cohen, A. Dimca and P. Orlik: Nonresonance conditions for arrangements. Ann. Institut Fourier (Grenoble) 53, 1883-1896 (2003).

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  Regular functions transversal at infinity, Tohoku Math. J. (2) Volume 58, Number 4 (2006), 549-564.

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- **Joint papers**, e.g.
  Regular functions transversal at infinity, Tohoku Math. J. (2) Volume 58, Number 4 (2006), 549-564.
- **Counter-example to a claim**: 'Tangent Cone Theorem':
  A. Libgober: First order deformations for rank one local systems with a non-vanishing cohomology, Topology Appl. 118 (2002), no. 1-2, 159-168.
  A. Dimca, S. Papadima and A. Suciu: Topology and geometry of cohomology jump loci, preprint, math.AT 0902.1250. (to appear in Duke Math. J.).

Line arrangements

A line arrangement $\mathcal{A}$ in the complex projective plane $\mathbb{P}^2$ is a finite collection of lines $L_1, ..., L_d$. Choose a linear equation $f_j = 0$ for each line L_j and set

$$Q(\mathcal{A}) = f_1 \cdot ... \cdot f_d \in \mathbb{C}[x, y, z].$$

Then the corresponding arrangement complement is

$$M(\mathcal{A}) = \mathbb{P}^2 \setminus \cup_{j=1,d} L_j.$$

The cohomology algebra $H^*(M(\mathcal{A}))$ with any coefficients is known (Orlik-Solomon, Invent. Math. 1980). In particular, $H^*(M(\mathcal{A}))$ is determined by the combinatorics of $\mathcal{A}$, expressed in the incidence lattice $L(\mathcal{A})$.

Milnor fibers and monodromy

The Milnor fiber of a line arrangement $\mathcal{A}$ is the smooth surface defined in $\mathbb{C}^3$ by the equation

$$F(\mathcal{A}) : Q(\mathcal{A})(x,y,z) = 1.$$

The monodromy automorphism $h : F(\mathcal{A}) \to F(\mathcal{A})$ is given by

$$h(x,y,z) = \alpha \cdot (x,y,z)$$

with $\alpha = \exp(2\pi i/d)$. It induces the algebraic monodromy $h^* : H^*(F(\mathcal{A})) \to H^*(F(\mathcal{A}))$. Since $h^d = Id$, we get an eigenspace decomposition

$$H^*(F(\mathcal{A}),\mathbb{C}) = \oplus_{\beta \in \mu_d} H^*(F(\mathcal{A}),\mathbb{C})_\beta$$

such that

$$H^*(F(\mathcal{A}),\mathbb{C})_1 = H^*(M(\mathcal{A}),\mathbb{C}).$$

For simplicity, we assume $\mathcal{A}$ known, and write simply M for $M(\mathcal{A})$, and F for $F(\mathcal{A})$.

Open questions

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- Similar question for the Hodge numbers $h^{p,q}(H^1(F, \mathbb{C}))$ of the Deligne MHS on $H^*(F, \mathbb{Q})$. Here $H^1(F, \mathbb{C}) = H^{1,0} \oplus H^{0,1} \oplus H^{1,1}$ (special case of Deligne splitting).

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- Similar question for the numbers $b_1(F)_\beta = \dim H^1(F, \mathbb{C})_\beta$. This is the same as computing the characteristic polynomial of $h^* : H^1(F) \to H^1(F)$, which is precisely the Alexander polynomial of the arrangement (**R. Randell**).

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- Similar question for $\dim(H^1(F, \mathbb{C})_\beta \cap H^{p,q})$ for $(p, q) = (1, 0), \ (0, 1), \ (1, 1)$.

Relation to the spectrum

If $Q = 0$ is the defining equation of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$, one defines as above the Milnor fiber F, the monodromy h and the spectrum of Q as the formal sum

$$Sp(Q) = \sum_{a \in \mathbb{Q}} n_{Q,a} t^a$$

where

$$n_{Q,a} = \sum_j (-1)^{j-n} \dim \mathrm{Gr}_F^p \tilde{H}^j(F, \mathbb{C})_\beta$$

with $p = [n + 1 - a]$ and $\beta = \exp(-2\pi i a)$.

