Invariants of Singularities of Pairs

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1. Introduction

Let X be a smooth complex variety and Y be a closed subscheme of X. We are interested in invariants attached to the singularities of the pair (X, Y). We discuss various methods to construct such invariants, coming from the log-resolutions of the pairs, and the geometry of the space of arcs. We present several applications of these invariants to algebra, higher dimensional birational geometry and to singularities. The general setup is to assume only that X is normal and $\mathbb{Q}$-Gorenstein, as in Kollár's paper [32]. However, several of the approaches we will discuss become particularly transparent if we assume, as we do, the smoothness of the ambient variety.

2. Multiplier ideals

Multiplier ideals were first introduced by J. Kohn for solving certain partial differential equations. Siu and Nadel introduced them to complex geometry. We discuss below these ideals in the context of algebraic geometry.

Let X be a smooth complex affine variety and Y be a closed subscheme of X. Suppose that the ideal of Y is generated by $f_1, \ldots, f_m$, and let λ be a positive real number. We define the multiplier ideal of (X, Y) of coefficient λ as follows:

$$\mathcal{J}(X, \lambda{\cdot}Y) = \left\{ g \in \mathcal{O}_X \mid \frac{|g|^2}{(\sum_{i=1}^{m} |f_i|^2)^{\lambda}} \text{ is locally} L^1 \right\}$$

Example 2.1. Let $X = \mathbb{C}^n$ and let Y be the closed subscheme of X defined by $f = x_1^{a_1} \cdots x_n^{a_n}$. Then

$$\mathcal{J}(X, \lambda \cdot Y) = (x_1^{\lfloor \lambda a_1 \rfloor} \ldots x_n^{\lfloor \lambda a_n \rfloor}),$$

where $\lfloor \alpha \rfloor$ denotes the integer part of α.

We can use a log resolution of singularities and the above example to give in general a more geometric description of the multiplier ideals of (X, Y). By Hironaka's Theorem there is a log resolution of singularities of the pair (X, Y), i.e. a proper birational morphism

$$\mu : X' \longrightarrow X$$

with the following properties. The variety X' is smooth, $\mu^{-1}(Y)$ is a divisor, and the union of $\mu^{-1}(Y)$ and the exceptional locus of μ has simple normal crossings. The relative canonical divisor $K_{X'/X}$ is locally defined by the determinant of the Jacobian $J(\mu)$ of μ, We write $\mu^{-1}(Y) = \sum_{i=1}^{N} a_i E_i$ and $K_{X'/X} = \sum_{i=1}^{N} k_i E_i$, where the E_i are distinct smooth irreducible divisors in X' such that $\sum_{i=1}^{N} E_i$ has only simple normal crossing singularities.

The local integrability of a function g on X can be expressed as a local integrability condition on X' via the change of variable formula. This reduces us to a monomial situation, similar to that in Example 2.1. On deduces that $g \in \mathcal{J}(X, \lambda \cdot Y)$ if and only if

$$\mathrm{ord}_{E_i} g \geq \lfloor \lambda a_i \rfloor - k_i$$

for every i. Equivalently, if we put $\lfloor \lambda \mu^{-1}(Y) \rfloor = \sum_i \lfloor \lambda a_i \rfloor E_i$, then
(1)
$\mathcal{J}(X, \lambda \cdot Y) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \lambda \mu^{-1}(Y) \rfloor)$.

Note that because of the original definition, it follows that this expression for $\mathcal{J}(X, \lambda \cdot Y)$ is independent of the choice of a resolution of singularities. We note that if $\lambda_1 \geq \lambda_2$, then

$$\mathcal{J}(X, \lambda_1 \cdot Y) \subseteq \mathcal{J}(X, \lambda_2 \cdot Y).$$

If λ is small enough, then $\lambda a_i < k_i + 1$ for $i = 1, \ldots, N$. This implies that

$$\mathrm{ord}_{E_i} 1 \geq \lfloor \lambda a_i \rfloor - k_{E_i},$$

hence $\mathcal{J}(X, \lambda \cdot Y) = \mathcal{O}_X$. This leads us to the definition of the log canonical threshold of the pair (X, Y): this is the smallest λ such that $\mathcal{J}(X, \lambda \cdot Y) \neq \mathcal{O}_X$, i.e.

$$c = \mathrm{lc}(X, Y) = \min_i \left\{ \frac{k_i + 1}{a_i} \right\}.$$

We may regard $\frac{1}{c}$ as a refined version of multiplicity. In general a singularity with a smaller log canonical threshold tends to be more complex.

