ON THE DETERMINANTAL REPRESENTATIONS OF SINGULAR PLANE CURVES

DMITRY KERNER AND VICTOR VINNIKOV

Abstract. Let $\mathcal{M}$ be a $d\times d$ matrix whose entries are linear forms in the homogeneous coordinates of $\mathbb{P}^2$. Then $\mathcal{M}$ is called a determinantal representation of the curve $\{\det(\mathcal{M})=0\}$. Such representations are well studied for smooth curves.

We study determinantal representations of curves with arbitrary singularities (mostly reduced). The kernel of $\mathcal{M}$ defines a torsion free sheaf on the curve. We classify torsion free sheaves arising as kernels of determinantal representations and study their properties.

Further, we study the local version of the problem: families of square matrices, depending on two parameters. In particular we give various criteria of local decomposability of $\mathcal{M}$ into a direct sum of determinantal representations.

Contents

1. Introduction

Let $\mathcal{M}$ be a square matrix with entries - linear forms in the homogeneous coordinates of $\mathbb{CP}^2$, i.e. $\mathcal{M}\in Mat\Big(d\times d, H^0(\mathcal{O}_{\mathbb{P}^2}(1))\Big)$. It is called a determinantal representation of the (complex, projective, possibly singular and non-reduced) curve $\{det(\mathcal{M})=0\}\subset\mathbb{P}^2$. (We always assume $\det(\mathcal{M})\not\equiv 0$ and $d>1$.)

Such representations were studied classically and many results are known (cf. [Beauville00] and some related works [Catanese81], [Wall78], [Room38]). For example, any plane curve admits a determinantal representation [Dixon1900],[Dickson21],[Arbarello-Sernesi1979] and the representation is determined (up to equivalence) by its cokernel sheaf and the curve [Cook-Thomas79, Thm 1.1].

Thus it's natural to study possible determinantal representations for a fixed curve. For smooth (irreducible, reduced) curves such representations have been classified in [Vinnikov89] (cf. also [Beauville00, prop.1.11 and cor.1.12]). In [Ball-Vinnikov96, theorem 3.2] the classification was extended to the case of maximal representations of multiple nodal curves i.e. curves of the form $\{f^p=0\}\subset\mathbb{P}^2$ for $p\in\mathbb{N}$ and $\{f=0\}$ irreducible, reduced, nodal curve. There are several reasons to consider non-reduced curves. First, they have more representations. And while the general hypersurface in $\mathbb{P}^n$ doesn't admit a determinantal representation(unless $(d,n)=(3,3)$), its higher multiples (i.e. $\{f^p=0\}\subset\mathbb{P}^n$) do.

Another reason comes from applications. The determinantal representations of plane curves are important not only in algebraic geometry. For example, the problem can be reformulated as the simultaneous classification of triples of matrices. Hence the applications in linear algebra, operator theory and dynamical systems (cf.

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[Ball-Vinnikov96]). In particular, these applications ask for the classification of determinantal representations of an *arbitrary* plane curve (i.e. with arbitrary singularities, possibly reducible and non-reduced).

In this paper local and global aspects of determinantal representations.

1.1. **Setup.** Let $C = \bigcup\limits_{i=1}^{k} p_i C_i$ be the decomposition of the projective plane curve into irreducible components (with miltiplicities) $C_i = \{f_i = 0\}$ of degrees $deg(f_i) = d_i$. such that $\sum p_i d_i = d$. The (reduced) components can have arbitrary singularities. Let $\mathcal{M} \in Mat\Big(d \times d, H^0(\mathcal{O}_{\mathbb{P}^2}(1))\Big)$ be a determinantal representation, i.e. $det\mathcal{M} = f$.

The representations are studied up to the (global) equivalence: $\mathcal{M} \sim U\mathcal{M}V$, where $U, V \in GL(d, \mathbb{C})$. Sometimes we face the questions of local equivalence: $U, V \in GL(d, \mathcal{O})$, here $\mathcal{O} = \mathbb{C}[[x, y]]$ is the ring of locally analytic functions (cf.§2.2). This is a typical singularity-theory-style problem (families of matrices) and is the content of §3.

It seems most attention was to the one variable case, or the homogeneous dependence on two variables and from the side of applied math.

The modern study started probably from the seminal paper [Arnol'd1971] (cf. the citing papers) and is mentioned in [Arnol'd-problems, 1975-26,pg.23]. Some recent works (for the arbitrary number of variables) from the singularity side are: [Bruce-Tari04, Bruce-Goryunov-Zakalyukin02, Goryunov-Zakalyukin03, Goryunov-Mond05], in particular additional motivation is given in [Bruce-Tari04].

Definition 1.1. *The representation $\mathcal{M}$ is called (globally) decomposable if by a $GL(d, \mathbb{C}) \times GL(d, \mathbb{C})$ conjugation it can be brought to the block-diagonal form $\oplus\mathcal{M}_i$ where $\mathcal{M}_i$ is the representation of $p_i C_i$.*

The representation is locally decomposable (near a singular point) if such a decomposition (associated to the branches of the curve) can be obtained locally by $GL(d, \mathcal{O}) \times GL(d, \mathcal{O})$ conjugation.

Note that at each point $corank\mathcal{M}|_{pt} \leq mult(C, pt)$ (cf. property 3.1). This motivates the following

Definition 1.2. • *The representation is called weakly maximal at a point $pt \in C$ (or locally weakly maximal) if $corank\mathcal{M}|_{pt} = mult(C, pt)$. If this holds for all points of C the representation is called weakly maximal.*
• *The representation is called maximal at a point if any entry of the adjoint matrix $\mathcal{M}^\vee$ belongs to the adjoint ideal $Adj_{(C,pt)}$ (cf.§2.2.3). The representation is maximal if it is maximal at any point.*

For example, weak maximality at a point means that $jet_1(M)$ is non-degenerate. For the adjoint ideal of non-reduced curve see definition 2.7.

Maximality appears to be the suitable strengthening of weak maximality, very useful for the classification of determinantal representations. Any determinantal representation of a smooth curve is maximal. For reduced curves with only ordinary multiple points (e.g. nodes), maximality coincides with weak maximality.

1.2. **Results.** We study the notions of (local) maximality and (local) decomposability. The criteria for global decomposability are obtained in terms of the adjoint $\mathcal{M}^\vee$ (proposition 3.11) and in terms of the matrix and its kernel (corollary 3.15).

Maximality implies weak maximality and local decomposability (lemma 3.16) but not the converse. We give several examples, e.g. a weakly maximal but non-maximal determinantal representation of a branch (example 3.17).

The definition of maximality above is in terms of the adjoint matrix, the applications ask for an equivalent definition in terms of the matrix itself or its kernel. We reduce this question to the maximality per branch (cf. remark 3.18).

We obtain a (quite unexpected) property: any weakly maximal determinantal representation is locally decomposable for tangential decomposition (proposition 3.4). In general, any weakly maximal determinantal representation can be brought to a canonical upper-block-triangular form with the blocks of quite a restricted form (proposition 3.7). A necessary and sufficient condition is also obtained for the local complete decomposability of the representation (corresponding to the local branch decomposition of the curve), proposition 3.9.

These results simplify the study of local determinantal representations significantly. Thus in examples 3.5 and 3.8 we re-derive some of the result of [Bruce-Tari04].

As mentioned above any determinantal representation is determined by its (co)kernel (a torsion-free sheaf on C). In §4 we classify torsion-free sheaves arising from determinantal representations and study their properties (propositions 4.1 and 4.2). This is a generalization of classifications for smooth curves [Vinnikov89],[Beauville00] or multiple nodal curves [Ball-Vinnikov96].

A natural question is to study the corresponding vector bundles on the normalization $\tilde{C} \to C$ (or some minimal modification $C' \stackrel{\nu}{\to} C$ such that the pull-back of the kernel is locally free.) We solve the case of maximal representation (propositions C.3,C.2). We hope to report soon about the general case.

As illustrative examples we describe the normal families of determinantal representations in some specific cases (singular conics/cubics, curves with high singularities), cf. appendix A.

The results are applied to real curves. In particular in Appendix B we prove that any self-adjoint positive-definite determinantal representation of a real plane curve with smooth branches is maximal. Recall that a self-adjoint positive-definite determinantal representation defines a hyperbolic curve, which has at most one singular point with non-smooth branches. In the latter case the region of hyperbolicity degenerates to the point.

In Appendix C we formulate some related questions and directions.

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2. Preliminaries and background

2.1. **The matrix and its adjoint.** We work with (square) matrices, their sub-blocks and particular entries. To avoid confusion we sometimes emphasize the dimensionality, e.g. $\mathcal{M}_{d\times d}$. Then $\mathcal{M}_{i\times i}$ denotes an $i\times i$ block in $\mathcal{M}_{d\times d}$ and $\det(\mathcal{M}_{i\times i})$ the corresponding minor. On the contrary by $\mathcal{M}_{ij}$ we mean a particular entry.

Let $\mathcal{M}$ be a determinantal representation of C, let $\mathcal{M}^\vee$ be the adjoint matrix of $\mathcal{M}$ (so $\mathcal{M}\mathcal{M}^\vee = \det(M)\mathbb{1}_{d\times d}$). Then $\mathcal{M}$ is non-degenerate outside the curve C and the corank over the curve satisfies:

$$1 \le corank(\mathcal{M}|_{pt\in C}) \le mult(C, pt) \tag{1}$$

(this is checked by taking the derivatives.) So, the adjoint matrix $\mathcal{M}^\vee$ is not zero at smooth points of C. In fact any determinantal representation of a smooth curve is weakly maximal. As $\mathcal{M}^\vee|_C \times \mathcal{M}|_C = 0$ we get that the rank of $\mathcal{M}^\vee$ at any smooth point of C is 1 (for the reduced curve). Note that $\mathcal{M}^{\vee\vee} = f^{d-2}\mathcal{M}$ and $\det \mathcal{M}^\vee = f^{d-1}$.

Property 2.1. *(cf. e.g.* [Vinnikov89, Lemma,pg. 114]*) Let $\mathcal{M} \in Mat(d\times d)$ and f irreducible. Suppose any $i\times i$ minor $\det(\mathcal{M}_{i\times i})$ is divisible by f^l. Then any $(i+1)\times(i+1)$ minor $\det(\mathcal{M}_{(i+1)\times(i+1)})$ is divisible by f^{l+1}.*

Proof. By the assumption, any entry of $\mathcal{M}^\vee{}_{(i+1)\times(i+1)}$ is divisible by f^l, thus $\det(\mathcal{M}^\vee{}_{(i+1)\times(i+1)})$ is divisible by $f^{l(i+1)}$. But $\det(\mathcal{M}^\vee{}_{(i+1)\times(i+1)}) = (\det \mathcal{M}_{(i+1)\times(i+1)})^i$. Hence $(\frac{\det \mathcal{M}_{(i+1)\times(i+1)}}{f^l})^i$ is divisible by f^l. As f is irreducible the statement follows. ■

Let $\mathcal{M}^\vee{}_{\alpha\beta}|_{C_i}$ be the restriction of a particular entry of $\mathcal{M}^\vee$ to the component C_i of C. For any point $pt \in C_i$ the vanishing order of $\mathcal{M}^\vee{}_{\alpha\beta}|_{C_i}$ at pt is defined as the local degree of intersection $\Big(C_i, \{\mathcal{M}^\vee{}_{\alpha\beta} = 0\}\Big)_{pt}$. If $\mathcal{M}^\vee{}_{\alpha\beta}|_{C_i} \equiv 0$ then $\Big(C_i, \{\mathcal{M}^\vee{}_{\alpha\beta} = 0\}\Big)_{pt} := \infty$.

2.1.1. *Kernels and cokernels.* Let $M_{d\times d}$ be the determinantal representation (of an arbitrary curve), the corresponding cokernel sheaf is defined by the sequence

$$0 \to \mathcal{O}^{\oplus d}_{\mathbb{P}^2}(-1) \stackrel{M}{\to} \mathcal{O}^{\oplus d}_{\mathbb{P}^2} \to Coker(M) \to 0 \tag{2}$$

The cokernel is supported on the curve. Restrict the sequence to the curve (and twist), then the kernel appears. From now on all the sheaves are considered on the curve.

$$\begin{aligned} &0 \to E_C \to \mathcal{O}^{\oplus d}(d-1)|_C \stackrel{M}{\to} \mathcal{O}^{\oplus d}(d)|_C \to Coker(M)_C \to 0, \\ &0 \to E^l_C \to \mathcal{O}^{\oplus d}(d-1)|_C \stackrel{M^T}{\to} \mathcal{O}^{\oplus d}(d)|_C \to Coker(M^T)_C \to 0 \end{aligned} \tag{3}$$

The sheaves E, E^l are spanned by the columns/rows of $\mathcal{M}^\vee$ (the precise statement in §4).

We impose the following conditions of linear independence. Given a point $pt \in C$, let $\{C_i\}$ be the local branches at this point. Let $E_i = \overline{E|_{C_i \setminus pt}}$ i.e. restrict to the branch outside the singular point then extend to the singular point by the closure. Let $E_i|_{pt}$ be the reduced fibres of the corresponding (embedded) kernel bundles.

We often ask for the "weak" independence: $Span(\cup E_i|_{pt}) = \oplus E_i|_{pt}$ or for the "strong" independence: $Span(\cup E_i) = \oplus Span(E_i)$. In the last case the span means the minimal linear subspace (of the ambient space) into which the bundle(s) embed, when restricted to some neighborhood of the point. The two notions are essentially distinct, cf. example 3.10.

2.1.2. *Local equivalence of determinantal representations.* It's often useful to study the local determinantal representations near a fixed point. Pass to the local coordinates $(x, y, z) \to (x, y, 1)$. For simplicity we always assume that the point is the origin $(0, 0) \in \mathbb{C}^2$. In such a case the equivalence is extended from the constant matrices $(GL(d, \mathbb{C})_L \times GL(d, \mathbb{C})_R)$ to locally analytic ones $(GL(d, \mathcal{O})_L \times GL(d, \mathcal{O})_R)$. Sometimes to emphasize the locality we denote such a matrix by $A(x, y)$. As $A(x, y) \in GL(d, \mathcal{O})$ one has $\frac{1}{\det(A(x,y))} \in \mathcal{O}$ and thus $A(0, 0) \in GL(d, \mathbb{C})$ is invertible.

Several natural objects are:

- $deg_x(A)$=the maximal degree of x in the entries of A. This is infinity unless all the entries of A are polynomials in x. Similarly for $deg_y(A)$ and $deg(A)$ (the total degree).
- $ord_x(A)$=the minimal degree of x appearing in A (if an entry of A doesn't depend on x the order is zero, if $A \equiv \mathbb{0}$ then $ord_x(A) := \infty$). Similarly $ord_y(A)$, $ord(A)$ and $ord_x(A_{ij})$ for a particular entry. So, e.g. $ord(A) \geq 1$ iff $A|_{(0,0)} = \mathbb{0}$
- $A^{(l)}$=the linear part of A, i.e. the image of A under the projection $\mathcal{O} \to \frac{\mathcal{O}}{m^2_{x,y}}$. More generally, $jet_k(A)$ is obtained from A by truncation of all the monomials with total degree higher than k.

2.2. **Singular curves and sheaves.**

2.2.1. *Plane curve singularities.* For local considerations we always assume the (singular) point to be at the origin and use the ring of locally converging power series $\mathcal{O} := \mathbb{C}[[x, y]]$. The maximal ideal of the origin is denoted by $m_{x,y}$.

Associated to any germ $(C, 0)$ there is a branch decomposition $(C, 0) = \cup(p_i C_i, 0)$ where each $(C_i, 0)$ is locally irreducible. The (reduced) tangent cone $T_{(C,0)} = \{l_1..l_k\}$ traces all the tangents of the branches. To this cone is associated the tangential decomposition: $(C, 0) = \cup(C_\alpha, 0)$. Here $C_\alpha = \{f_\alpha = 0\}$ consists of all the branches with the tangent line l_α, in general C_α is reducible and non-reduced. Let $mult(C, 0) = m$ and $mult(C_\alpha, 0) = m_\alpha$.

For any reduced curve-germ the normalization $(\tilde{C}, 0) \to (C, 0)$ is a multi-germ, corresponding to the branches.

Some standard names for singularity types are: the node (A_1: $x^2 = y^2$), the ordinary multiple point of multiplicity p: $x^p = y^p$.