Theorem BS. (N. Budur, M. Saito 2009) The spectrum $Sp(Q)$ is determined by the combinatorics described in the lattice $L(\mathcal{A})$.

Rank one local systems and characteristic varieties

The rank one local systems $\mathcal{L}$ on M are parametrized by the affine algebraic torus

$$\mathbb{T}(M) = Hom(\pi_1(M), \mathbb{C}^*) = H^1(M, \mathbb{C}^*) = (\mathbb{C}^*)^{d-1}.$$

The first characteristic varieties of M are defined by

$$\mathcal{V}_k(M) = \{\rho \in \mathbb{T}(M) \ | \ \dim H^1(M, \mathcal{L}_\rho) \geq k\}.$$

To know the dimension $\dim H^1(M, \mathcal{L})$ means exactly to know the position of $\mathcal{L} \in \mathbb{T}(M)$ with respect to the subvarieties $\mathcal{V}_k(M)$.

Open question: Are the first characteristic varieties of M determined by the combinatorics described in the lattice $L(\mathcal{A})$?

Rank one local systems and monodromy

A rank 1 local system on M is determined by given a set of d complex numbers $\lambda_1, ..., \lambda_d$ such that $\lambda_1 \cdot ... \cdot \lambda_d = 1$. ($\lambda_j$ is the monodromy about the line L_j). For $\beta \in \mu_d$, we denote by $\mathcal{L}_\beta$ the rank 1 local system corresponding to the choice

$$\lambda_1 = ... = \lambda_d = \beta.$$

One has the isomorphism

$$H^1(F, \mathbb{C})_\beta = H^1(M, \mathcal{L}_\beta)$$

for any $\beta \in \mu_d$.

Example.

Theorem L. (A. Libgober, 2002, hyperplane arrangement case)
Let $\beta \in \mu_d$ and assume that there is a line in $\mathcal{A}$, say L_1, such that the multiplicity m_p of any multiple point p of $\mathcal{A}$ situated on L_1 satisfies either $m_p = 2$ or $\beta^{m_p} \neq 1$. Then $H^1(F, \mathbb{C})_\beta = 0$.

For a generalization, see D. Cohen, A. Dimca and P. Orlik (2004).

Remark If $H^1(F, \mathbb{C})_\beta = 0$ for all $\beta \neq 1$, then $H^1(F, \mathbb{C}) = H^1(M, \mathbb{C})$ and the answer to the first set of open questions above is affirmative.

Some non trivial examples

D. Cohen and A. Suciu (1995)

- **The A_3-arrangement**: $Q = xyz(x-y)(x-z)(y-z)$ and $\mathcal{A}$ consists of the 3 reducible fibers of the pencil of conics $(x(y-z), y(z-x))$. Then $b_1(F)_\beta = 1$ for $\beta = \alpha^2$ and $\beta = \alpha^4$.

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- **The Pappus configuration** $(9_3)_1$:

$$Q = xyz(x-y)(y-z)(x-y-z)(2x+y+z)(2x+y-z)(-2x+5y-z)$$

consists of the 3 reducible fibers of a cubic pencil (find it!). One has $b_1(F)_\beta = 1$ for $\beta = \alpha^3$ and $\beta = \alpha^6$.

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- **The Hesse arrangement** consists of the 4 reducible fibers of the pencil $(x^3 + y^3 + z^3, xyz)$, 12 lines in all. One has $b_1(F)_\beta = 2$ for $\beta = \alpha^k$ and $k = 3, 6, 9$.

Mixed Hodge Structure on $H^1(F)$

Consider the natural direct sum decomposition

$$H^1(F,\mathbb{Q}) = H^1(F,\mathbb{Q})_1 \oplus H^1(F,\mathbb{Q})_{\neq 1}$$

where $H^1(F,\mathbb{Q})_1 = \ker(h^* - 1) = p^*H^1(M,\mathbb{Q})$ is the eigenspace corresponding to the eigenvalue $\lambda = 1$ of the monodromy operator $h^* : H^1(F,\mathbb{Q}) \to H^1(F,\mathbb{Q})$, and $H^1(F,\mathbb{Q})_{\neq 1} = \ker((h^*)^{d-1} + ... + 1)$.