The first appearance of the log canonical threshold was in the work of Arnold, Gusein-Zade and Varchenko (see [2] and [48]), in connection with the behavior of certain integrals over vanishing cycles.

In the last decade this invariant has enjoyed renewed interest due to its applications to birational geometry. The following is probably the most interesting open problem about log canonical thresholds.

Conjecture 2.2. (Shokurov's ACC Conjecture) For every n, Consider the set

$$T_n = \{\mathrm{lc}(X, Y) \mid X \dim X = n, Y \subset X\}$$

where X is a log-canonical variety. Then T_n satisfies the Ascending Chain Condition: The set T_n contains no infinite strictly increasing sequences.

This conjecture attracted considerable interest due to its implications to the Termination of Flips Conjecture (see [Bir] for a result in this direction). For this talk, we'll only discuss about

$$T_n^{sm} = \{\mathrm{lc}(X, Y) \mid X \dim X = n, Y \subset X\}$$

where X is smooth variety The first unconditional results on sequences of log canonical thresholds on smooth varieties of arbitrary dimension have been obtained deFernex and Mustaţă in [dFM] using nonstandard methods from model theory, and they were subsequently reproved and strengthened by Kollár in [Kol2] using the recent spectacular results of Birkir, Cascini, Hacon and McKernan [BCHM] on existence of minimal models. In particular, Kollár proves that the set of accumulation points of T_n^{sm} is exactly the set T_{n-1}^{sm}.

Theorem 2.3. (de Fernex, Ein and Mustaţă) *T_n^{sm} satisfies the ascending chain condition.*

The proof is in fact rather elementary and does not need to use the results from [BCHM]. I understand that Mustaţă will discuss the problem in a later talk. I would leave the details to his talk.

We can consider also higher jumping numbers. In general, we say that λ is a jumping number of (X, Y), if

$$\mathcal{J}(X, \lambda \cdot Y) \subsetneq \mathcal{J}(X, (\lambda - \epsilon) \cdot Y)$$

for all $\epsilon > 0$. If λa_i is not an integer, then $\lfloor \lambda a_i \rfloor = \lfloor (\lambda - \epsilon) a_i \rfloor$ for sufficiently small positive ϵ. We see that a necessary condition for λ

to be a jumping number is that λa_i is an integer for some i. In particular, if λ is a jumping number, then it is rational and has a bounded denominator.

The following theorem gives a periodicity property of the jumping numbers.

Theorem 2.4. (i) *If $Y = D$ is a hypersurface in X, then*

$$\mathcal{J}(X, \lambda \cdot D) \cdot \mathcal{O}_X(-D) = \mathcal{J}(X, (\lambda + 1) \cdot D).$$

(ii) (Ein and Lazarsfeld [16]) *For every Y defined by the ideal I_Y, if $\lambda \geq \dim X - 1$, then*

$$\mathcal{J}(X, \lambda \cdot D) \cdot I_Y = \mathcal{J}(X, (\lambda + 1) \cdot Y).$$

Corollary 2.5. *Suppose that $\lambda > \dim X - 1$. Then λ is a jumping number for (X, Y) if and only if so is $(\lambda + 1)$.*

It follows from the sub-additivity theorem of multiplier ideals (Demailly, Ein and Lazarsfeld [13] that the following result holds.

Theorem 2.6. *(*Ein, Lazarsfeld, Smith and Varolin*) Suppose c is the log-canonical threshold of* (X, Y)*. Let* $\lambda \geq 0$ *be a non-negative real number. Then there is a jumping* λ' *in the interval* $(\lambda, \lambda + c]$*.*

We conclude that the set of jumping numbers of the pair (X, Y) is a discrete subset of $\mathbb{Q}$ and it is eventually periodic with period one.