2.2.2. *Sheaves on a singular curve.* We often use the Riemann-Roch for torsion free sheaves on curves (????? [Hartshorne-book, §IV.I exercise 1.9]):

$$h^0(F) - h^1(F) = c_1(F) + 1 - p_a \tag{4}$$

Here c_1 is the first Chern number, p_a is the arithmetic genus, for plane curve of degree d: $p_a = \binom{d-1}{2}$ regardless of the singularities.

Recall that also any torsion free sheaf on a smooth curve is locally free.

Recall that any plane curve singularity is Gorenstein, hence the dualizing sheaf w_C is invertible. Adjunction formula for the plane is: $w_C = \mathcal{O}(d-3)|_C$. Hence $h^0(w_C) = p_a$ and $\deg(w_C) = 2(p_a - 1)$.

For torsion free sheaves the usual Serre duality is valid: $H^1(F) = H^0(F^* \otimes w_C)^*$.

2.2.3. *Adjoint and conductor ideals.* Let $(C,0)=\{f(x,y)=0\}\subset(\mathbb{C}^2,0)$ be a reduced singular curve and $\tilde{C}\stackrel{\nu}{\to}(C,0)$ its normalization. Here $\tilde{C}$ is a multi-germ. Namely, for the branch decomposition $(C,0)=\cup(C_i,0)$ one has $\tilde{C}=\coprod(\tilde{C}_i,0_i)$ with the branch-wise normalization: $(\tilde{C}_i,\tilde{0}_i)\stackrel{\nu_i}{\to}(C_i,0)$.

Let $\vec{v}$ be the generic tangent direction, not tangent to any of the branch.

Definition-Proposition 2.2. $\bullet$ *The adjoint divisor of* $(C,0)$ *on* $\tilde{C}$ *is* $D:=\sum m_i\tilde{0}_i$ *where* $m_i=-ord(\nu_i^*\frac{d\vec{v}}{\partial_{\vec{v}}f(x,y)})=\sum\limits_{j\neq i}\langle C_j,C_i\rangle+\mu(C_i,0)$ *(here* μ *is the Milnor number). In particular* $\sum m_i=2\delta(C)$*.*
$\bullet$ *The adjoint ideal is* $Adj_{(C,0)}:=\{g\in\mathcal{O}|\ \nu^*div(g_C)\geq D\}\subset\mathcal{O}$

Proof. Note that

$$\partial_{\vec{v}}f|_{C_i}=(\prod_{j\neq i}f_j)(\partial_{\vec{v}}f_i)|_{C_i}\ \Rightarrow\ ord(\partial_{\vec{v}}f|_{C_i})=\sum_{j\neq i}(C_j,C_i)+\kappa(C_i) \tag{5}$$

where $\kappa(C_i)$ is the classical invariant, in particular $\kappa(C_i)=\mu(C_i)+mult(C_i)-1$. Also $ord(d\vec{v}|_{C_i})=mult(C_i)-1$, hence: $m_i=\sum\limits_{j\neq i}(C_j,C_i)+\mu(C_i)$. ■

An equivalent definition is that of conductor ideal.

Definition 2.3.

$$I^{cd}:=Ann_{\mathcal{O}_{(C,0)}}(\mathcal{O}_{\tilde{C}})=\{g|\ \forall i:\ \nu_i^*(g)\mathcal{O}_{\tilde{C}_i}\subset\nu_i^*(\mathcal{O}_{(C,0)})\}\subset\mathcal{O}_{(C,0)} \tag{6}$$

By duality I^{cd} is the pullback of Adj to C [Serre-book, §IV.11].

Its properties are [GLS-book1, I.3.4, pg 214]:

$$I^{cd}=\{g|\ \forall i:\ div(\nu_i^*(g))\geq 2\delta(C_i)+\sum_{j\neq i}(C_j,C_i)\},\ \dim\frac{\mathcal{O}_{(C,0)}}{I^{cd}}=\delta,\ \dim\frac{\mathcal{O}_{\tilde{C}}}{I^{cd}}=2\delta \tag{7}$$

Property 2.4. $\bullet$ *Let* $f=\prod f_i$*. Then* $g\in Adj$ *iff* $g=\sum\frac{f}{f_i}g_i$ *with* $ord(g_i)|_{C_i}\geq\mu_i$*. In particular, if all the branches are smooth then* $Adj=\langle\frac{f}{f_1}...\frac{f}{f_r}\rangle$
$\bullet$ *By the direct check:* $\langle f,\vec{df}\rangle\subset Adj$ *and* $m_{xy}^n\subset Adj$ *iff* $n\geq\max_i\frac{\sum_{j\neq i}(C_j,C_i)+\mu_i}{mult(C_i)}$
$\bullet$ [GLS-book2, Lemma 1.26] *The adjoint ideal is a cluster ideal and* $adj\subset m_{x,y}^{p-1}$ *for* $p=mult(C,0)$*.*

Example 2.5. Let $C=\{y^{ar}-x^{br}=0\}\subset(\mathbb{C}^2,0)$ where $a\leq b$ and $gcd(a,b)=1$. For $a=1=b$ this is an ordinary multiple point, other choices give e.g. A_k singularities (y^2-x^{k+1}) etc.

The normalization $\coprod\limits_{i=1}^{r}(\tilde{C}_i,\tilde{0}_i)\to(C,0)$ is defined by: $t_i\to(t_i^a,\omega_i t_i^b)$ where ω_i is an appropriate root of unity.

Alternatively, for the corresponding local rings: $\mathcal{O}_C$ and $\mathcal{O}_{\tilde{C}}=\mathbb{C}[t_1..t_r]/\langle t_it_j|i\neq j\rangle$ the homomorphism is defined by: $x\to\sum t_i^a\alpha_i$ and $y\to\sum t_i^b\beta_i$ (where $\alpha_i\beta_i$ are some numbers).

By choosing the generic coordinates of $(\mathbb{C}^2,0)$ and calculating the order of the pole of $\frac{dx}{\partial_y f(x,y)}$ one has: $D=(abr+1-a-b)\sum\tilde{0}_i\subset\coprod(\tilde{C}_i,\tilde{0}_i)$.

Therefore the adjoint ideal is generated by $\{x^iy^j\}$ for $\frac{i+1}{b}+\frac{j+1}{a}\geq r+\frac{1}{ab}$. So, e.g.
$\bullet$ for an ordinary multiple point $(x^r+y^r=0)$ have: $I_D=m_{xy}^{r-1}$ =all functions with vanishing order at the origin at least $(r-1)$.
$\bullet$ for an A_k : y^2+x^{k+1} have: $I_D=\langle y,x^{\lfloor k\rfloor}\rangle\mathbb{C}[x,y]$
$\bullet$ For the cusp $y^{d-1}+x^d$ the adjoint ideal is $m_{x,y}^{d-2}$ where $m_{x,y}$ is the local maximal ideal (of the reduced point).

Example 2.6. For $(y-x^{\frac{d}{2}})^2-y^d$ with d even one has: $D=\frac{d^2}{4}(0_1+0_2)$ and thus $Adj=\langle y-x^{\frac{d}{2}},y^{\frac{d}{2}},x^{\frac{d^2}{4}}\rangle$.

Note that $D\subset\coprod(\tilde{C}_i,0_i)$ depends on the topological type of the singularity only, while $Adj_{(C,0)}\subset\mathcal{O}_{(\mathbb{C}^2,0)}$ depends essentially on the particular germ $(C,0)$.

Finally, for the purposes of the current paper we define the adjoint ideal of the *non-reduced* curve-germs. The property 2.4 is the motivation for the definition.

Definition 2.7. *Let* $(C,0)=\cup(p_iC_i,0)$ *be the local decomposition into irreducible reduced branches with multiplicities* $\{p_i\}$*, corresponding to the factorization* $f=\prod f_i^{p_i}\in\mathcal{O}$*. The adjoint ideal is:*

$$Adj_{(C,0)}:=\{\sum\frac{f}{f_i}g_i|\ g_i\in Adj_{(C_i,0)}\}$$

2.2.4. *Factorizations of the normalization.* If the singularity $(C,0)$ is not a node, the normalization $(\tilde{C},0)\stackrel{\nu}{\to}C$ can be (nontrivially) factorized: $(\tilde{C},0)\to(C',0)\to(C,0)$. Here both maps are birational morphisms. Usually this can be done in many distinct ways. All the possible intermediate steps form an oriented graph, usually not a tree. Algebraically, the intermediate steps correspond to embeddings of rings:

$$\nu^*_{\tilde{C}/C}(\mathcal{O}_{(C,0)})\subset\nu^*_{\tilde{C}/C'}(\mathcal{O}_{(C',0)})\subset\mathcal{O}_{(\tilde{C},0)} \tag{8}$$

For any such intermediate modification $(C',0)\to(C,0)$ the relative conductor ideal is defined, similarly to that of normalization:

$$I^{cd}_{C'/C}:=\{g\in\mathcal{O}_{(C,0)}|\ \nu^*(g)\mathcal{O}_{(C',0)}\subset\nu^*(\mathcal{O}_{(C,0)})\}\subset\mathcal{O}_{(C,0)} \tag{9}$$

Similarly, the relative adjoint ideal $Adj_{C'/C}\subset\mathcal{O}_{(\mathbb{C}^2,0)}$ is the maximal such ideal whose restriction to the curve gives $I^{cd}_{C'/C}$.

Example 2.8. Consider the ordinary triple point i.e. the germ of the type $xy(x-y)=0$. The normalization is defined by the embedding of local rings:

$$\frac{\mathbb{C}[x,y]}{\langle xy(x-y)\rangle}\stackrel{i}{\hookrightarrow}\frac{\mathbb{C}[t_1]\oplus\mathbb{C}[t_2]\oplus\mathbb{C}[t_3]}{\langle 1_1 1_2,1_2 1_3,1_3 1_1\rangle},\qquad \begin{array}{l}1_{xy}\to 1_1+1_2+1_3\\ x\to t_1+t_2\\ y\to t_2+t_3\end{array} \tag{10}$$

Hence, in this case the graph of the possible modifications is:
(11)

$$\begin{tikzcd}
 & Spec\Big(\frac{\frac{\mathbb{C}[t_1,t_2]}{<t_1,t_2>}\oplus\mathbb{C}[t_3]}{\langle 1_{12}1_3\rangle}\Big) \arrow[dr] & & \\
Spec\Big(\frac{\mathbb{C}[t_1]\oplus\mathbb{C}[t_2]\oplus\mathbb{C}[t_3]}{\langle 1_1 1_2,1_2 1_3,1_3 1_1\rangle}\Big) \arrow[ur] \arrow[r] \arrow[dr] & Spec\Big(\frac{\frac{\mathbb{C}[t_1,t_3]}{<t_1,t_3>}\oplus\mathbb{C}[t_2]}{\langle 1_{13}1_2\rangle}\Big) \arrow[r] & Spec\Big(\frac{\mathbb{C}[t_1,t_2,t_3]}{\langle t_1t_2,t_2t_3,t_3t_1\rangle}\Big) \arrow[r] & Spec\Big(\frac{\mathbb{C}[x,y]}{\langle xy(x-y)\rangle}\Big) \\
 & Spec\Big(\frac{\frac{\mathbb{C}[t_2,t_3]}{<t_3,t_2>}\oplus\mathbb{C}[t_1]}{\langle 1_{23}1_1\rangle}\Big) \arrow[ur] & &
\end{tikzcd}$$

Note that in this example we have at an intermediate step a *non-planar* singularity whose embedding dimension is 3.

2.2.5. *The minimal lifting of torsion free sheaves.* Let F_C be a torsion free sheaf and $\tilde{C}\stackrel{\nu}{\to}C$ the normalization. Then $\nu^*(F)_{\tilde{C}}:=F\overset{\mathcal{O}_{(C,0)}}{\otimes}\mathcal{O}_{(\tilde{C},0)}$ is locally free (being torsion free on a smooth curve). It can happen that already for some intermediate lifting $C'\stackrel{\nu}{\to}C$ the sheaf $\nu^*_{C'/C}(F)$ is locally free. There always exists the *minimal* such lifting.

Lemma 2.9. *Given a torsion free sheaf F_C of rank 1 there exists a modification $\tilde{C}\to C'\to C$ such that:*
- *$\nu^*_{C'/C}(F)$ is locally free.*
- *If for some modification $\tilde{C}\to C''\to C$ the pullback $\nu^*_{C''/C}(F)$ is locally free then the modification factorizes as $\tilde{C}\to C''\to C'\to C$.*

Proof. The question is local, hence can pass to modules over the local ring. So, we prove that for a torsion free module M over the local ring $\mathcal{O}_{(C,0)}$ the minimal extension needed $\mathcal{O}_{(C,0)}\hookrightarrow\mathcal{O}_{(C',0)}\hookrightarrow\mathcal{O}_{(\tilde{C},0)}$ is unique.

Preliminaries.
Let $(C,0)=\underset{i}{\cup}(C_i,0)$ be the branch decomposition. Then the normalization is a multi-germ: $(\tilde{C},0):=\coprod_i(\tilde{C}_i,0_i)$ and the ring $\mathcal{O}_{(\tilde{C},0)}=\underset{i}{\oplus}\mathcal{O}_{(\tilde{C}_i,0_i)}$ is localized at the points $\{0_i\}_i$.

Let $\mathcal{O}_{(\tilde{C},0)}\stackrel{\pi_i}{\to}\mathcal{O}_{(\tilde{C}_i,0_i)}$ be the projections, so $\underset{i}{\oplus}\pi_i=\mathbb{1}$. The module $M_{(C,0)}$ extends to $M_{(\tilde{C},0)}:=\mathcal{O}_{(\tilde{C},0)}\underset{\mathcal{O}_{(C,0)}}{\otimes}M$, with the natural embedding: $M\stackrel{i}{\hookrightarrow}M_{(\tilde{C},0)}$ given by $m\to m\otimes 1$. Identify M with its image inside $M_{(\tilde{C},0)}$.

As $M_{(C,0)}$ is torsion free, its extension $M_{(\tilde{C},0)}$ satisfies: if $a\in\mathcal{O}_{(\tilde{C},0)}$ and $m\in M_{(\tilde{C},0)}$ then $am=0$ causes $a=0$ or $m=0$.

Construction.
As $\mathcal{O}_{(\tilde{C},0)}$ is regular, $M_{(\tilde{C},0)}$ is locally free, i.e. generated by one element. Can choose this element as $m\in M_{(C,0)}$. Let $\{m,m_1..m_k\}\in M_{(C,0)}$ be the generators of M over $\mathcal{O}_{(C,0)}$. Then for any i exists $a_i\in\mathcal{O}_{(\tilde{C},0)}$ such that $m_i=a_im$. As was proven above such an element is unique.

Consider now the extension $\mathcal{O}_{(C,0)}\subset\mathcal{O}_{(C',0)}\subset\mathcal{O}_{(\tilde{C},0)}$ of $\mathcal{O}_{(C,0)}$ by all such elements $\{a_i\}$. By construction the lifted module $M_{(C',0)}$ is locally free over $\mathcal{O}_{(C',0)}$ and the extension is minimal. ■

3. Local determinantal representations

Here we provide some criteria of local decompositions. The case of matrix family depending on one variable is elementary (e.g. [Gantmakher-book, chapter VI]): $\mathcal{M}(t)$ is locally equivalent to the diagonal matrix $diag(a_1(t)...a_n(t))$ where $a_i(t)$ is divisible by a_{i+1}.

The case of more variables is more interesting. It seems most of the results of §3.2 are new.

3.1. **Localization.**

Property 3.1. *Suppose the multiplicity of $(C,0)$ is $m\geq 1$ and $\mathcal{M}_{d\times d}$ is the (global) determinantal representation.*

- *Locally $\mathcal{M}_{d\times d}$ is equivalent to $\begin{pmatrix}\mathbb{1}_{(d-p)\times(d-p)} & \mathbb{0}\\ \mathbb{0} & \mathcal{M}_{p\times p}\end{pmatrix}$ with $\mathcal{M}_{p\times p}|_{(0,0)}=\mathbb{0}$ and $1\leq p\leq m$.*
- *If the representation is weakly maximal, i.e. $rank(\mathcal{M}_{d\times d}|_{(0,0,1)})=d-m$ then $p=m$ and $det(jet_1\mathcal{M}_{p\times p})\not\equiv 0$ and $det(jet_{p-1}{\mathcal{M}^\vee}_{p\times p})\not\equiv 0$.*

The first claim is proved e.g. in [Bruce-Tari04, Proposition 4.1] or [Piontkowski2006, lemma 1.7] for the symmetric case. Both bounds are sharp, regardless of the singularity of curve.