Theorem A. (A.D. and S. Papadima, N. Budur, A.D. and M. Saito, 2009)
The mixed Hodge structure on $H^1(F,\mathbb{Q})$ is split, i.e., the subspaces $H^1(F,\mathbb{Q})_1$ and $H^1(F,\mathbb{Q})_{\neq 1}$ inherit pure Hodge structures from $H^1(F,\mathbb{Q})$, such that $H^1(F,\mathbb{Q})_1$ (respectively $H^1(F,\mathbb{Q})_{\neq 1}$) has weight 2 (respectively 1).

Monodromy and pencils

Let $\mathcal{A}_1 \cup ... \cup \mathcal{A}_k$ be a partition of the set $\{1, 2, \ldots, d\}$ into $k \geq 3$ subsets of the same cardinality $e > 0$. Set $Q_j = \prod_{i \in \mathcal{A}_j} f_i$, for $j = 1, ..., k$. Clearly, $Q = Q_1 \cdots Q_k$. The relation between such (multi)nets and the characteristic varieties has been explored first by **M. Falk and S. Yuzvinsky, Compositio Math 2007**.

Theorem B.(A.D. and S. Papadima, 2009) With the above notation, assume that the vector space $< Q_1, ..., Q_k >$ of degree e homogeneous polynomials has dimension 2. Then $b_1(F)_\beta \geq k - 2$, for any β with $\beta^k = 1$.

In **[N. Budur, A.D. and M. Saito, 2009]** a sufficient condition to have equality is given, and a more general setting is discussed (though interesting examples beyond nets are still missing).

Some notations

Set $\Sigma = \{y \in Z \mid m_y = \mathrm{mult}_y Z \geq 3\}$ where $Z : Q(x,y,z) = 0$ in $Y = \mathbb{P}^2$. For a fixed k, $1 \leq k \leq d-1$, we set

$$\Sigma(k) = \{y \in \Sigma \mid m_y k/d \in \mathbb{Z}\}.$$

For $y \in \Sigma$, let $\mathcal{I}_{\{y\}} \subset \mathcal{O}_Y$ be the reduced ideal of $\{y\} \subset Y$, and define

$$\mathcal{J}^{(k)} := \bigcap_{y\in\Sigma} \mathcal{I}_{\{y\}}^{\lceil m_y k/d \rceil -2}, \quad \mathcal{J}^{(>k)} := \bigcap_{y\in\Sigma} \mathcal{I}_{\{y\}}^{\lfloor m_y k/d \rfloor -1}.$$

Here $\lceil a \rceil := \min\{k \in \mathbb{Z} \mid k \geq a\}$, $\lfloor a \rfloor := \max\{k \in \mathbb{Z} \mid k \leq a\}$, and $\mathcal{I}_{\{y\}}^j = \mathcal{O}_Y$ for $j \leq 0$. Let $\mathbb{C}[X]_j$ denote the space of homogeneous polynomials of degree j. This is identified with $\Gamma(Y, \mathcal{O}_Y(j))$. Define

$$J_j^{(k)} := \Gamma(Y, \mathcal{O}_Y(j) \otimes_{\mathcal{O}_Y} \mathcal{J}^{(k)}) \subset \Gamma(Y, \mathcal{O}_Y(j)) = \mathbb{C}[X]_j.$$

Theorem C.