Example 2.7. If Y is a smooth subvariety of X of codimension e, then the set of jumping numbers of the pair (X, Y) is $\{e, e+1, \cdots\}$. In particular $\mathrm{lc}(X, Y) = e$.

Example 2.8. (Howald) Let $X = \mathbb{C}^n$ and let Y be the closed subscheme defined by a monomial ideal $\mathfrak{a}$. If $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, we denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$ by x^a. Consider the Newton polyhedron $P_{\mathfrak{a}}$ associated with $\mathfrak{a}$: this is the convex hull of those

$a \in \mathbb{N}^n$ such that $x^a \in \mathfrak{a}$. Using toric geometry Howald showed in [27] that

$$\mathcal{J}(X, \lambda \cdot Y) = (x^a \mid a + e \in \lambda \cdot \mathrm{Int}(P_{\mathfrak{a}})),$$

where $e = (1, \ldots, 1)$. In particular, the log canonical threshold c of (X, Y) is characterized by the fact $c \cdot e$ lies on the boundary of $P_{\mathfrak{a}}$.

Example 2.9. (Howald) Let $X = \mathbb{C}^n$ and let Y be the closed subscheme defined by the monomial ideal $\mathfrak{a} = (x_1^{a_1}, \ldots, x_n^{a_n})$. In this case, the boundary of the Newton polyhedron $P_{\mathfrak{a}}$ is

$$\{u = (u_1, \ldots, u_n) \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} \frac{u_i}{a_i} = 1\}.$$

Applying Howald's theorem , one sees that $\mathrm{lc}(X, Y) = \sum_i \frac{1}{a_i}$.

One reason that multiplier ideals have been very powerful in studying questions in higher dimensional algebraic geometry is that they appear naturally in a Kodaira type vanishing theorem. The following statement is the algebraic version of a result due to Nadel. In our context, it can be deduced from the Kawamata-Viehweg Vanishing Theorem (see [34]). Let X be a smooth complex variety and Y be a closed subscheme of X. Let $\mu : X' \longrightarrow X$ be a log resolution of (X, Y). Let $E = \mu^{-1}Y$

Theorem 2.10. *(*Nadel's vanishing theorem*)* *Suppose that L is a divisor on X and λ is a positive real number. Assume that $\mu^*L - \lambda \cdot E$ is nef and big, then for every $i > 0$*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(X, \lambda \cdot Y)) = 0.$$

3. Applications of multiplier ideals

One of the most important applications of multiplier ideals is the following theorem of Siu (see [44] and [45]) on the deformation invariance of plurigenera.

Theorem 3.1. *Let $f : X \longrightarrow T$ be a smooth projective morphism of relative dimension n between two smooth irreducible varieties. If we denote by X_t the fiber $f^{-1}(t)$ for each $t \in T$, then for every fixed $m > 0$, the dimension of the cohomology group $H^0(X_t, (\Omega^n_{X_t})^{\otimes m})$ is independent of t.*

The techniques of extending sections of line bundle from a divisor involved in the proof of this theorem have been recently extended by Hacon and McKernan to study existence of flips. Using these results and other important techniques introduced by Shokurov, we have the following theorem.

Theorem 3.2. *(*Birkir, Cascini, Hacon and McKernan*) If X is a smooth complex projective variety. Then the canonical ring of X is finitely generated.*

Remark 3.3. Siu independently using more analytic techniques proved the same result for varieties of general type.

The following extension theorem is essentially due to Hacon-McKernan.

Theorem 3.4. (Hacon-McKernan, Ein-Popa) *Let (X, Δ) be a log-pair, with X a normal projective variety and Δ an effective $\mathbb{Q}$-divisor with $[\Delta] = 0$. Let $S \subset X$ be an irreducible normal effective Cartier divisor such that $S \not\subset \mathrm{Supp}(\Delta)$, and A a big and nef $\mathbb{Q}$-divisor on X such that $S \not\subseteq \mathbf{B}_+(A)$. Let k be a positive integer and M a Cartier divisor such that $M \sim_{\mathbb{Q}} k(K_X+S+A+\Delta)$. Assume the following:*

- $(X, S + \Delta)$ *is a plt pair.*
- M *is pseudo-effective.*

- *the restricted base locus* $\mathbf{B}_-(M)$ *does not contain any irreducible closed subset* $W \subset X$ *with minimal log-discrepancy* $\mathrm{mld}(\mu_W; X, \Delta) < 1$ *which intersects* S*.*
- *the restricted base locus* $\mathbf{B}_-(M_S)$ *does not contain any irreducible closed subset* $W \subset S$ *with minimal log-dicrepancy* $\mathrm{mld}(\mu_W; S, \Delta_S) < 1$.[1]

Then the restriction map

$$H^0(X, \mathcal{O}_X(mM)) \longrightarrow H^0(S, \mathcal{O}_S(mM_S))$$

is surjective for all $m \geq 1$.