The second claim is immediate.

Definition 3.2. *In the notations as above, the localization of $\mathcal{M}_{d\times d}$ (or the local representation) is $\mathcal{M}_{p\times p}$.*

So, the matrix of localization vanishes at the point. Note that the localization $\mathcal{M}_{p\times p}$ is non-unique, but any two results of localization are locally equivalent, in particular p =the dimension of the matrix is well defined.

It appears that any matrix with *polynomial* entries (vanishing at the origin) is the localization of some global determinantal representation.

Lemma 3.3. *For any $\mathcal{M}_{local}\in Mat(p\times p,\mathbb{C}[x,y])$, such that $\mathcal{M}_{local}|_{(0,0)}=\mathbb{0}$, there exists $\mathcal{M}_{global}\in Mat(d\times d,H^0(\mathcal{O}_{\mathbb{P}^2}(1)))$ which is locally equivalent to $\begin{pmatrix}\mathbb{1}_{(d-p)\times(d-p)} & \mathbb{0}\\ \mathbb{0} & \mathcal{M}_{p\times p}\end{pmatrix}$.*

This property is useful: in various examples we'll give just the local form, without proving that it actually arises from some representation.

Proof. Let x^ay^b be a monomial in $\mathcal{M}_{local}$ with the highest total degree. By permutation assume it belong to the entry $\mathcal{M}_{11}$. Consider the augmented matrix: $\begin{pmatrix}1 & 0 & & \\ 0 & x^ay^b+.. & \mathcal{M}_{12} & ..\\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & ..\\ 0 & .. & .. & \end{pmatrix}$. It is locally equivalent to $\begin{pmatrix}1 & x & 0 & \\ -x^{a-1}y^b & 0+.. & \mathcal{M}_{12} & ..\\ 0 & \mathcal{M}_{21} & \mathcal{M}_{22} & ..\\ 0 & .. & .. & \end{pmatrix}$. For the new matrix the number of monomials with highest total degree is less by one. Continue in the same way till all the monomials of the highest total degree $(a+b)$ are removed. Continue by induction till one gets a matrix with entries of degree at most 1. ■

3.2. **Local decomposability.**

It is a very useful property for proving global statements. For the tangent cone $T_{(C,0)}=\{l_\alpha\}$ of the germ of curve, consider the local *tangential decomposition*: $(C,0)=\underset{\alpha}{\cup}(C_\alpha,0)$. Here $C_\alpha=\{f_\alpha=0\}$ consists of all the branches with the tangent line l_α, in general C_α can be reducible and non-reduced. Let $mult(C,0)=m$ and $mult(C_\alpha,0)=m_\alpha$

Proposition 3.4. *Let $\mathcal{M}_{m\times m}$ be the localized weakly maximal determinantal representation of C. Corresponding to the tangential decomposition of $(C,0)$, the representation $\mathcal{M}$ is locally equivalent to:*

$$\begin{pmatrix}\mathcal{M}_{m_1\times m_1} & \mathbb{0} & & \\ \mathbb{0} & \mathcal{M}_{m_2\times m_2} & \mathbb{0} & ..\\ \mathbb{0} & .. & .. & \mathcal{M}_{m_k\times m_k}\end{pmatrix}$$

Here $\mathcal{M}_{m_\alpha \times m_\alpha}$ is the local representation of $(C_\alpha, 0)$.

Proof. By the property 3.1 $\mathcal{M}$ vanishes at the origin, while $\det(jet_1\mathcal{M}) \not\equiv 0$.

• First, by $GL(m,\mathbb{C}) \times GL(m,\mathbb{C})$ one can bring $jet_1(\mathcal{M})$ to the Jordan form. For that, let $jet_1(\mathcal{M}) = xP + yQ$ with P, Q constant matrices. Then P can be assumed as $\begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$. Let $Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$, then the remaining $GL(m) \times GL(m)$ transformations that preserve P are

$$Q \to \begin{pmatrix} U_1 & U_2 \\ \mathbb{0} & U_4 \end{pmatrix} Q \begin{pmatrix} V_1 & \mathbb{0} \\ V_3 & V_4 \end{pmatrix} \to \begin{pmatrix} *** & (U_1Q_2 + U_2Q_4)V_4 \\ U_4(Q_3V_1 + Q_4V_3) & U_4Q_4V_4 \end{pmatrix} \tag{12}$$

Then can assume Q_4 is diagonal: $Q_4 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$. Use this identity $\mathbb{1}$ submatrix to remove from Q all other entries in the rows and columns of $\mathbb{1}$. Consider rows and columns of Q corresponding to the rows and columns of Q_4 outside $\mathbb{1}$. Recall that $jet_1(\mathcal{M})$ is nondegenerate, hence Q can be brought to the form

$$\begin{pmatrix} \multicolumn{3}{c}{Q_1} & 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix} \\ \multicolumn{3}{c}{0} & 1 & 0 \\ \begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} & & & 0 & 0 \end{pmatrix} \tag{13}$$

Now use the last several rows and columns of Q, to remove everything from the corresponding rows and columns of Q_1, so that $Q_1 = \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & * \end{pmatrix}$. Bring the remaining (non-vanishing) part of Q_1 to the Jordan form. Finally, consider the so obtained matrix $jet_1(\mathcal{M})$, its only problematic part consists of sets of submatrices $\begin{pmatrix} x & y \\ y & 0 \end{pmatrix}$. Permute the two columns to convert such sub-matrix to $\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$. After treating all such submatrices the matrix $jet_1(\mathcal{M})$ is in the Jordan form. By further permutations (of rows and columns) can group equal eigenvalues together.

Thus $\mathcal{M}$ can be assumed as: $\begin{pmatrix} l_1 & * & 0 & .. \\ 0 & l_1 & * & 0 \\ .. & .. & & * \\ 0 & .. & 0 & l_k \end{pmatrix} + \tilde{\mathcal{M}}$ where $ord(\tilde{\mathcal{M}}) \geq 2$ and $\{l_i\}$ are the linear forms defining the lines of the tangent cone.

• The matrix $\mathcal{M}$ is naturally subdivided into the blocks B_{ij}, which are $m_i \times m_j$ rectangles (corresponding to the fixed eigenvalues of $jet_1(\mathcal{M})$). So, need to remove the off-diagonal blocks, B_{ij} for $i \neq j$. We do this by induction, at the q'th step removing all the terms whose order is $\leq q$.

Let $q = min_{i \neq j}(ord\mathcal{M}_{ij})$ for (ij) not in a diagonal block (thus $q > 1$). Consider $jet_q(\mathcal{M})$, i.e. truncate all the monomials whose total degree is bigger than q. Suppose the block $B_{12} \subset jet_q(\mathcal{M})$ is non-zero, i.e. there is an entry of order q.

As l_1, l_2 are linearly independent, by a linear change of coordinates in $(\mathbb{C}^2, 0)$ can assume $l_1 = x$, $l_2 = y$. Decompose: $B_{12} = xT + yR$, where T, R are $m_1 \times m_2$ matrices, with $ord(T) \geq q-1$ and $ord(R) \geq q-1$. From the last row of B_{12} subtract the rows $jet_q\mathcal{M}_{m_1+1,*}, jet_q\mathcal{M}_{m_1+2,*}, .., jet_q\mathcal{M}_{m_1+m_2,*}$ of $jet_q(\mathcal{M})$ multiplied by R_{m_11}, $R_{m_12}..R_{m_1m_2}$. By the assumptions this doesn't change $jet_q(\mathcal{M})$ outside the block B_{12}. After this procedure every entry of the last row of B_{12} is divisible by x. Thus subtract from the columns of B_{12} the column $jet_q\mathcal{M}_{*,m_1}$ multiplied by the appropriate factors.

Now the last row of B_{12} consists of zeros, while $jet_q(\mathcal{M})$ is unchanged outside B_{12}. Do the same procedure for the row $jet_q\mathcal{M}_{m_1-1,*}$ of B_{12} (using the rows $jet_q\mathcal{M}_{m_1+1,*}, jet_q\mathcal{M}_{m_1+2,*}, .., jet_q\mathcal{M}_{m_1+m_2,*}$ and the column $jet_q\mathcal{M}_{*,m_1-1}$). And so on.

After this process one has a refined matrix $jet_q(\mathcal{M}')$ which coincides with $jet_q(\mathcal{M})$ outside the block B_{12} and has zeros inside this block. Do the same thing for all other (off-diagonal) blocks. Then one has a block diagonal matrix $jet_q(\mathcal{M}')$.

• Now repeat all the computation starting from non-truncated version $\mathcal{M}$. This results in the increase of q. Continue by induction. Thus, for each q the matrix $jet_q(\mathcal{M})$ is locally equivalent to a block diagonal one.

Take the limit of this process. Namely, suppose the increase $q \to q+1$ is done by the equivalence $\mathcal{M} \to U^{(q)}\mathcal{M}V^{(q)}$. Consider the equivalence transformation: $\mathcal{M} \to \lim\Big(\prod\limits_{i\geq q} U^{(i)}\Big)\mathcal{M}\Big(\prod\limits_{j\geq q} V^{(j)}\Big)$. The limit is well defined e.g. because $jet_{q-1}(U^{(q)}) = \mathbb{1} = jet_{q-1}(V^{(q)})$. So, for each q only finitely many terms of total degree q are involved in $\prod_i U^{(i)}$ or $\prod_j V^{(j)}$. This finishes the proof. ■

Example 3.5. The proposition reduces the classification of local weakly maximal determinantal representations of plane curve singularities (i.e. families of matrices depending on 2 parameters) to the case of singularity with one tangent line.

Consider the singularity of D_k type: $\{y^2x + x^{k-1} = 0\}$, the union of an A_{k-3} part $(y^2 + x^{k-2})$ and a non-tangent smooth branch. The (non-trivial) local determinantal representations of such singularity are either 2×2 or 3×3. We get that in the later case the representation is decomposable and the classification problem is reduced to that of A_{k-3}. Compare to the classification in [Bruce-Tari04, table 4].

Example 3.6. The assumption of weak maximality is necessary. Consider $\begin{pmatrix} x^2 & x^4+x^3y+x^2y^2+xy^3+y^4 \\ -x+y & y^2 \end{pmatrix}$ which is a determinantal representation of the singularity of type $x^2y^2 + x^5 - y^5 \sim (x^2+y^3)(y^2-x^3)$ (two singular, non-tangent branches). It is indecomposable, otherwise it would be equivalent to a diagonal matrix one of whose nonzero entries has linear term (and thus defines a smooth branch).

The last proposition reduces the problem of local decomposition to the case of curves $(C, 0)$ whose tangent cone is just one line. Suppose $(C, 0)$ is such, choose $\hat{x}$ axis as the unique tangent line, so $(C, 0) = \{y^p + .. = 0\}$, with the dots for higher order terms. Let $(C, 0) = \bigcup\limits_{\alpha=1}^{k}(C_\alpha, 0)$ be a decomposition into some curves without common components. Here each C_α can be further reducible and non-reduced.

A weakly maximal determinantal representation of such a curve can be brought to an almost canonical form.

Proposition 3.7. *Let $\mathcal{M}$ be a weakly maximal local determinantal representation of the curve-germ as above. Then $\mathcal{M}$ is locally equivalent to an upper-block-triangular matrix:*

$$\begin{pmatrix} \mathcal{M}_{m_1\times m_1}(x,y) & \mathcal{M}_{m_1\times m_2}(x) & * & \\ \mathbb{0} & \mathcal{M}_{m_2\times m_2}(x,y) & \mathcal{M}_{m_2\times m_3}(x) & * \\ .. & .. & .. & .. \\ \mathbb{0} & \mathbb{0} & .. & \mathcal{M}_{m_k\times m_k}(x,y) \end{pmatrix}$$

Here the blocks $\{\mathcal{M}_{m_i\times m_i}(x,y)\}_i$ are local determinantal representations of $\{(C_\alpha, 0)\}$, while the blocks $\mathcal{M}_{m_i\times m_j}(x)$ for $i<j$ depend on x only.

Moreover, the blocks $\{\mathcal{M}_{m_i\times m_i}(x,y)\}_i$ can be assumed of the form: $Diag_{m_i\times m_i}(x,y) + B_{m_i\times m_i}(x)$, where $B_{m_i\times m_i}(x)$ depends on x only.

Proof. The proof consists of several steps. Choose some C_α.

• *Step1.* Adjust the coordinates to arrive at the Weierstraß form: $f_\alpha = y^{m_\alpha} + g_1(x)y^{m_\alpha-1} + ... + g_{m_\alpha}(x) = 0$. Here $g_i(x)$ are locally analytic functions. This transformation certainly lifts to the transformation of $\mathcal{M}_{(m\times m)}$.

Let $E_\alpha = Ker\mathcal{M}_{(m\times m)}|_{C_\alpha}$ and s_α a local section. So, s_α is a vector of functions in (x, y) such that $\mathcal{M}_{(m\times m)}s_\alpha$ is divisible by f_α. Can assume that the entries of s_α have no common divisor, thus at least one entry contains the monomial y^{n-1}. Reduce $s_\alpha mod f_\alpha$, so can assume that the only powers of y appearing in s_α are $< m_\alpha$. Thus by local $GL(d, \mathcal{O})$ transformations can bring s_α to the form $(y^{n-1} + xh_1(x, y), y^{n-2}x^a + h_2(x, y), .., .., h_n(x), 0, .., 0)$ where h_i are some locally analytic functions with $deg_y(h_i) \leq n-1-i$

• *Step2.* According to the choice of s_α, consider the first n columns of $\mathcal{M}$ (with $n \leq m_\alpha$), denote this matrix by $\mathcal{M}_{m\times n}$. Let s be the corresponding truncation of s_α, so that $\mathcal{M}_{m\times n}s$ is divisible by f_α.

By lemma 3.1 one can choose n rows in $\mathcal{M}_{m\times n}$ to form a submatrix $A_{n\times n}$ such that $\det(jet_1 A_{n\times n}) \not\equiv 0$. As $A_{n\times n}s$ is divisible by f_α, get: $\det A_{n\times n}$ is divisible by f_α. But $ord \det A_{n\times n} = n \leq m_\alpha$. Therefore: $n = m_\alpha$ and $\det A_{m_\alpha\times m_\alpha} = f_\alpha$ (up to a constant) and $\det(jet_1 A_{m_\alpha\times m_\alpha}) = y^{m_\alpha}$. So, by $GL(m_\alpha, \mathbb{C})_L$ can assume (cf. the beginning of the proof of proposition 3.4) that $jet_1 A_{m_\alpha\times m_\alpha} = y\mathbb{1} + x\tilde{A}$, for $\tilde{A}$ strictly upper triangular.

Return now to $\mathcal{M}_{m\times m_\alpha}=\begin{pmatrix}A_{m_\alpha\times m_\alpha}\\ C_{(m-m_\alpha)\times m_\alpha}\end{pmatrix}$. By $GL(m,\mathbb{C})_L$ can assume that $C_{(m-m_\alpha)\times m_\alpha}$ has no linear y-terms. In fact, now we show that by the action of $GL(m,\mathcal{O})_L$ all the y-dependence of $C_{(m-m_\alpha)\times m_\alpha}$ can be removed. It's enough to prove this for any particular row.

Consider the row $(\beta_1(x)+y\gamma_1(x,y),...\beta_{m_\alpha}(x)+y\gamma_{m_\alpha}(x,y))$. Let $q=\min_j(ord(\gamma_i(x,y)))$. By the assumption $q\geq 1$. By a permutation of the indices can assume: $ord(\gamma_1(x,y))=q$. Subtract from this row the first row of $A_{m_\alpha\times m_\alpha}$ multiplied by γ_1. Recall that $A_{m_\alpha\times m_\alpha}=y\mathbb{1}+x\tilde{A}+B(x,y)$, where $\tilde{A}$ is a constant strictly upper triangular matrix and $ordB(x,y)\geq 2$. Thus one gets the row (omitting the monomials containing x only):

$$(-B_{11}\gamma_1,y\gamma_2-B_{12}\gamma_1-x\gamma_1\tilde{A}_{12},..y\gamma_{m_\alpha}-B_{1m_\alpha}\gamma_1-x\gamma_1\tilde{A}_{1m_\alpha}) \tag{14}$$

Again, omit all the monomials containing x only, then the row is: $(y\tilde{\gamma}_1,..y\tilde{\gamma}_{m_\alpha})$ where: $ord\tilde{\gamma}_i\geq ord\gamma_i$ and $ord\tilde{\gamma}_1>ord\gamma_1$. Continue by induction.