Theorem C. (N. Budur, A.D. and M. Saito, 2009)
For $k \in [1, d-1]$, let $k' = d - k$ and $\beta = \alpha^k$. Then

$$\dim \mathrm{Gr}_F^0 H^1(F_f)_\beta = \dim \mathrm{Coker}\left(\rho^{(k)} : J_{k-3}^{(k)} \to \bigoplus_{y \in \Sigma(k)} \mathcal{J}_y^{(k)} / \mathcal{J}_y^{(>k)}\right)$$

$$= \dim \mathrm{Coker}\left(\tilde{\rho}^{(k)} : \mathbb{C}[X]_{k-3} \to \bigoplus_{y \in \Sigma} \mathcal{O}_{Y,y} / \mathcal{J}_y^{(>k)}\right),$$

$$\dim \mathrm{Gr}_F^1 H^1(F_f)_\beta = \dim \mathrm{Coker}\left(\rho^{(k')} : J_{k'-3}^{(k')} \to \bigoplus_{y \in \Sigma(k)} \mathcal{J}_y^{(k')} / \mathcal{J}_y^{(>k')}\right)$$

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and $b_1(F_f)_\beta = \dim \mathrm{Gr}_F^1 H^1(F_f)_\beta + \dim \mathrm{Gr}_F^0 H^1(F_f)_\beta$.

An example

Assume that d is divisible by 3 and that Z has only double and triple points. By **Theorem L.** only the cubic roots of unity β may give a nonzero $b_1(F)_\beta$. Set $k = 2d/3$, $k' = d/3$. Then the target of $\rho^{(k')}$ vanishes, and $\rho^{(k)}$ coincides with $\tilde{\rho}^{(k)}$ which is identified with the evaluation map

$$\bigoplus_{y\in\Sigma(k)} \mathrm{ev}_y^{k-3} : \mathbb{C}[X]_{k-3} \to \bigoplus_{y\in\Sigma(k)} \mathbb{C}_y.$$

In particular, for $\beta = \exp(4\pi i/3)$, one has $H^1(F)_\beta \subset H^{0,1}$ and its dimension is given by the superabundance or defect of the linear system $\mathbb{C}[X]_{k-3}$ with respect to the finite set Σ. If $\gamma = \exp(2\pi i/3)$, then $H^1(F)_\gamma \subset H^{1,0}$, as this eigenspace is the complex conjugate of $H^1(F)_\beta$.

Idea of proof for Theorem C.

Using a result by **M. Mustaţă**, we get the following.

Key Lemma.
For $\lambda = \exp(2\pi ik/d)$, we have a canonical isomorphism

$$\mathrm{Gr}_F^0 H^1(F_f, \mathbb{C})_\lambda = H^1(Y, \mathcal{O}_Y(k-3) \otimes_{\mathcal{O}} \mathcal{J}(k/dZ))^\vee.$$

Results of similar flavour were obtained by **E. Artal-Bartolo(1991), H. Esnault (1982), A. Libgober (adjunction ideals),..,** and myself (rational differential forms), book 1992.

Other (open) questions

One may ask which properties of the line arrangement complement M continue to hold for the Milnor fiber F, e.g.

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- Is the first homology group $H_1(F, \mathbb{Z})$ torsion free?
- Is F a minimal CW-complex? **Is F a formal space?**
- **Does the Tangent Cone Theorem hold for F?** This means: does the resonance variety $\mathcal{R}_1(F)$ of F equal the tangent cone of the characteristic variety $\mathcal{V}_1(M)$ at the trivial local system $\mathbb{C}_F$? Here $\alpha \in \mathcal{R}_1(F)$ if and only if $\alpha \in H^1(F)$ and there is $\beta \in H^1(F)$ not a multiple of α such that $\alpha \wedge \beta = 0$.

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- **Anatoly Libgober** has shown that

$$TC_1\mathcal{V}_1(X) \subset \mathcal{R}_1(X)$$

for X a finite CW-complex and claimed equality for X a smooth quasiprojective variety X. **S. Papadima, A. Suciu and A.D.** gave counterexamples to this claim (some configuration spaces) and showed that if X is in addition 1-formal, then

$$TC_1\mathcal{V}_1(X) = \mathcal{R}_1(X).$$

A result by Hugues Zuber, Nice

Theorem (H. Zuber, arXiv:0906.3658)

Let $F : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 1$
Then the inclusion $TC_1\mathcal{V}_1(F) \subset \mathcal{R}_1(F)$ is strict. In particular, F is not even a 1-formal space.

The proof uses the description by D. Arapura of the irreducible components of $\mathcal{V}_1(F)$ and the properties of MHS on $H^1(F, \mathbb{Q})$ given above.