In a different direction, there are applications of multiplier ideals to singularities of theta divisors on abelian varieties. Let (X, Θ) be a principally polarized abelian variety, i.e. Θ is an ample divisor on an abelian variety X such that $\dim H^0(X, \mathcal{O}_X(\Theta)) = 1$. The following result is due to Ein and Lazarsfeld [17].

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[1] By inversion of adjunction, these last two conditions can be expressed in a unified was as saying that $\mathbf{B}_-(M)$ does not contain any irreducible closed subset $W \subset X$ with $\mathrm{mld}(\mu_W; X, \Delta + S) < 1$ which intersects S but is different from S itself.

Theorem 3.5. *Let (X, Θ) be a principally polarized abelian variety.*

(i) *(*Kollár*)* (X, Θ) *is log-canonical.*
(ii) *(*Ein and Lazarsfeld*)If* Θ *is irreducible, then* Θ *has at most rational singularities.*

Proof. (i) Observe that (X, Θ) is log-canonical is equivalent to $\mathcal{J}(X, (1-\epsilon)\Theta) = \mathcal{O}_X$ for all $\epsilon > 0$. Suppose for contradiction that $I_Z = \mathcal{J}(X, (1-\epsilon)\Theta)$ is a nontrivial ideal for some $\epsilon > 0$. Observe that Z is closed subscheme contains in the singular locus of Θ. Observe that K_X is trivial. Now Nadel's vanishing says that $H^i(I_Z \otimes \mathcal{O}_X(\Theta + x_0)) = 0$ for $i > 0$ and any $x_0 \in X$. Observe that $\chi(I_Z \otimes \mathcal{O}_X(\Theta)) = h^0(I_Z \mathcal{O}_X(\Theta)) > 0$, since $Z \subset \Theta$. It follows from the vanishing theorem that $\chi(I_Z \otimes \mathcal{O}_X(\Theta + x_0)) = h^0(I_Z \otimes \mathcal{O}_X(\Theta +$

$x_0)) > 0$. We conclude that

$$Z \subset \bigcap_{x_0 \in X} (\Theta + x_0) = \emptyset$$

This is a contradiction. $\square$

Corollary 3.6. *Let (X, Θ) be a principally polarized abelian variety of dimension g, with Θ irreducible. If*

$$\Sigma_k(\Theta) = \{x \in X \mid \mathrm{mult}_x(\Theta) \geq k\},$$

then for every $k \geq 2$ we have $\mathrm{codim}(\Sigma_k(\Theta), X) \geq k+1$. In particular, Θ is a normal variety and $\mathrm{mult}_x(\Theta) \leq g-1$ for every singular point x on Θ.

Remark 3.7. The fact that Θ is normal was first conjectured by Arbarello, De Concini and Beauville. When X is the Jacobian of a curve, the fact that Θ has only rational singularities was proved by Kempf. It was Kollár who first observed in [31] that one can use vanishing theorems to study the singularities of the theta divisor.

Multiplier ideals have been applied in several other directions: to Fujita's problem on adjoint linear systems [3], to Effective Nullstellensatz, to Effective Artin-Rees Theorem [20]. Building on work of Tsuji, recently Hacon and McKernan and independently, Takayama have used multiplier ideals to prove a very interesting result on boundedness of pluricanonical maps for varieties of general type (see [24] and [47]). We end this section with an application to commutative algebra due to Ein, Lazarsfeld and Smith [19].