So, by the $GL(m,\mathcal{O})_L$ action all the monomials x^ay^b with $b<N$ can be removed, for any given N. By taking the limit of this procedure (i.e. taking the product of all the $GL(m,\mathcal{O})_L$ actions), one has:

$\mathcal{M}_{m\times m_\alpha}$ is locally equivalent to $\begin{pmatrix}A_{m_\alpha\times m_\alpha}(x,y)\\ C_{(m-m_\alpha)\times m_\alpha}(x)\end{pmatrix}$.

Note that this is achieved by multiplication from the left only (the permutations of columns can be undone at the end), so the form of s is not changed.

• *Step3.* We prove that in fact $C_{(m-m_\alpha)\times m_\alpha}(x)\equiv\mathbb{0}$. Indeed, by construction $C_{(m-m_\alpha)\times m_\alpha}(x)s=f_\alpha$.. and y appears in s only in powers $<m_\alpha$. Thus, in fact $C_{(m-m_\alpha)\times m_\alpha}(x)s\equiv 0$ and the claim follows by considering the highest power of y, then the next etc.

By applying the procedure as above for each α we arrive at the upper-block-triangular matrix:

$$\mathcal{M}=\begin{pmatrix}\mathcal{M}_{m_1\times m_1}(x,y) & * & * & ..\\ \mathbb{0} & \mathcal{M}_{m_2\times m_2}(x,y) & * & ..\\ \mathbb{0} & .. & .. & \mathcal{M}_{m_k\times m_k}(x,y)\end{pmatrix} \tag{15}$$

Here the linear part of each $\mathcal{M}_{m_i\times m_i}$ is non-degenerate and can be brought to the form: $y\mathbb{1}+xT$.. where T is a constant strictly upper triangular matrix whose only possibly non-zero values are right over the diagonal.

• *Step4.* Apply now the "matrix chasing" from the proof of proposition 3.4. Present each entry of $\mathcal{M}$ as $\alpha_{ij}(x)+y\beta_{ij}(x,y)$. Let $q=\min_{i\neq j} ord\beta_{ij}$. Suppose this minimum is obtained in the entry $\alpha_{ij}(x)+y\beta_{ij}(x,y)$.

$\star$ If $i<j$ subtract from the row i the row j multiplied by $\beta_{ij}(x,y)$. Now the $y-$order of the (i,j)'th entry is at least $q+1$, while the $y-$orders of all others entries are non-decreased, except possibly for the entry $(i,j+1)$, where the order might have dropped to q. If this is the case, do the same for the entry $(i,j+1)$ and so on till one gets to the entry (i,i).

By this procedure the number of non-diagonal entries in $\mathcal{M}$ with $y-$order q decreases at least by 1, the $y-$orders of all other entries is non-decreased, the linear part of the matrix remains constant.

$\star$ If $i>j$ do similarly, now by subtracting get from the entry (i,j) to the entry $(i,j+1)$ and so on till the entry (i,p). This last is killed by the $p-$'th row.

Thus after a finite number of steps the minimal $y-$order of the non-diagonal entries increases by at least one. This is the step of the induction. Take the limit of such transformations to eliminate any non-diagonal dependence on y. Note that during the whole process the matrix remains upper-block-triangular. ■

A simple application of the last proposition is to weakly maximal determinantal representations of the singularity of type $\prod_{i=1}^{k}(y+\alpha_ix^{l_i})$ for $l_k\geq l_{k-1}\geq..\geq l_1>1$. These are k smooth branches with various mutual tangency.

Corollary 3.8. Any weakly maximal determinantal representation of the curve as above is locally equivalent to:

$$\begin{pmatrix} y+\alpha_1 x^{l_1} & x^{n_1} & h_{13}(x) & .. & .. & h_{1n}(x) \\ 0 & y+\alpha_2 x^{l_2} & x^{n_2} & h_{24}(x) & .. & h_{2n}(x) \\ .. & .. & .. & .. & & \\ 0 & .. & .. & .. & 0 & y+\alpha_k x^{l_k} \end{pmatrix}$$

with $1 \leq n_i < l_i$ and $h_{ij}(x)$ a polynomial in x such that $ord_x(h_{ij}) \geq 1$ and $deg(h_{ij}) < l_i$ (or $h_{ij} \equiv 0$).

So, for example for the singularity A_{2n-1} ($y^2 - x^{2n} = 0$) any (non-trivial) determinantal representation is locally equivalent to

$$\mathcal{M} = \begin{pmatrix} y - x^k & x^l \\ 0 & y + x^{2n-k} \end{pmatrix}$$

for $1 \leq l < \min(k, 2n-k)$. Compare to [Bruce-Tari04, Table 2]

Proof. Use the proposition 3.7 for the curve decomposition $\prod\limits_{i=1}^{k}(y+\alpha_i x^{l_i})$ to achieve the upper triangular form, such that elements over the main diagonal depend on x only.

Consider the diagonal $(i, i+1)$. Represent each element $\mathcal{M}_{i,i+1}(x)$ as $x^{n_i}\tilde{\mathcal{M}}_{i,i+1}$, where $\mathcal{M}_{i,i+1}|_{(0,0)} \neq 0$, i.e. is locally invertible. If $n_i \geq l_i$ then by adding the i'th column to the column $(i+1)$ and subtracting the row $(i+1)$ from the row i the $x-$order can be increased. Continue this process inductively, as in proofs of the previous propositions. Hence, if for some element $\mathcal{M}_{i,i+1}$ the $x-$order is at least l_i the element can be just set to zero. The remaining elements $x^{n_i}\tilde{\mathcal{M}}_{i,i+1}$ are set to x^{n_i} by the conjugation $\mathcal{M} \to U^{-1}\mathcal{M}U$ with

$$U = \begin{pmatrix} \prod\limits_{i\geq 1} \tilde{\mathcal{M}}_{i,i+1} & 0 & .. & 0 & \\ 0 & \prod\limits_{i\geq 2} \tilde{\mathcal{M}}_{i,i+1} & .. & 0 & \\ 0 & .. & .. & 0 & \tilde{\mathcal{M}}_{k-1,k} \end{pmatrix} \tag{16}$$

Regarding the remaining entries $h_{ij}(x)$ with $j - i \geq 2$, bring them to the needed form diagonal-by-diagonal. This is again done by the standard procedure: add $y + x^{l_i}$, subtract $y + x^{l_j}$ etc. ■

Weakly maximal determinantal representations are *not completely decomposable* in general. For example $\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$ is indecomposable (as it is not equivalent to $\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$).

We can only give a cheap criterion.

Proposition 3.9. • *In the notations and assumptions of the proposition 3.7 consider the kernels $E_\alpha = Ker\mathcal{M}_{m\times m}|_{C_\alpha}$ in some small neighborhood of the origin $(C, 0)$. Then $Span(\bigcup\limits_\alpha E_\alpha) = \oplus Span(E_\alpha)$ iff the representation decomposes, i.e. $\mathcal{M}_{m\times m}(x, y)$ is locally equivalent to $\begin{pmatrix} \mathcal{M}_1 & 0.. & \\ 0 & \mathcal{M}_2 & 0.. \\ ... & & \end{pmatrix}$.*

• *Let $(C, 0) = \cup(p_i C_i, 0)$ be the local decomposition into branches $C_i = \{f_i = 0\}$ with $f = \prod f_i^{p_i}$. The weakly maximal determinantal representation decomposes locally ($\mathcal{M} \sim \oplus\mathcal{M}_i$) iff the adjoint matrix can be written as $\mathcal{M}^\vee = \sum \frac{f}{f_i^{p_i}}\mathcal{M}^\vee{}_i$.*

Proof. • (the non-trivial direction).

By the proposition 3.7 the matrix can be brought to the upper-block-triangular form, so we show how to remove the blocks $\mathcal{M}_{m_i\times m_j}(x)$ for $i < j$. Start from $\mathcal{M}_{m_1\times m_2}(x)$. By the assumption on spans of the kernels any section of E_2 is of the form: $(\underbrace{0, .., 0}_{m_1}, \underbrace{?, ?.., ?}_{m_2}, 0..0)$. Therefore, from the matrix $\mathcal{M}_{m\times m}(x, y)$ can extract the columns numbered (m_2),..,$(m_3 - 1)$ and one has: $\mathcal{M}_{m\times m_2}s \sim f_2$ where s is the truncation of the section (all the zeros are cut). Now proceed as in the steps 2,3 to kill all the entries of $\mathcal{M}_{m\times m_2}$ outside $\mathcal{M}_{m_2\times m_2}$.

Do the same for all the other blocks.

• Its enough to prove the decomposability for the case $(C, 0) = (C_1, 0) \cup (C_2, 0)$, where $(C_i, 0)$ are possibly reducible, non-reduced, but without common components (i.e. $C_1 \cap C_2$ is finite). Let f, f_1, f_2 be the defining functions of the germs, so f_1, f_2 are relatively prime and $f = f_1 f_2$. Let m, m_1, m_2 be the corresponding multiplicities at the origin, so $ord(\mathcal{M}^\vee) \geq (m-1)$ and $ord(\mathcal{M}^\vee{}_i) \geq (m_i - 1)$.

Multiply $\mathcal{M}^\vee = \frac{f}{f_1}\mathcal{M}^\vee{}_1 + \frac{f}{f_2}\mathcal{M}^\vee{}_2$ by $\mathcal{M}$, then one has:

$$f\mathbb{1} = \mathcal{M}\mathcal{M}^\vee = \frac{f}{f_1}\mathcal{M}\mathcal{M}^\vee{}_1 + \frac{f}{f_2}\mathcal{M}\mathcal{M}^\vee{}_2 \tag{17}$$

Therefore can define the matrices $\{A_i\}$, $\{B_i\}$ by $f_iA_i := \mathcal{M}\mathcal{M}_i$ and $f_iB_i := \mathcal{M}_i\mathcal{M}$. By definition: $\sum A_i = \mathbb{1}$ and $\sum B_i = \mathbb{1}$. We prove that in fact $\oplus A_i = \mathbb{1}$ and $\oplus B_i = \mathbb{1}$. The key ingredient is the identity:

$$\mathcal{M}^\vee{}_j f_i A_i = \mathcal{M}^\vee{}_j \mathcal{M}\mathcal{M}^\vee{}_i = f_j B_j \mathcal{M}_i \tag{18}$$

It follows that $\mathcal{M}^\vee{}_j A_i$ is divisible by f_j and thus $jet_{p_j-1}(\mathcal{M}^\vee{}_j A_i) = 0$ for $i \neq j$. Hence, due to the orders of $\mathcal{M}^\vee{}_j$,$\mathcal{M}$ we get: $jet_{p_j}(\mathcal{M}\mathcal{M}^\vee{}_j A_i) = jet_{p_j}(f_jA_jA_i) = 0$, implying:

$$jet_0(A_j)jet_0(A_i) \overset{for\ i\neq j}{=} \mathbb{0}, \text{ and } \sum jet_0(A_i) = \mathbb{1} \;\Rightarrow\; \mathbb{1} = \oplus jet_0(A_i) \tag{19}$$

The equivalence $\mathcal{M} \to U\mathcal{M}V$ results in: $A_i \to UA_iU^{-1}$ and $B_j \to VB_jV^{-1}$. So, by the conjugation by (constant) matrices can assume the block form: $jet_0(A_1) = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$ and $jet_0(A_2) = \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$.

Apply further conjugation to remove the terms of A_i in the columns of the i'th block to get:

$$A_1 = \begin{pmatrix} \mathbb{1} & * \\ \mathbb{0} & * \end{pmatrix}, \; A_2 = \begin{pmatrix} * & \mathbb{0} \\ * & \mathbb{1} \end{pmatrix} \tag{20}$$

Finally, use $A_1 + A_2 = \mathbb{1}$ to obtain $A_1 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$ and $A_2 = \begin{pmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$.

Do the same procedure for B_i's, this keeps A_i's intact. Now use the original definition, to write: $\mathcal{M}^\vee{}_i = \frac{1}{f}\mathcal{M}^\vee f_iA_i$ and $\mathcal{M}^\vee{}_i = f_iB_i\frac{1}{f}\mathcal{M}^\vee$. This gives:

$$\mathcal{M}^\vee = \oplus \frac{f}{f_i}\mathcal{M}^\vee{}_i \tag{21}$$

■

A natural question is: whether it's enough to ask the linear independence $Span(\underset{\alpha}{\cup} E_\alpha) = \oplus Span(E_\alpha)$ for the *fibres* at the singular point only, and not for the local trivialization in some neighborhood as in the conditions of the proposition. The following example shows, that consideration in the neighborhood is necessary.

Example 3.10. Consider a representation of the singularity $\det(\mathcal{M}) = (y^2 + x^{a+b})(y^2 - x^{a+b})$, $a < b$:

$$\mathcal{M} = \begin{pmatrix} -y & x^b & x & 0 \\ x^a & y & 0 & x \\ 0 & 0 & y & x^b \\ 0 & 0 & x^a & y \end{pmatrix}, \tag{22}$$

Below are the generators of the kernels of restriction to branches (obtained using the fact that E is spanned by the columns of $\mathcal{M}^\vee$):

(23)

$$E_1 = Ker(\mathcal{M}|_{y^2+x^{a+b}=0}) = \left\langle \begin{pmatrix} y(y^2 - x^{a+b}) \\ -x^a(y^2 - x^{a+b}) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x^b(y^2 - x^{a+b}) \\ y(y^2 - x^{a+b}) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2x^by \\ x(y^2 - x^{a+b}) \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad E_1|_{(0,0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$E_2 = Ker(\mathcal{M}|_{y^2-x^{a+b}=0}) = \left\langle \begin{pmatrix} -x(y^2 + x^{a+b}) \\ 0 \\ -y(y^2 + x^{a+b}) \\ x^a(y^2 + x^{a+b}) \end{pmatrix}, \begin{pmatrix} 2x^by \\ 0 \\ x^b(y^2 + x^{a+b}) \\ -y(y^2 + x^{a+b}) \end{pmatrix} \right\rangle, \quad E_2|_{(0,0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that $E_1|_{(0,0)} \neq E_2|_{(0,0)}$. But the minimal vector spaces into which E_1, E_2 embed locally are:

$$Span(E_1) = Span \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix} \approx \mathbb{C}^2 \subset \mathbb{C}^4, \; Span(E_2) = Span \begin{pmatrix} * \\ 0 \\ * \\ * \end{pmatrix} \approx \mathbb{C}^3 \subset \mathbb{C}^4 \tag{24}$$

and thus $Span(E_1) \cap Span(E_2) \neq \{0\}$. Hence the determinantal representation is not decomposable.

The second part of the proposition above is particularly useful, we give a reformulation of the necessary condition.

Lemma 3.11. *The determinantal representation $\mathcal{M}$ of $(C,0)=\cup(p_iC_i,0)$ satisfies $\mathcal{M}^\vee=\sum_i \frac{f}{f_i^{p_i}}{\mathcal{M}^\vee}_i$ iff for each decomposition $C=p_iC_i\cup\bigcup\limits_{j\neq i}p_jC_j$, for each intersection point $pt\in p_iC_i\cap\bigcup\limits_{j\neq i}p_jC_j$ and for each entry ${\mathcal{M}^\vee}_{\alpha\beta}$ of $\mathcal{M}^\vee$ any of the following equivalent conditions holds:*
• ${\mathcal{M}^\vee}_{\alpha\beta}\in\langle f_i^{p_i},\prod'_{j\neq i}f_j^{p_j}\rangle\mathcal{O}$. *(Here x,y are local coordinates around pt) and the right hand side is the ideal inside the ring of formal power series.*
• *the local degree of intersections at pt satisfies: $\Big((p_iC_i,pt),\{{\mathcal{M}^\vee}_{\alpha\beta}=0\}\Big)\ \geq\ \Big((p_iC_i,pt),\bigcup\limits_{j\neq i}p_jC_j\Big)$*

First check that the two later conditions are indeed equivalent. A chain of reformulations: ${\mathcal{M}^\vee}_{\alpha\beta}\in\langle f_i^{p_i},\prod\limits_{j\neq i}{}' f_j^{p_j}\rangle$ iff ${\mathcal{M}^\vee}_{\alpha\beta}|_{p_iC_i}\in\Big(\langle\prod\limits_{j\neq i}{}' f_j^{p_j}\rangle\mathcal{O}\Big)|_{p_iC_i}$ iff $\forall pt\in p_iC_i$ the vanishing order of ${\mathcal{M}^\vee}_{\alpha\beta}|_{p_iC_i}$ at pt is at least that of $\prod'_{j\neq i}f_j^{p_j}$. Which means $(C_i,\{{\mathcal{M}^\vee}_{\alpha\beta}=0\})_{pt}\geq(p_iC_i,\bigcup'_{j\neq}p_jC_j)_{pt}$.

Example 3.12. • If the curve is reduced with transversal self-intersections (i.e. nodes) only, then the order of each f_i at a singular point is 1. Thus the condition means: at each singular point of C all the entries of $\mathcal{M}^\vee$ vanish. This means: at each singular point of C the corank of $\mathcal{M}$ is 2.
• Let $mult(f)|_x=m$ then $rk\mathcal{M}\geq d-m$. If component of C at x is reduced and smooth then $rk\mathcal{M}|_x=d-m$ so, $corank\mathcal{M}|_x=$the number of components of C at x.

The criterion is based on the Noether fundamental theorem:

Proposition 3.13. *Let $C_{f_i}=\{f_i=0\}$ for $i=1,2,3$ be the algebraic projective curves such that*
• *C_1,C_2 have no common components*
• *for every intersection point $pt\in C_1\cap C_2$ locally: $f_3\in\langle f_1,f_2\rangle\mathcal{O}$*
Then (globally) $f_3\in\langle f_1,f_2\rangle\mathbb{C}[x,y,z]$, i.e. there exist homogeneous polynomials $a_1,a_2\in\mathbb{C}[x,y,z]$ such that $f_3=a_1f_1+a_2f_2$.

Proof. of 3.11

and compare the results. At the first step have ${\mathcal{M}^\vee}_{\alpha\beta}=A_if_i^{p_i}+{\mathcal{M}^\vee}_{\alpha\beta}(i)\prod\limits_{j\neq i}{}' f_j^{p_j}$. At the second:

$$ {\mathcal{M}^\vee}_{\alpha\beta}=\tilde{A}_if_i^{p_i}f_k^{p_k}+{\mathcal{M}^\vee}_{\alpha\beta}(i)\prod_{j\neq i}{}' f_j^{p_j}+{\mathcal{M}^\vee}_{\alpha\beta}(k)\prod_{j\neq k}{}' f_j^{p_j} $$

and so on. ■

3.3. **Global decomposability.** The local decomposability at each point implies the global one.

Theorem 3.14. *Let $C=\cup p_iC_i$ be the global decomposition into irreducible components. For a point $pt\in C$ let $\{p_iC_i\}_{i\in I_{pt}}$ be the components passing through the point. $\mathcal{M}$ is globally decomposable iff for each point $pt\in C$ locally: $\mathcal{M}\sim\oplus_{i\in I_{pt}}\mathcal{M}_i$*

Proof. The nontrivial direction is the global version of the proof of proposition 3.9.

By the assumption at each intersection point have $\mathcal{M}^\vee=\sum_{i\in I_{pt}}\frac{f}{f_i^{p_i}}{\mathcal{M}^\vee}_i$.

Note that f_i,f_j are relatively prime, thus: ${\mathcal{M}^\vee}_i\mathcal{M}=f_i^{p_i}A_i$ where A_i is a $d\times d$ matrix with *constant* coefficients. Similarly $\mathcal{M}{\mathcal{M}^\vee}_i=f_i^{p_i}B_i$.

Note that $\sum_i A_i=\mathbb{1}=\sum_i B_i$. In addition $A_iA_j=\mathbb{0}=B_iB_j$ (for $i\neq j$). Indeed: $f_i^{p_i}A_i{\mathcal{M}^\vee}_j={\mathcal{M}^\vee}_i\mathcal{M}{\mathcal{M}^\vee}_j=f_j^{p_j}{\mathcal{M}^\vee}_iB_j$. So $A_i{\mathcal{M}^\vee}_j$ is divisible by $f_i^{p_i}$ and ${\mathcal{M}^\vee}_iB_j$ is divisible by $f_i^{p_i}$. But f_i is prime and the degree of entries in ${\mathcal{M}^\vee}_i$ is p_id_i-1. Therefore: $A_i{\mathcal{M}^\vee}_j=0={\mathcal{M}^\vee}_iB_j$ (for $i\neq j$). And this causes $A_i({\mathcal{M}^\vee}_j\mathcal{M})=0=(\mathcal{M}{\mathcal{M}^\vee}_iB_j)$.

Thus $\{A_i\}$ and $\{B_i\}$ form a partition of $\mathbb{1}$, i.e. $\oplus A_i = \mathbb{1} = \oplus B_i$. So, by the multiplication $\mathcal{M} \to U\mathcal{M}V$ (and accordingly $\mathcal{M}^\vee \to V^{-1}\mathcal{M}^\vee U^{-1}$), which acts on A, B as: $A_i \sim \mathcal{M}^\vee{}_i\mathcal{M} \to UA_iU^{-1}$ and $B_i \sim \mathcal{M}\mathcal{M}^\vee{}_i \to VB_iV^{-1}$ can bring the collections $\{A_i\}$, $\{B_i\}$ to the block-diagonal form:

$$\begin{pmatrix} \tilde{A}_1 & 0 & .. & 0 \\ 0 & \tilde{A}_2 & 0 & \\ .. & .. & .. & \\ 0 & .. & 0 & \tilde{A}_k \end{pmatrix} = \mathbb{1} = \begin{pmatrix} \tilde{B}_1 & 0 & .. & 0 \\ 0 & \tilde{B}_2 & 0 & \\ .. & .. & .. & \\ 0 & .. & 0 & \tilde{B}_k \end{pmatrix} \tag{25}$$

Then from the definitions $\mathcal{M}^\vee{}_i\mathcal{M} \sim A_i$ and $\mathcal{M}\mathcal{M}^\vee{}_i \sim B_i$ the statement follows.

Finally note that from $\mathcal{M}^\vee{}_i\mathcal{M} = f_i^{p_i}A_i$ it follows that $\det(\mathcal{M}^\vee{}_i)f_i^{p_i} = f_i^{p_i d_i} \times$const. So the multiplicities are determined uniquely. ■

An immediate consequence of the proposition 3.4 is:

Corollary 3.15. If at each intersection point of the components the tangent cones of the components are disjoint then any weakly maximal determinantal representation of C is decomposable.

3.4. **Local (weak) maximality.** The local maximality implies the local weak maximality and local decomposability.

Lemma 3.16. *Let $\mathcal{M}_{p\times p}$ be a local maximal determinantal representation of $(C,0)$.*
• *The representation is weakly maximal, i.e. $p = mult(C,0)$.*
• *The representation is decomposable, i.e. for the branch decomposition $(C,0) = \cup(p_iC_i,0)$ one has: $\mathcal{M} \approx \oplus\mathcal{M}_i$ with $\mathcal{M}_i$ the (maximal) representation of $(p_iC_i,0)$.*

Proof. • By the maximality assumption and property 2.4: $ord(\mathcal{M}^\vee) \geq mult(C) - 1$.

Thus $ord(\det\mathcal{M}^\vee) \geq p(m-1)$. But $\det\mathcal{M}^\vee = f^{p-1}$, giving: $p(m-1) \leq (p-1)m$, i.e. $m \leq p$. As $\mathcal{M}|_{(0,0)} = \mathbb{0}$ have $m \geq p$ and the claim follows.
• Immediate corollary of the property the property 2.4 and the proposition 3.9. ■

The following example shows that weak maximality implies neither local maximality (even for a branch) nor local decomposability.

Example 3.17. • The case of two branches. Consider the weakly maximal local representation of $\{y^2 + yx^2 - x^4 = 0\}$

$$\mathcal{M} = \begin{pmatrix} y & x \\ x^3 & y + x^2 \end{pmatrix}, \ \mathcal{M}^\vee{}_{32} = x \notin Adj \tag{26}$$

• The case of one branch. Consider the following representation of $\det\mathcal{M} = x^2z^{d-2} + y^d$

$$\mathcal{M} = \begin{pmatrix} x & y & 0 & .. & 0 & \\ 0 & x & y & .. & & \\ 0 & 0 & z & y & 0.. & \\ .. & .. & .. & .. & .. & 0 \\ 0 & 0 & .. & .. & z & y \\ y & 0 & 0 & .. & 0 & z \end{pmatrix}, \overset{locally}{\rightsquigarrow} \begin{pmatrix} x & y & 0 \\ y^{d-1} & x & 0 \\ 0 & 0 & \mathbb{1}_{(d-2)\times(d-2)} \end{pmatrix} \Rrightarrow \langle\mathcal{M}^\vee\rangle \not\subset Adj) \tag{27}$$

Remark 3.18. • Weak maximality plus local decomposability reduces the question to "per branch" consideration. Therefore it's maximal iff each part $\mathcal{M}_\alpha$ is maximal. In particular if all the branches are smooth then weak maximality together with the local decomposability implies maximality.
• Let $\mathcal{M}$ be a weakly maximal representation of a multiple curve $\{f^p = 0\}$. Then each entry of $\mathcal{M}^\vee$ is divisible by f^{p-1}. Indeed, the assumption is $corank(\mathcal{M}|_C) = p$, i.e. any minor $\det(\mathcal{M}_{(pd-p+1)\times(pd-p+1)})$ is divisible by f. Then the claim follows by the property 2.1.

3.5. **On the types of local determinantal representations.** For a given curve the set of all the local determinantal representations can be subdivided into types by the following criteria. (The minimal modification is defined in §2.2.5, the relative conductor in §2.2.4).
• The determinantal representation $(\mathcal{M}, E)$ is called of C'/C type if $(C',0) \xrightarrow{\nu} (C,0)$ is the minimal modification such that $\nu^*_{C'/C}(E)$ is locally free.

For the adjoint matrix $\mathcal{M}^\vee$ let $\langle\mathcal{M}^\vee\rangle$ denote the ideal in $\mathbb{C}[[x,y]]$ generated by the entries of $\mathcal{M}^\vee$.

• The determinantal representation $(\mathcal{M},E)$ is called of $I^{cd}_{C'/C}$ type if $(C',0)\xrightarrow{\nu}(C,0)$ is the minimal modification such that $\langle\mathcal{M}^\vee\rangle\supset I^{cd}_{C'/C}$.

Note that for $(C'',0)\to(C',0)\xrightarrow{\nu}(C,0)$ the conductors embed: $I^{cd}_{C''/C}\subset I^{cd}_{C'/C}$.

The natural question is the equivalence of the two notions. Below are some relevant results.

Lemma 3.19. *Suppose the determinantal representation is locally decomposable $\mathcal{M}\sim\mathcal{M}_1\oplus\mathcal{M}_2$ according to the local decomposition $(C,0)=(C_1,0)\cup(C_2,0)$ (here $(C_i,0)$ can be further locally reducible). Let $(C',0)\xrightarrow{\nu}(C,0)$ be the minimal modification such the the pullback $\nu^*(E_C)$ is locally free. Then $(C',0)=\nu^{-1}(C_1,0)\coprod\nu^{-1}(C_2,0)$.*

In particular, if $\mathcal{M}$ is maximal then $(C',0)\to(C,0)$ separates all the branches.

Proof. Associated to $(C_i,0)\overset{i}{\hookrightarrow}(C,0)$ define the kernels of $\mathcal{M}_{C_i}$ as the closure: $E_i=i_*(Ker\mathcal{M}_{C_i\setminus\{0\}})$. Let s_i be a local section of E_i, normalized such that $s_i|\not=0$. The local decomposability implies: $s_1|_0\not\sim s_2|_0$. Hence if for the modification $C'\xrightarrow{\nu}C$, the pullback $\nu^*(E)=E\underset{\mathcal{O}_{(C,0)}}{\otimes}\mathcal{O}_{(C',0)}$ becomes locally free, then in C' the parts C_1,C_2 must be separated. ■

Remark 3.20. • A converse to the last lemma does not hold in full generality. Consider the determinantal representation from example 3.10. As the local section have distinct limits ($E_1|_0\not\sim E_2|_0$) the minimal modification $C'\to C$ should separate the branches. But the determinantal representation is indecomposable. In fact, if $a+b$ is even, then the singularity has only smooth branches, so the converse doesn't hold even in this case.

• If in the definition of the $I^{cd}_{C'/C}$ type one asks for the equality $\langle\mathcal{M}^\vee\rangle=I^{cd}_{C'/C}$ then the two notions are not equivalent. Here is a simple example with $\langle\mathcal{M}^\vee\rangle\not\subset I^{cd}_{C'/C}$.

Consider a (not weakly maximal) determinantal representation $\begin{pmatrix}x & y\\ 0 & y(x+y)\end{pmatrix}$ of an ordinary triple point $xy(x+y)=0$. In this case $\langle\mathcal{M}^\vee\rangle=\langle x,y\rangle$, the maximal ideal, which is the conductor ideal for the normalization $(\tilde{C},0)\to(C,0)$.

But the kernel becomes locally free already for the lifting: $Spec(\frac{\mathbb{C}[x,y]}{y(x+y)})\oplus\mathbb{C}[z]\to C$. Indeed, E_C is generated by $\begin{pmatrix}y(x+y)\\ 0\end{pmatrix},\begin{pmatrix}-y\\ x\end{pmatrix}$ as an $\mathcal{O}_{(C,0)}$ module. Consider the embedding

$$\frac{\mathbb{C}[x,y]}{xy(x+y)}\overset{i}{\hookrightarrow}\frac{\mathbb{C}[t_1,t_2]}{t_1t_2}\oplus\mathbb{C}[t_3],\qquad\begin{matrix}x\to t_1+t_3\\ y\to t_2-t_3\end{matrix}\tag{28}$$

The direct gives that the extended kernel module $E_C\underset{\mathcal{O}_{(C,0)}}{\otimes}\mathcal{O}_{(C',0)}$ is generated by $\begin{pmatrix}t_3-t_2\\ 0t_1+t_3\end{pmatrix}$, hence is locally free.

• Not every modification $(C',0)\to(C,0)$ is the minimal modification for some determinantal representation.

As an example, consider the ordinary triple point (singularity of the type $xy(x-y)=0$). The graph of all the modification is given in example 2.8.

In this case $corank(\mathcal{M}|_0)\le 3$, hence all the local determinantal representations are realizable as the global determinantal representations of a plane cubic. Those are classified in Appendix, proposition A.2. In particular, an intermediate step is the non-planar triple point.

If $corank(\mathcal{M}|_0)=3$ (weakly maximal representation) then the representation is completely decomposable, hence the minimal lifting is the full normalization.

If $corank(\mathcal{M}|_0)=1$ then the kernel E_C is already locally free.

There are two case with $corank(\mathcal{M}|_0)=2$: $\begin{pmatrix}x & 0\\ 0 & y(x+y)\end{pmatrix}$ and $\begin{pmatrix}x & y\\ 0 & y^2-x(x+y)\end{pmatrix}$. For both cases the minimal lifting $C'\to C$ implies that C' has separated branches.

Thus, the non-planar triple point does not correspond to any determinantal representation.

4. Properties of (co-)kernels bundles

Here we write in detail the classification sketched in [Ball-Vinnikov96, §3.2].

4.1. **(Co-)Kernels on the plane curve.** Recall (from §2.1.1) that the (co)kernels are defined by:

$$0 \to E_C \to \mathcal{O}^{\oplus d}(d-1)|_C \xrightarrow{M} \mathcal{O}^{\oplus d}(d)|_C \to Coker(M)_C \to 0,$$
$$0 \to E^l_C \to \mathcal{O}^{\oplus d}(d-1)|_C \xrightarrow{M^T} \mathcal{O}^{\oplus d}(d)|_C \to Coker(M^T)_C \to 0 \tag{29}$$

Let $Col(\mathcal{M}^\vee|_C)$ be the $\mathbb{C}[x_0,x_1,x_2]/(f)$ module, generated by the columns of $\mathcal{M}^\vee|_C$.