Let X be a smooth n-dimensional variety and $Y \subseteq X$ defined by the reduced sheaf of ideals $\mathfrak{a}$. The m^{th}th symbolic power of $\mathfrak{a}$ is the sheaf $\mathfrak{a}^{(m)}$ of functions on X that vanish with multiplicity at least m at the generic point of every irreducible component of Y. If Y is smooth, then the symbolic powers of $\mathfrak{a}$ agree with the usual powers, but in general they are very different.

Theorem 3.8. *If X is a smooth n-dimensional variety and if $\mathfrak{a}$ is a reduced sheaf of ideals, then $\mathfrak{a}^{(mn)} \subseteq \mathfrak{a}^m$ for every m.*

Multiplier ideals have been applied in several other directions: to Fujita's problem on adjoint linear systems [3], to Effective Nullstellensatz, to Effective Artin-Rees Theorem [20] and to the symbolic powers of ideals. [19].

4. Bounds on log canonical thresholds and birational rigidity

In this section we compare the log canonical threshold with the classical Samuel multiplicity. We give then an application of the inequality between these two invariants to a classical question on birational rigidity. Let X be a smooth complex variety and $x \in X$ a point. Denote by R the local ring of X at

x, and by $\mathfrak{m}$ its maximal ideal. The following result was proved by de Fernex, Ein and Mustaţă in [10].

Theorem 4.1. *Let $\mathfrak{a}$ be an ideal in R that defines a subscheme Y supported at x. Let c be the log canonical threshold of (X, Y), $l(R/\mathfrak{a})$ be the length of $R/\mathfrak{a}$ and $e(\mathfrak{a})$ be the Samuel multiplicity of R along $\mathfrak{a}$. If $n = \dim R$, then we have the following inequalities.*

(i) $l(R/\mathfrak{a}) \geq \frac{n^n}{n! \cdot c^n}$.
(ii) $e(\mathfrak{a}) \geq \frac{n^n}{c^n}$. *Furthermore, this is an equality if and only if the integral closure of $\mathfrak{a}$ is equal to $\mathfrak{m}^k$ for some k.*

Example 4.2. Suppose that $\mathfrak{a} = (x_1^{a_1}, \ldots, x_n^{a_n})$. In this case $e(\mathfrak{a}) = \prod_{i=1}^n a_i$ and $\mathrm{lc}(\mathfrak{a}) = \sum_{i=1}^n \frac{1}{a_i}$. The inequality in Theorem 4.1(ii) becomes

$$\prod_{i=1}^n a_i \geq \frac{n^n}{\left(\sum_{i=1}^n \frac{1}{a_i}\right)^n}.$$

This is equivalent to

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right)^n \geq \prod_{i=1}^{n}\frac{1}{a_i},$$

which is just the classical inequality between the arithmetic and the geometric mean.

Theorem 4.1 is used in [11] to study the behavior of the log canonical threshold under a generic projection. Using the above theorems and some beautiful geometric ideas of Pukhlikov [42], one gives in [11] a simple uniform proof for the following result.

Theorem 4.3. *If X is a smooth hypersurface of degree N in $\mathbb{CP}^N$, with $4 \leq N \leq 12$, then X is birationally super-rigid. In particular, every birational automorphism of X is bi-regular.*

Remark 4.4. Consider the group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(X))$, the automorphism group of the field of rational functions of X. This is naturally isomorphic to $\mathrm{Bir}_{\mathbb{C}}(X)$, the group of birational automorphisms of X. If X is birationally superrigid, then $\mathrm{Bir}_{\mathbb{C}}(X) \simeq \mathrm{Aut}_{\mathbb{C}}(X)$, the automorphism group of X. When X is a hypersurface of degree N in $\mathbb{P}^N$, X has no nonzero vector fields and therefore $\mathrm{Aut}_{\mathbb{C}}(X)$ is a finite group. If $\mathbb{C}(X)$ is purely transcendental, then $\mathrm{Aut}_{\mathbb{C}}(X)$ will contain a subgroup isomorphic to the general linear group GL_n. In particular this shows that these hypersurfaces are not rational. In a recent preprint, de Fernex has overcame some of the technical difficulties that we encountered in [11]. He is now able to extend the arguments to all $N \geq 4$. When N=4, it is a classical theorem of Iskovskikh and Manin that X is birationally rigid [28]. They used this to show that the function field of a suitable quartic threefold provides a counterexample to the classical Luroth's problem.