Proposition 4.1. • *The sheaf E_C is associated to the module $Col(\mathcal{M}^\vee|_C)$. Similarly, E^l_C is associated to $Col\Big((\mathcal{M}^\vee|_C)^T\Big)$. In particular $h^0(E_C) = d = h^0(E^l_C)$ and $h^0(E_C(-1)) = 0 = h^0(E^l_C(-1))$.*
• *The sheaves E_C, E^l_C, $Coker(M)_C$, $Coker(M^T)_C$ are torsion free and satisfy:*

$$E^*_C \otimes \mathcal{O}(2d-1)|_C = Coker(M^T)_C, \qquad (E^l_C)^* \otimes \mathcal{O}(2d-1)|_C = Coker(M)_C$$

.
• *If the curve is reduced, the sheaves E_C, E^l_C are of rank=1 and satisfy: $deg(E_C) = deg(E^l_C) = \frac{d(d-1)}{2}$, $h^1(E_C) = 0 = h^1(E^l_C)$, $h^1(E_C(-1)) = 0 = h^1(E^l_C(-1))$.*
• *The sheaves E_C, E^l_C are locally free iff $rank(\mathcal{M}|_C) = const$. In particular if the curve is smooth the sheaves are invertible.*

Proof. • As $(\mathcal{M}\mathcal{M}^\vee)|_C = 0$ the sheaf associated to $Col(\mathcal{M}^\vee|_C)$ is a subsheaf of E_C. Compare their sections near an arbitrary point. Let s be a local section of E_C, i.e. s is a d-tuple with entries in $\mathcal{O}(d-1)|_C$. So, by projective normality s is the restriction to C of some d-tuple S with entries in $\mathbb{C}[[x,y]]$ such that $\mathcal{M}S \sim f$ on $(\mathbb{C}^2,0)$. Let $v_1..v_d$ be the columns of $\mathcal{M}^\vee$, then for some linear combination $\mathcal{M}(S - \sum \alpha_i v_i) = 0$ on $(\mathbb{C}^2,0)$. (This is because $\mathcal{M}\mathcal{M}^\vee = f\mathbb{1}$.) As $\mathcal{M}$ is non-degenerate on $(\mathbb{C}^2,0)$ one has: $S \in Span(v_1..v_d)$.

Finally, the columns of $\mathcal{M}^\vee|_C$ are linearly independent (for linear combinations with constant coefficients). Otherwise by a $GL(d,\mathbb{C})$ transformation can bring $\mathcal{M}^\vee{}_{\mathbb{P}^2}$ to a form with one column-a multiple of f, this is impossible by the degree. Hence $h^0(E_C) = d$.

The vanishing $h^0(E_C(-1)) = 0$ is a consequence of projective normality of $C \subset \mathbb{P}^2$. Let $s \in H^0(E_C(-1))$, i.e. s is a d-tuple whose entries are sections of $\mathcal{O}_C(d-2)$. By projective normality $\Big(H^0(\mathcal{O}_{\mathbb{P}^2}(d-2)) \to H^0(\mathcal{O}_C(d-2)) \to 0\Big)$ we get a global section $S \in H^0(\mathcal{O}_{\mathbb{P}^2}(d-2))$ which restricts to s on the curve. Thus $M_{\mathbb{P}^2}S$ is proportional to f. But the total degree on the left is $(d-1)$, less than $deg(f) = d$.

The same for E^l_C.
• Note that E, E^l are torsion-free as sub-sheaves of the torsion-free: $\mathcal{O}^{\oplus d}(d-1)|_C$.

Regarding the co-kernels, consider the dual of the second exact sequence and twist it by $\mathcal{O}(2d-1)$:

$$\begin{array}{ccccccccc} 0 \to & E_C & \to & \mathcal{O}^{\oplus d}(d-1)|_C & \xrightarrow{M} & \mathcal{O}^{\oplus d}(d)|_C & \to & Coker(M)_C & \to 0 \\ & \wr\wr & & \| & & \| & & \downarrow & \\ 0 \to & Coker(M^T)^*_C(2d-1) & \to & \mathcal{O}^{\oplus d}(d-1)|_C & \xrightarrow{M} & \mathcal{O}^{\oplus d}(d)|_C & \to & (E^l_C)^*(2d-1) & \to Ext^1.. \end{array} \tag{30}$$

Thus $Coker(M)_C$ is a subsheaf of the torsion free sheaf $(E^l_C)^*(2d-1)$ and hence is itself torsion-free. Similarly for $Coker(M^T)_C$, hence the equalities follow.
• A reduced curve is smooth at its generic point, hence generically $corank(\mathcal{M}) = 1$ on C.

Use the exact sequences in (3) and the identifications E, E^l vs $Coker(M^T), Coker(M)$ to get:

$$deg(E_C) + deg(E^l_C) = deg(\mathcal{O}^d(d-1)|_C) - deg(\mathcal{O}^d(d)|_C) + deg(\mathcal{O}(2d-1)|_C) = d(d-1) \tag{31}$$

Note that $deg(E_C), deg(E^l_C) \geq 0$ since $h^0(E_C), h^0(E^l_C) > 0$. Hence also $deg(E_C), deg(E^l_C) \leq d(d-1)$. The cohomology of (3) is:

$$0 \to d \to d\binom{d+1}{2} \to d\Big(\binom{d+2}{2} - 1\Big) \to h^0\Big((E^l)^* \otimes \mathcal{O}(2d-1)\Big) \to h^1(E) \to 0 \tag{32}$$

Note that $h^1\Big((E^l)^* \otimes \mathcal{O}(2d-1)\Big) = 0$. In fact, Serre duality: $h^1\Big((E^l)^* \otimes \mathcal{O}(2d-1)\Big) = h^0\Big(E^l \otimes \mathcal{O}(-d-2)\Big) = 0$ because $deg(E^l \otimes \mathcal{O}(-d-2) < 0)$ (because $deg(E^l) \leq d(d-1)$).

Therefore Riemann-Roch for $(E^l)^* \otimes \mathcal{O}(2d-1)$ relates h^0 and the degree. Substituting all this to (32) one has $deg(E^l) + h^1(E) = \binom{d}{2}$. Similarly $deg(E) + h^1(E^l) = \binom{d}{2}$. Thus, finally:

$$d(d-1) = deg(E) + deg(E^l) \leq deg(E) + deg(E^l) + h^1(E) + h^1(E^l) = d(d-1) \tag{33}$$

Hence $h^1(E) = 0 = h^1(E^l)$ and Riemann-Roch for E, E^l gives their degrees. Finally, RR for $E_C(-1)$,$E^l_C(-1)$ gives $h^1(E_C(-1)) = 0 = h^1(E^l_C(-1))$.

• First, note that the dimension of the fibers of E, E^l at any point equals $corank(M)$. To see this localize the eq. (2) at any point of the plane:

$$\mathcal{O}^{\oplus d}_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_{pt} \xrightarrow{M} \mathcal{O}^{\oplus d}_{\mathbb{P}^2} \otimes \mathcal{O}_{pt} \to Coker(M) \otimes \mathcal{O}_{pt} \to 0 \tag{34}$$

Now we have a torsion free sheaf over the curve whose fibers have constant dimension, hence it is locally free. ■

It is possible to give a simple criterion for a torsion free sheaf to be the kernel of some determinantal representation. The following statement is a generalization from smooth case ([Vinnikov89], [Beauville00]). For singular curves it is probably known in folklore but we couldn't find any reference.

Proposition 4.2. *Let C be a reduced curve and E_C a torsion-free sheaf of rank=1. Assume $deg(E_C(-1)) = (p_a - 1) = \frac{d(d-3)}{2}$ and $h^0(E_C(-1)) = 0$. Then there exists $\mathcal{M} \in Mat(d \times d, H^0(\mathcal{O}(1)_{\mathbb{P}^2}))$ such that*

$$0 \to E_C \to \mathcal{O}^{\oplus d}_{\mathbb{P}^2}(d-1)|_C \xrightarrow{\mathcal{M}} \mathcal{O}^{\oplus d}_{\mathbb{P}^2}(d)|_C \to ..$$

Proof.
Step 1. We construct a candidate for $\mathcal{M}^\vee$. Define the associated (torsion-free) sheaf: $E^l_C(-1) := E_C(-1)^* \otimes w_C$. Its degree is: $-deg(E_C) + deg(w_C) = p_a - 1$. Thus Riemann-Roch and Serre duality give:

$$\begin{array}{l} h^0(E^l_C(-1)) = h^1(E_C(-1)) = h^0(E_C(-1)) - deg(E_C(-1)) + (p_a - 1) = 0, \\ h^1(E^l_C(-1)) = h^0(E^l_C(-1)) - deg(E^l_C(-1)) + (p_a - 1) = 0, \\ h^0(E_C) = d = h^0(E^l_C) \end{array} \tag{35}$$

Let $H^0(E_C, C) = Span(e_1..e_d)$ and $H^0(E^l_C, C) = Span(e^l_1..e^l_d)$.

By the construction of E_C, E^l_C one has the non-degenerate pairing: $\Big(E_C, E^l_C\Big) \to w_C(2) \approx \mathcal{O}_{\mathbb{P}^2}(d-1)|_C$. (The last isomorphism is due to adjunction.)

Hence, by the projective normality $(H^0(\mathcal{O}_{\mathbb{P}^2}(d-1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(d-1)|_C) \to 0)$ the chosen basis of global sections defines a matrix of homogeneous polynomials:

$$< e_i, e^l_j > \to \{V_{ij}\} \in Mat\Big(pd \times pd, H^0(\mathcal{O}_{\mathbb{P}^2}(d-1)(-D))\Big) \tag{36}$$

The entries V_{ij} are defined up to the global sections of $I_C(d-1)$. As $I_C =< f >$ and $deg(f) = d$ we obtain that the homogeneous polynomials V_{ij} are defined uniquely.

Step 2. We prove that V is globally non-degenerate. Let $L \subset \mathbb{P}^2$ be the generic line, such that $L \pitchfork C = \{pt_1..pt_d\}$ (distinct reduced points). By linear transformations applied to $Span(e_1..e_d)$ and $Span(e^l_1..e^l_d)$ can choose the sections satisfying the conditions:

$$C \supset div(e_i) \geq \sum_{j \neq i}{}' pt_j \ngeq pt_i, \qquad C \supset div(e^l_i) \geq \sum_{j \neq i}{}' pt_j \ngeq pt_i \tag{37}$$

Indeed, suppose $e_1|_{pt} \neq 0$, then can assume $e_{i>1}|_{pt_1} = 0$ and continue with the remaining points $\{pt_{i>1}\}$ and sections $\{e_{i>1}\}$. At each step there is at least one section which doesn't vanish at the given point (otherwise get a section vanishing at all the points, thus producing a section of $E(-1)$).

By the choice of sections, one has $\forall k, \forall i \neq j$: $V_{ij}|_{pt_k} = 0$. So, $deg(V_{ij}|_L) \geq d$. But $deg(V_{ij}(x, y, z)) = d - 1$ hence $V_{ij}|_L \equiv 0$. On the contrary: $V_{ii}|_{pt_i} \neq 0$, so $V_{ii}|_L \not\equiv 0$. Therefore, $V|_{X_p}$ is a diagonal matrix, none of whose diagonal entries vanishes identically on the curve X_p. Thus $\det(V) \not\equiv 0$.

Step 3. Construction of $\mathcal{M}$. By construction of V, all its 2×2 minors are degenerate on C, i.e. $\det(V_{2\times2})$ is divisible by f. Thus by the property 2.1 all the entries of $V^\vee \in Mat\Big(d \times d, H^0(\mathcal{O}(d-1)(d-1))\Big)$ are divisible by f^{d-2}. Hence the natural candidate for determinantal representation is: $\mathcal{M} := \frac{V^\vee}{f^{d-2}} \in Mat\Big(d \times d, H^0(\mathcal{O}(1)\Big)$. From here:

$$\det(\mathcal{M}) = \frac{\det(V^\vee)}{f^{d(d-2)}} = (\frac{V}{f^{d-1}})^{d-1} f \tag{38}$$

As $det(V)$ is of degree $(d-1)$ and f is irreducible one has (scaling by a constant): $\det(V) = f^{d-1}$ and $\det(\mathcal{M}) = f$.

Finally, by construction $MV = f\mathbb{1}$, while $V = \{(e_i, e^l_j)\}$. Thus $\mathcal{M}(e_i, e^l_j) \equiv \mathbb{0}(f)$, causing: $\mathcal{M}e_i \equiv 0(f)$. Therefore $Span(e_1..e_{pd}) \subset Ker(\mathcal{M}|_C)$ and $E = Ker(\mathcal{M}|_C)$ (by the equality of dimensions). ■

Appendix A. Normal families in some particular cases

The presentations are classified up to the equivalence $\mathbb{P}GL(3) \times GL(d) \times GL(d)$. Here the first factor corresponds to the projective transformations of the plane, the rest is multiplication of $\mathcal{M}$ by invertibles.

Here we consider some specific cases and give the *normal families*, i.e. the parameterizations $\mathcal{M}(\alpha_1..\alpha_n)$ such that the projection map $\mathcal{M}(\alpha_1..\alpha_n) \to \frac{Mat\left(d\times d, H^0(\mathcal{O}_{\mathbb{P}^2}(1))\right)}{\mathbb{P}GL(3)\times GL(d)\times GL(d)}$ is surjective and has finite fibres.

A nice way to parameterize determinantal representations of reduced curves of a given degree was given in [Dickson21, Theorem 6]. We include it for completeness.

Proposition A.1. *Let $C \subset \mathbb{P}^2$ a reduced projective curve of degree d, not a line arrangement. By $\mathbb{P}GL(3) \times GL(d) \times GL(d)$ any of its determinantal representations can be brought to the form: $\mathcal{M} = z\mathbb{1} + yD + x\mathcal{M}_x$, where D is a diagonal matrix, $tr(D) = 0 = tr(\mathcal{M}_x)$ and $(\mathcal{M}_x)_{11} = 0$. The remaining $GL(d) \times GL(d)$ transformations (preserving the given choice) are of the form: $\mathcal{M} \to U\mathcal{M}U^{-1}$ with U diagonal.*

Proof. Consider two points $pt_1, pt_2 \in \mathbb{P}2\setminus C$. Assume the spanned line $\overline{pt_1, pt_2}$ is generic, i.e. intersects the curve at d *distinct* points. Choose the coordinates such that $pt_1 = (0, 0, 1)$ and $pt_2 = (0, 1, 0)$. Let $\mathcal{M} = x\mathcal{M}_x + y\mathcal{M}_y + z\mathcal{M}_z$ be a determinantal representation. By the choice of points we have: $\det(\mathcal{M}_z) \neq 0 \neq \det(\mathcal{M}_y)$. Hence by $GL(d) \times GL(d)$ can set $\mathcal{M}_z = \mathbb{1}$ and bring $\mathcal{M}_y$ to its Jordan form.

Note that $\mathcal{M}_y$ has all the eigenvalues distinct (and thus is diagonalizable). Indeed, restrict $\mathcal{M}$ to the line $\overline{pt_1, pt_2}$, i.e. $\{x = 0\}$. Then $\det(\mathcal{M}|_{x=0})$ vanish at d distinct points. Therefore can assume $\mathcal{M} = z\mathbb{1} + yD + x\mathcal{M}_x$, with $D = diag(\lambda_1..\lambda_d)$ with entries ordered as $|\lambda_i| \leq |la_{i+1}|$.

Now use the remaining transformations of $PGL(3)$: $z \to z + x + y$ and $y \to y + x$ to set: $tr(D) = 0$, $tr(\mathcal{M}_x) = 0$ and $(\mathcal{M}_x)_{11} = 0$.

Finally observe that the $GL(d) \times GL(d)$ transformations $\mathcal{M} \to U\mathcal{M}V$ preserving the given structure must satisfy: $U\mathbb{1}V = \mathbb{1}$ and $UDV = D$ and all the diagonal entries of D are distinct. Hence $V = U^{-1}$ must be diagonal. ■

For our needs we use other ways to classify.

We constantly use the following observations. Let $\mathcal{M} = x\mathcal{M}_x + y\mathcal{M}_y + z\mathcal{M}_z$, with $\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z \in Mat(d\times d)$ constant matrices be a determinantal representation of the curve $\{f_d = 0\}$.

• Let k be the highest power of x that appear in f_d. Then $rank(\mathcal{M}_x) \geq k$. Similarly for $(y, \mathcal{M}_y)$ and $(z, \mathcal{M}_z)$.
To see this, apply the conjugation $\mathcal{M} \to U\mathcal{M}V$ to diagonalize $\mathcal{M}_x$.

• Suppose $\mathcal{M}_z$ is of $rank = k$. By conjugations $\mathcal{M} \to U\mathcal{M}V$ bring it to the form: $\mathcal{M}_z = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$. Then the conjugations that preserve $\mathcal{M}_z$ are precisely of the form:

$$\mathcal{M} \to \begin{pmatrix} U_{k\times k} & U^{(1)}_{k\times(d-k)} \\ \mathbb{0} & U^{(2)}_{(d-k)\times(d-k)} \end{pmatrix} \mathcal{M} \begin{pmatrix} U^{-1}_{k\times k} & \mathbb{0} \\ V^{(1)}_{(d-k)\times k} & V^{(2)}_{(d-k)\times(d-k)} \end{pmatrix} \tag{39}$$

• Suppose $\mathcal{M}_z$ is as above and write $\mathcal{M}_y$ in the corresponding block-form

$$\mathcal{M}_z = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}, \ \mathcal{M}_y = \begin{pmatrix} (\mathcal{M}_y)_1 & (\mathcal{M}_y)_2 \\ (\mathcal{M}_y)_3 & (\mathcal{M}_y)_4 \end{pmatrix} \tag{40}$$

Assume $(\mathcal{M}_y)_4$ is of full rank (d-k). By conjugations preserving $\mathcal{M}_z$ one can bring $\mathcal{M}_y$ to the following form:

$$\mathcal{M}_y \to \begin{pmatrix} U((\mathcal{M}_y)_1 - (\mathcal{M}_y)_2(\mathcal{M}_y)_4^{-1}(\mathcal{M}_y)_3)U^{-1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{41}$$

We restrict to the singular curves only, for the classifications for smooth conics and cubics cf. e.g. [Vinnikov89].

A.1. **Small degree cases.**

A.1.1. *Conics.*

• Two lines: $f = xy$, $\mathcal{M} = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ (not weakly maximal) or $\mathcal{M} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ (weakly maximal).

• The double line: $f = x^2$, $\mathcal{M} = \begin{pmatrix} x & z \\ 0 & x \end{pmatrix}$ (not weakly maximal) or $\mathcal{M} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ (weakly maximal).

A.1.2. *Cubics.*

• Three generic lines: $f = xyz$. Restrict to weakly maximal representations. By checking at the singular points get: $rank(\mathcal{M}_x) = rank(\mathcal{M}_y) = rank(\mathcal{M}_z) = 1$. Then: $\mathcal{M} = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$

• Consider cubics with an ordinary triple point, i.e. triples of (distinct) lines intersecting at one point.

Proposition A.2. *Let $\mathcal{M}$ be a determinantal representation of a cubic with an ordinary triple point at $(0,0,1) \in \mathbb{P}^2$.*

$\star$ *For $rank(\mathcal{M}|_{(0,0,1)}) = 2$ the normal families are:*

$$\mathcal{M} = \begin{pmatrix} z & 0 & y \\ ..x + ..y & z + ..y & 0 \\ 0 & x & 0 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} z & 0 & y \\ 0 & z + ..x + ..y & -y \\ x & x & 0 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} z & ..y & x \\ ..x & z & -y \\ y & x & 0 \end{pmatrix} \tag{42}$$

$\star$ *For $rank(\mathcal{M}|_{(0,0,1)}) = 1$ the normal families are:*

$$\mathcal{M} = \begin{pmatrix} z & 0 & y \\ 0 & x & 0 \\ x + y & 0 & 0 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} z & y & ..x + ..y \\ 0 & x & y \\ x & 0 & 0 \end{pmatrix} \tag{43}$$

$\star$ *If $\mathcal{M}|_{(0,0,1)} = \mathbb{0}$ (a maximal determinantal representation), then $\mathcal{M}$ is equivalent to a diagonal matrix.*

Here at each place the dots mean some numerical coefficients, they are generic enough (so that the ordinary triple point doesn't degenerate to a singularity with a double line).

Proof. As the curve has a triple point at $(0,0,1)$ the defining equation depends on the variables x, y only.

$\star$ By the rank assumption we have: $\mathcal{M} = \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} .. & .. & \gamma \\ .. & .. & \delta \\ \alpha & \beta & 0 \end{pmatrix}$ where the second matrix depends on x, y only. (The zero in the second matrix is forced as z^2 is not in the equation.)

Using the remaining freedom can assume $\beta \neq 0$ and in fact set $\beta = x$. As z is not in the equation we have: $x\delta + \gamma\alpha = 0$. Note that at least one of α, γ, δ is not proportional to x (otherwise the curve has a double line).

Thus the following cases are possible (using $x\delta + \gamma\alpha = 0$)

$$\begin{aligned} &\alpha \sim x: \begin{pmatrix} z + .. & .. & y \\ .. & z + .. & 0 \\ 0 & x & 0 \end{pmatrix} \ \textit{or} \ \begin{pmatrix} z + .. & .. & y \\ .. & z + .. & -y \\ x & x & 0 \end{pmatrix} \\ &0 \neq \alpha \sim y: \begin{pmatrix} z + .. & .. & x \\ .. & z + .. & -y \\ y & x & 0 \end{pmatrix} \end{aligned} \tag{44}$$

Now using the 3'd row and column can remove some x, y terms from the upper-left 2×2 block to get the stated forms.

$\star$ By the rank assumption can assume: $\mathcal{M} = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} .. & .. & .. \\ .. & x & .. \\ .. & 0 & 0 \end{pmatrix}$. Here the second matrix depends on x, y only, the zeros are forced by the absence of z in the determinant.

By subtracting the second column from the third (if necessary) can assume the second matrix either $\begin{pmatrix} .. & .. & \alpha \\ .. & x & 0 \\ .. & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} .. & .. & .. \\ .. & x & y \\ .. & 0 & 0 \end{pmatrix}$. In the first case α is not proportional to x, otherwise the curve has a double line. So can set $\alpha = y$. Hence the possibilities are:

$$\begin{pmatrix} z+..x & 0 & y \\ ..y & x & 0 \\ ..x+..y & 0 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} z+..x+..y & ..y & ..x+..y \\ 0 & x & y \\ .. & 0 & 0 \end{pmatrix} \tag{45}$$

Note that in the first case the entry $\mathcal{M}_{3,1}$ is not proportional to x, neither to y (to avoid the double line). Hence by further rescaling can set it to $x+y$.

In the second case can assume $\mathcal{M}_{3,1} = x+y$ or $\mathcal{M}_{3,1} = x$ or $\mathcal{M}_{3,1} = y$. All the three options are equivalent by the coordinate change: $x+y \to x$ or $y \to x$ and further addition of the 2,3 rows/columns.

At the end apply the relevant transformation $z \to z+..x+..y$. ■

A.2. **A curve with a singularity of type** $y(y^{d-2}+x^{d-1})$**.** It is immediate, that in this case the curve is necessarily reducible. So, start from the general expression $y(..zy^{d-2}+..x^{d-1}+..+..y^{d-1})$ (with some coefficients). Apply $\mathbb{P}GL(3)$ to bring this to $y(zy^{d-2}+x^{d-1}+..+..x^2y^{d-3})$ (with some coefficients). There remains no further $PGL(3)$ transformations.

Proposition A.3. *The weakly maximal determinantal representations are parameterized by families:*

$$\mathcal{M} = \begin{pmatrix} z & 0 & \alpha_1 x & .. & \alpha_{d-2}x \\ 0 & y & x & 0 & \\ 0 & 0 & y & x & .. \\ .. & & & & x \\ x & 0 & .. & 0 & y \end{pmatrix}, \quad or\ \mathcal{M} = \begin{pmatrix} z & \alpha_1 x & .. & & & & & & \alpha_{d-1}x \\ 0 & y & x & 0 & .. & & & .. & 0 \\ .. & 0 & y & x & 0.. & & & .. & .. \\ .. & .. & .. & .. & x & 0 & .. & & .. \\ 0 & .. & .. & .. & y & \mathbf{0} & .. & & 0 \\ 0 & & & & 0 & y & x & & 0 \\ 0 & & & & & & y & & x \\ x & 0 & & & .. & .. & 0 & & y \end{pmatrix} \tag{46}$$

Proof. Let $f = \det(x\mathcal{M}_x + y\mathcal{M}_y + z\mathcal{M}_z)$ where $\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z$ are $d \times d$ constant matrices. By the observations above and weak maximality: $rank(\mathcal{M}_z) = 1$, $rank(\mathcal{M}_y) = d-1$, $rank(\mathcal{M}_x) = d-1$. Further, as the term zy^{d-1} appears in f, can assume:

$$\mathcal{M}_z = \begin{pmatrix} \mathbb{1}_{1\times 1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix}, \ \mathcal{M}_y = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{(d-1)\times(d-1)} \end{pmatrix} \tag{47}$$

Let $\mathcal{M}_x = \begin{pmatrix} a & \bar{a} \\ a^\dagger & (\mathcal{M}_x)_{(d-1)\times(d-1)} \end{pmatrix}$. As zx^i is absent for any i, one has: $rank((\mathcal{M}_x)_{(d-1)\times(d-1)}) < d-1$. Bring $(\mathcal{M}_x)_{(d-1)\times(d-1)}$ to the Jordan form and use the absence of xy^iz^j in $\det \mathcal{M}$.

- Assume $rank((\mathcal{M}_x)_{(d-1)\times(d-1)}) = d-2$, i.e. there is precisely one cell in the Jordan form. Then:

$$\mathcal{M} = z\begin{pmatrix} \mathbb{1}_{1\times 1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{(d-1)\times(d-1)} \end{pmatrix} + x\begin{pmatrix} a & & \bar{a} & & \\ & 0 & 1 & & \\ a^\dagger & 0 & 0 & 1 & \\ & 0 & .. & 0 & 1 \end{pmatrix} \tag{48}$$

Check the remaining conjugations, i.e. $\mathcal{M} \to U\mathcal{M}V$ such that $\mathcal{M}_z = U\mathcal{M}_zV$, $\mathcal{M}_y = U\mathcal{M}_yV$ and the Jordan form of $(\mathcal{M}_x)_{(d-1)\times(d-1)}$ is preserved. We get: $U = \begin{pmatrix} 1 & 0 \\ 0 & U_J \end{pmatrix} = V^{-1}$, where $U_J = \sum\limits_{i=0}^{d-1} \alpha_i((\mathcal{M}_x)_{(d-1)\times(d-1)})^i$ (with the convention: $\alpha_0((\mathcal{M}_x)_{(d-1)\times(d-1)})^0 = \mathbb{1}$).

Therefore $a^\dagger$ can be brought to the form $(0,...0,1)$ or $(0,...0,0)$. The last case implies $det\mathcal{M} = z(..)$ and thus is not realized. At this step all the freedom of conjugations is exhausted. Indeed, if $U_J\begin{pmatrix} 0 \\ .. \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ .. \\ 1 \end{pmatrix}$ then $U_J = \mathbb{1}$.

Finally by checking the determinant of the so obtained matrix one has: $a = 0$ and $\bar{a} = (0, *..*)$.

- Suppose $rank((\mathcal{M}_x)_{(d-1)\times(d-1)}) < d-2$. As $yx^{d-1} \in \det \mathcal{M}$ one has: $rank((\mathcal{M}_x)_{(d-1)\times(d-1)}) = d-3$.

Again, as $xy^iz \notin \det\mathcal{M}$ all the eigenvalues of $A_{(d-1)\times(d-1)}$ are zero and the matrix has precisely 2 Jordan cells: $(\mathcal{M}_x)_{(d-1)\times(d-1)} = \begin{pmatrix}(\mathcal{M}_x)_1 & \mathbb{0}\\ \mathbb{0} & (\mathcal{M}_x)_2\end{pmatrix}$. The remaining freedom now is $\mathcal{M}\to U\mathcal{M}U^{-1}$ with $U = \begin{pmatrix}1 & 0 & 0\\ \mathbb{0} & U_1 & \mathbb{0}\\ \mathbb{0} & \mathbb{0} & U_2\end{pmatrix}$ where $U_i = \sum_{j\geq 0}((\mathcal{M}_x)_i)^j a_j^{(i)}$.

Proceed as previously to get:

$$\mathcal{M} = \begin{pmatrix} z & \alpha_1 x & .. & .. & & .. & .. & \alpha_{d-1}x\\ 0 & y & x & 0 & .. & & 0 & \\ 0 & 0 & y & x & ... & & & \\ .. & . & & & & & & \\ \beta_1 x & 0 & .. & & y & 0.. & & \\ 0 & .. & & & 0 & y & x & 0..\\ & 0 & .. & & & & & \\ \beta_2 x & 0 & .. & & & & 0 & y \end{pmatrix} \tag{49}$$

Where $\beta_i = 0$ or 1. As the total determinant is not divisible by z, at most one of β_i can vanish.

Assume $\beta_1 = 0$, $\beta_2 = 1$. By checking explicitly the determinant of $\mathcal{M}$ one has no further restrictions. The case $\beta_2 = 0$, $\beta_1 = 1$ is symmetric.

Suppose $\beta_1\beta_2 \neq 0$. This case is reduced to the previous once by the additions of rows:

$$\begin{aligned}\mathcal{M} = \det\begin{pmatrix} z & \alpha_1 x & .. & .. & & .. & .. & \alpha_{d-1}x\\ 0 & y & x & 0 & .. & & 0 & \\ 0 & 0 & y & x & ... & & & \\ .. & . & & & & & & \\ x & 0 & .. & & y & 0.. & & \\ 0 & .. & & & 0 & y & x & 0..\\ & 0 & .. & & & & & \\ x & 0 & .. & & & & 0 & y \end{pmatrix} = \det\begin{pmatrix} z & \alpha_1 x & .. & .. & & .. & .. & \alpha_{d-1}x\\ 0 & y & x & 0 & .. & & 0 & \\ 0 & 0 & y & x & ... & & & \\ .. & . & & & & & & \\ 0 & 0 & .. & & y & 0.. & & -y\\ 0 & .. & & & 0 & y & x & 0..\\ & 0 & .. & & & & & \\ x & 0 & .. & & & & 0 & y \end{pmatrix} = \\ = \det\begin{pmatrix} z & \alpha_1 x & .. & .. & & .. & .. & (\alpha_{d-1}-\alpha_{d_1})x\\ 0 & y & x & 0 & .. & & 0 & \\ 0 & 0 & y & x & ... & & & \\ .. & . & & & & & & \\ 0 & 0 & .. & & y & 0.. & & 0\\ 0 & .. & & & 0 & y & x & 0..\\ & 0 & .. & & & & & \\ x & 0 & .. & & & & 0 & y \end{pmatrix}\end{aligned} \tag{50}$$

■

A.3. **A curve with a singularity of type $x^{d-1}+y^d$.** Let $(\mathcal{M}_z)_d \subset \mathbb{P}^2$ be a curve with the singularity of type $x^{d-1}+y^d$. Apply $\mathbb{P}GL(3)$ to put the point to origin, let $\hat{y}$ axis be the tangent line, so locally: $x^{d-1}+y^d+...x^d$ (with some coefficients). Finally, by the transformations $y\to y+\alpha x$ and $z\to z+\alpha x$ and scaling one obtains: $zx^{d-1}+y^d+\sum\limits_{i\geq 2}^{d-1}\alpha_i x^{d-i}y^i$. At this step all the $\mathbb{P}GL(3)$ transformations are exhausted.

Proposition A.4. *The normal family for weakly maximal determinantal representations of $(\mathcal{M}_z)_d$ with a point of type $x^{d-1}+y^d$ is:* $\mathcal{M} = \begin{pmatrix} z & ..y & ..y & & ..y\\ 0 & x & y & 0.. & 0\\ 0 & 0 & x & y & 0\\ 0 & .. & .. & .. & \\ y & 0 & 0.. & 0 & x\end{pmatrix}$

Proof. For $f = \mathcal{M}_x x+\mathcal{M}_y y+\mathcal{M}_z z$ by weak maximality have $rank(\mathcal{M}_z) = 1$, so $\mathcal{M}_z$ can be assumed as $\begin{pmatrix}1 & \mathbb{0}\\ \mathbb{0} & \mathbb{0}\end{pmatrix}$. The remaining freedom of conjugation is now $\mathcal{M}\to\begin{pmatrix}1 & \vec{u}\\ 0 & U\end{pmatrix}\mathcal{M}\begin{pmatrix}1 & 0\\ \vec{v} & V\end{pmatrix}$. Note that $rank(\mathcal{M}_x) = d-1$ so by the freedom can partially diagonalize by the transformation

$$\mathcal{M}_x \to U\mathcal{M}_x V = \begin{pmatrix} a+\vec{u}a^{\downarrow}+\vec{v}\vec{a}+.. & (a^{\downarrow}+\vec{u}\mathcal{M}_x)V \\ U(a^{\downarrow}+\mathcal{M}_x\vec{v}) & U\mathcal{M}_x V \end{pmatrix} \tag{51}$$

Then get $\mathcal{M}_x = \begin{pmatrix} a & \vec{a} \\ a^{\downarrow} & \mathbb{1} \end{pmatrix}$ (that the lower right block is of full rank follows from the presence of zx^{d-1} term). Then can kill the off-block terms and arrive at $\mathcal{M}_x = \begin{pmatrix} a & 0 \\ 0 & \mathbb{1} \end{pmatrix}$. Here $a=0$ as there's no term x^d. The remaining freedom is $\mathcal{M} \to \begin{pmatrix} 1 & \mathbb{0} \\ \mathbb{0} & U \end{pmatrix} \mathcal{M} \begin{pmatrix} 1 & \mathbb{0} \\ \mathbb{0} & U \end{pmatrix}^{-1}$. Now bring the lower block of $\mathcal{M}_y$ to the Jordan form. Since no terms zx^jy^i appear, have $\mathcal{M}_y = \begin{pmatrix} .. & & ... & \\ & 0 & 1 & \\ .. & 0 & 0 & 1 \end{pmatrix}$. Since y^d appears in the polynomial get: there is only one Jordan cell.

The remaining freedom: all the matrices commuting with the lower block of $\mathcal{M}_y$. They are: $\sum \lambda_i \mathcal{M}_y^i$. So $\mathcal{M}_y$ can be brought to the form: $\begin{pmatrix} b & \vec{b} \\ 0 & Jordan(\mathcal{M}_y) \\ 1 & \end{pmatrix}$.

Here $b=0$ as yx^{d-1} does not appear in $\det \mathcal{M}$. The determinant computation shows that there is no further requirement on $\vec{b}$. This gives the statement. ■

Appendix B. On the maximality of self-adjoint positive-definite determinantal representations

Let $\mathcal{M} \in Mat(d \times d, H^0(\mathcal{O}_{\mathbb{RP}^2}(1)))$ be a self-adjoint positive definite determinantal representation of a real projective plane curve. Namely, for any $pt \in \mathbb{RP}^2$ one has: $\mathcal{M}|_{pt} = \mathcal{M}|^{\dagger}_{pt}$ and $\mathcal{M}|_{pt}$ is positive definite. Hence one can choose homogeneous coordinates to represent: $\mathcal{M} = xA + yB + zC$ with $A, B, C \in Mat(d \times d, \mathbb{R})$ positive definite and self-adjoint.

Note that a self-adjoint positive-definite determinantal representation defines a hyperbolic curve (cf. e.g. [Vinnikov93, §6]). Thus such a curve can have at most one singular point with a non-smooth branch. In the later case the region of hyperbolicity degenerates to this singular point. Hence, if a determinantal representation is positive definite (and not just semidefinite) then the region of hyperbolicity has a non-empty interior and thus all the branches of the curve are smooth.

Proposition B.1. *Let $\mathcal{M}$ be any self-adjoint positive-definite representation of the curve $C = \{\det \mathcal{M} = 0\} \subset \mathbb{RP}^2$ which has only smooth branches. The representation is maximal at all the real points.*

Proof. As the curve has smooth branches only, it's enough to prove that the representation is weakly maximal and locally decomposable at any point of the curve (cf. remark 3.18). Can assume that the region of hyperbolicity is in the affine part of the plane.

Let $x \in \mathbb{RP}^2$ be a singular point of the curve. We want to prove that $corank(\mathcal{M}|_x) = mult(C, x)$ and the corresponding kernel is the direct sum of the kernels arising from the branches.

Take the generic point y at infinity such that:
- the line $l = \overline{xy}$ is not tangent to any of the branches of the curve at the point x.
- the line l passes through the region of hyperbolicity.

By hyperbolicity the local (real) intersection degree of the line with the curve is $l \cap (C, x) = mult(C, x)$. Hence, can choose the line l_ϵ obtained by (arbitrarily) small rotation around y (i.e. $y \in l_\epsilon$) such that:
- in the small neighbourhood of x the line l_ϵ intersects C transversally at $mult(C, x)$ points.
- the line l_ϵ passes through the region of hyperbolicity.

Restrict the matrix $\mathcal{M}$ to this line: $(1-t)\mathcal{M}|_y + t\mathcal{M}|_z$. Here $z \in l_\epsilon$ belongs to the region of hyperbolicity. The choice of l_ϵ implies that for any small ϵ the equation $\det((1-t)\mathcal{M}|_y + t\mathcal{M}|_z) = 0$ has $mult(C, x)$ solutions for t such that $(1-t)y + tz$ is close to x. As $\epsilon \to 0$ these solutions converge to x.

Finally, note that $\mathcal{M}|_z$ is self-adjoint positive-definite, so can be written as: $\mathcal{M}|_z = U^T U$. Then the determinantal equation can be written as: $\det((1-t)U^{-T}\mathcal{M}|_y U^{-1} + t\mathbb{1}) = 0$ and is now the equation on eigenvalues of self-adjoint matrix. Thus the corresponding eigenvectors are *orthogonal*.

In the limit $\epsilon \to 0$, these eigenvectors converge to some (mutually orthogonal) vectors of the kernel of $\mathcal{M}|_x$. Therefore the kernel is the direct sum of (one-dimensional) kernels coming from the branches (i.e. the determinantal representation is locally decomposable at x) and in particular $corank(\mathcal{M}|_x) = mult(C, x)$ i.e. the determinantal representation is weakly maximal at x. Hence the statement. ■

Appendix C. Further questions/directions

C.1. **(Co-)Kernels on the normalization.** Let $\mathcal{M}$ be a maximal determinantal representation of some projective curve. By the decomposability (corollary 3.11) can assume the curve to be of the form $pC = \{f^p = 0\} \subset \mathbb{P}^2$, where C is irreducible reduced.

We need two related objects: kernel E and the left-kernel E^l (i.e. the kernel of M^T). As explained in §2.1.1 we consider here the torsion free sheaves, they lift to vector bundles on the normalization $\tilde{E}_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}}(-D)$, $\tilde{E}^l_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}}(-D)$. At each point E,E^l are spanned (as vector spaces) by the columns/rows of the adjoint matrix $\mathcal{M}^\vee$. This however leaves the freedom of twisting by an invertible sheaf. By the remark 3.18 each entry of $\mathcal{M}^\vee$ is divisible by f^{p-1}, so $\mathcal{M}\frac{\mathcal{M}^\vee}{f^{p-1}} = f\mathbb{1}$. Therefore we fix the "normalization":

$$\begin{aligned} &0 \to E_C \to \mathcal{O}^{\oplus dp}(d-1)|_C \xrightarrow{M} \mathcal{O}^{\oplus dp}(d)|_C \to Coker(M)_C \to 0, \\ &0 \to E^l_C \to \mathcal{O}^{\oplus dp}(d-1)|_C \xrightarrow{M^T} \mathcal{O}^{\oplus dp}(d)|_C \to Coker(M^T)_C \to 0 \end{aligned} \tag{52}$$

i.e. E, E^l are spanned as sheaves by the columns/rows of $\frac{\mathcal{M}^\vee}{f^{p-1}}$.

Remark C.1. We often use the completeness of adjoint linear system (cf. e.g. [ACGH-book, Appendix A§2]). Let C be an irreducible reduced projective plane curve of degree d and $\tilde{C}$ the normalization. Can restrict the twisting sheaf $\mathcal{O}_{\mathbb{P}^2}(r)$ to C and then lift it to the normalization: $\mathcal{O}_{\tilde{C}}(r)$. Then: $h^0(K_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}}(r-d+3)) = Adj(r)$. Here the r.h.s. is the r'th graded component of the adjoint ideal of $(C, 0)$.

The typical use of this fact is: a given section of some bundle on $\tilde{C}$ is in fact the restriction of a global section on $\mathbb{P}^2$.

Proposition C.2. *Let $\mathcal{M}$ be a maximal determinantal representation of pC and E_C the corresponding (twisted) kernel and $D \subset \tilde{C}$ the adjoint divisor. Then $h^0\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = 0 = h^1\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big)$ and $deg\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = p(g-1)$. Similarly for E^l. In particular, both sheaves are arithmetically Cohen Macaulay.*

Proof. Suppose $h^0\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) \neq 0$, let $s \not\equiv 0$ a corresponding holomorphic section. By construction s is a column whose entries are sections of the restriction $\mathcal{O}(d-2)|_C$.

By completeness (remark C.1) the section s is the vector of pullbacks of some section of $\mathcal{O}_{\mathbb{P}^2}(d-2)$, i.e. a homogeneous polynomial of degree $(d-2)$. Therefore we get a contradiction: $\mathcal{M}s \equiv 0(f)$ on $\mathbb{P}^2$, while $\mathcal{M}$ is non-degenerate and $deg(f) = d$. The same applies to E^l.

The definition of E, E^l provides the natural non-degenerate pairing: $\langle E, E^l \rangle \to \mathcal{O}(d-1)(-D) = Adj(d-1)$. On sections it is given by $\langle e_i, e^l_j \rangle \to {\mathcal{M}^\vee}_{ij}$. Use the completeness of the adjoint linear system and the non-degeneracy of the pairing $\langle \tilde{E}^l_{\tilde{C}}(-1)(-D), \tilde{E}_{\tilde{C}}(-1)(-D) \rangle \to K_{\tilde{C}}$ to get $\tilde{E}_{\tilde{C}}(-1)(-D) \approx (\tilde{E}^l_{\tilde{C}}(-1)(-D))^\vee \otimes K_{\tilde{C}}$. Therefore:

$$h^1\Big(\tilde{E}^l_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = h^0\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = 0, h^1\Big(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = h^0\Big(\tilde{E}^l_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)\Big) = 0, \tag{53}$$

Finally Riemann-Roch for vector bundles on the normalization gives: $0 = h^0 - h^1 = \deg -p(g-1)$. ■

Proposition C.3. *Let E_C be a torsion-free, rank-p sheaf. Assume $h^0(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)) = 0$ and $deg(\tilde{E}_{\tilde{C}}(-1) \otimes \mathcal{O}_{\tilde{C}}(-D)) = p(g_C - 1)$. Then there exists $\mathcal{M} \in Mat(pd \times pd, H^0(\mathcal{O}(1)_{\mathbb{P}^2}))$ such that $\det(\mathcal{M}) = f^p$ and $\tilde{E}_{\tilde{C}}$ is the kernel bundle of $\mathcal{M}$.*

Proof.

Step 1. First, Riemann-Roch gives:

$$h^1\Big(\tilde{E}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)=h^0\Big(\tilde{E}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)-deg\Big(\tilde{E}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)+p(g-1)=0 \tag{54}$$

Define the auxiliary bundle of rank p: $\tilde{G}_{\tilde{C}}$ by $\tilde{G}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D)=\Big(\tilde{E}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)^*\otimes K_{\tilde{C}}(2)$. Then:

$$deg\Big(\tilde{G}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)=p(g-1),\qquad h^0\Big(\tilde{G}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)=0=h^1\Big(\tilde{G}_{\tilde{C}}(-1)\otimes\mathcal{O}_{\tilde{C}}(-D)\Big) \tag{55}$$

Apply Riemann-Roch again to get:

$$h^0\Big(\tilde{G}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)=pd=h^1\Big(\tilde{G}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D)\Big) \tag{56}$$

Let $\{e_1..e_{pd}\}$ and $\{g_1..g_{pd}\}$ be the spanning holomorphic sections.

By the construction of $\tilde{E},\tilde{G}$ one has the non-degenerate pairing: $\Big(\tilde{E}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D),\tilde{G}_{\tilde{C}}\otimes\mathcal{O}_{\tilde{C}}(-D)\Big)\to K_{\tilde{C}}(2)$.

By the completeness of the adjoint series (remark C.1) $K_{\tilde{C}}(2)=\mathcal{O}_{\tilde{C}}(d-1)(-D)$ and one has $h^0(K_{\tilde{C}}(2))=h^0(\mathcal{O}_{\mathbb{P}^2}(d-1)(-D))$. Here the r.h.s. is the $(d-1)$'th graded piece of the adjoint ideal.

Therefore the pairing (e_i,g_j) gives a homogeneous polynomial $V_{ij}=V_{ij}(x,y,z)$ on $\mathbb{P}^2$. This defines the matrix $V\in Mat\Big(pd\times pd,H^0(\mathcal{O}_{\mathbb{P}^2}(d-1)(-D))\Big)$.

Now from V we construct the needed determinantal representation $\mathcal{M}$.

Step 2. Note that V is non-degenerate. To prove this, choose the sections $\{e_i\},\{g_i\}$ in a special way. Consider the set of points $\{pt_1..pt_{pd}\}\subset\tilde{C}\setminus D$. It is possible to chose the sections satisfying the conditions:

$$\tilde{C}\supset div(e_i)\geq\sum_{j\neq i}{}'pt_j+D,\ div(e_i)\not\geq pt_i,\qquad \tilde{C}\supset div(g_i)\geq\sum_{j\neq i}{}'pt_j+D,\ div(g_i)\not\geq pt_i \tag{57}$$

Indeed, suppose $e_1|_{pt}\neq0$, then can assume $e_{i>1}|_{pt_1}=0$ and continue with the remaining points $\{pt_{i>1}\}$ and sections $\{e_{i>1}\}$. At each step there is at least one section which doesn't vanish at the given point (otherwise get a section vanishing at all the points, thus producing a section of $E(-1)(-D)$).

Note that the sections so obtained are linearly independent, e.g. only e_i,g_i don't vanish at pt_i. Assume that the points $pt_1..pt_{pd}$ lie on the intersection of the curve C_d with some (generic, smooth) curve X_p.

By the choice of sections, one has $\forall k,\forall i\neq j$: $V_{ij}|_{pt_k}=0$. So, $deg(V_{ij}|_{X_p})\geq pd$. But $deg(V_{ij}(x,y,z))=d-1$ and X_p is irreducible, hence $V_{ij}|_{X_p}\equiv0$. On the contrary: $V_{ii}|_{pt_i}\neq0$, so $V_{ii}|_{X_p}\not\equiv0$. Therefore, $V|_{X_p}$ is a diagonal matrix, none of whose diagonal entries vanishes identically on the curve X_p. Thus $\det(V)\not\equiv0$.

Step 3. By construction of V, all its $(p+1)\times(p+1)$ minors are degenerate on C, i.e. $\det(V_{(p+1)\times(p+1)})$ is divisible by f. Thus by the property 2.1 all the entries of $V^\vee\in Mat\Big(pd\times pd,H^0(\mathcal{O}(d-1)(pd-1))\Big)$ are divisible by f^{pd-1-p}. Hence the natural candidate is: $\mathcal{M}:=\frac{V^\vee}{f^{pd-1-p}}\in Mat\Big(pd\times pd,H^0(\mathcal{O}(1)\Big)$. From here:

$$\det(\mathcal{M})=\frac{\det(V^\vee)}{f^{pd(pd-1-p)}}=(\frac{V}{f^{pd-p}})^{pd-1}f^p \tag{58}$$

As $det(V)$ is of degree $p(d-1)$ and f is irreducible one has (scaling by a constant): $\det(V)=f^{pd-p}$ and $\det(\mathcal{M})=f^p$.

Finally, by construction $MV=f^p\mathbb{1}$, while $V=\{(e_i,g_j)\}$. Thus $\mathcal{M}(e_i,f_j)\equiv\mathbb{0}(f)$, causing: $\mathcal{M}e_i\equiv0(f)$. Therefore $Span(e_1..e_{pd})\subset Ket(\mathcal{M}|_C)$ and $E=Ker(\mathcal{M}|_C)$ (by the equality of dimensions). ■

C.2. **Deformations of determinantal representations.** In general the deformations of singular curves don't lift to those of the determinantal representations without base change. As an example (for maximal representation), take an ordinary multiple point, e.g. $f=x^d+y^d$, and its simplest diagonal representation: $\mathcal{M}=\begin{pmatrix}l_1&&\\&\cdots&\\&&l_d\end{pmatrix}$. Consider the deformation $f_\epsilon=f+\epsilon$, suppose it lifts to the deformation $\mathcal{M}_\epsilon=M+\epsilon M'$. Then have $det(\mathcal{M}_\epsilon)=\det(\mathcal{M})\det(\mathbb{1}+\epsilon M^{-1}M')=f+\epsilon trace(\mathcal{M}^\vee\mathcal{M}')+..$ causing the contradiction as the second term is of multiplicity (d-1) (or zero).

If $\mathcal{M}$ is the generic representation (i.e. the most non maximal, $corank(\mathcal{M})=1$), then any deformation of the curve lifts to that of $\mathcal{M}$. It is interesting to classify the curve deformations which lift to those of (weakly) maximal representations.

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Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel.
E-mail address: kernerdm@math.bgu.ac.il

Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel.
E-mail address: vinnikov@math.bgu.ac.il