5. Spaces of arcs and contact loci

Let X be a smooth n-dimensional complex variety. Given $m \geq 0$, we denote by

$$X_m \;=\; \mathrm{Hom}\big(\mathrm{Spec}\,\mathbb{C}[t]/(t^{m+1})\,,\,X\,\big)$$

the space of m^{th} order jets on X. This carries a natural scheme structure. Similarly we define the space of formal arcs on X as

$$X_\infty = \mathrm{Hom}\big(\mathrm{Spec}\,\mathbb{C}[[t]]\,,X\,\big).$$

These constructions are functorial, hence to every morphism $\mu\colon X' \to X$ we associate corresponding morphisms μ_m and μ_∞. Thanks to the work of Kontsevich, Denef, Loeser and others on motivic integration, in recent years these spaces have been very useful in constructing invariants of singular algebraic varieties. For instance, the following is one of the applications.

Definition 5.1. Suppose that X_1 and X_2 are two smooth projective varieties. We say that X_1 is $K-$equivalent to X_2, if there are proper birational morphisms from a smooth projective variety Y , $\phi_i : Y \longrightarrow X_i$ for $i = 1$ and 2 with property $\phi_1^*(K_{X_1}) \sim \phi_2^*(K_{X_2})$ on Y.

Theorem 5.2. *(*Kontsevich*) Suppose that X_1 is $K-$ equivalent to X_2. Then the Hodge number $h^{p,q}(X_1) = h^{p,q}(X_2)$ for all p and q.*

In what follows we describe some applications of these spaces to the singularities of pairs.

Consider a divisorial valuation of the form val_E with center $c_X(E)$ in X (the center is the image of E in X). The *log discrepancy* of the pair $(X, \lambda \cdot Y)$ along E is

$$a(E, X, \lambda \cdot Y) = k_E + 1 - \lambda \cdot \mathrm{val}_E(I_Y),$$

where I_Y is the ideal of Y in X. The idea is to measure the singularities of the pair $(X, \lambda \cdot$

Y) along B using the log discrepancies along divisors with center contained in B.

Definition 5.3. Let $B \subset X$ be a nonempty closed subset. The minimal log discrepancy of $(X, \lambda \cdot Y)$ over B is defined by
(2)

$$\mathrm{mld}(B; X, \lambda \cdot Y) := \inf_{c_X(E) \subseteq B} \{a(E; X, \lambda \cdot Y)\}.$$

Theorem 5.4. (Ein, Mustaţă, and Yasuda)(Inversion of Adjunction) *Let D be a smooth divisor on the smooth variety X and let B be a nonempty proper closed subset of D. If Y is a closed subscheme of X such that $D \not\subseteq Y$, and if $\lambda \in \mathbb{R}_+$, then*

$$\mathrm{mld}(B; X, D + \lambda \cdot Y) = \mathrm{mld}(B; D, \lambda \cdot Y|_D).$$

Remark 5.5. The notion of minimal log discrepancy plays an important role in the Minimal Model Program. It can be defined under weak assumptions on the singularities of X:

one requires only that X is normal and $\mathbb{Q}$-Gorenstein. Kollár and Shokurov have conjectured the statement of Theorem 5.4 with the assumption that X and D are only normal and $\mathbb{Q}$-Gorenstein. It is easy to see that the inequality "$\leq$" holds in general, and the opposite inequality is known as Inversion of Adjunction. Theorem 5.4 has been generalized in [21] to the case when both X and D are normal locally complete intersections.

We have natural projection maps induced by truncation $X_{m+1} \to X_m$. Since X is smooth, this is locally trivial in the Zariski topology, with fiber $\mathbb{A}^n$. We similarly have projection maps $X_\infty \to X_m$. A subset C of X_∞ is called a *cylinder* if it is the inverse image of a constructible set S in some X_m. If C is a closed cylinder that is the inverse image of a closed subset $S \subset X_\infty$, its codimension in X_∞ is equal to the codimension of S in X_m.

Consider a nonzero ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ defining a subscheme $Y \subset X$. Given a finite jet or an arc γ on X, the order of contact of the corresponding scheme Y — along γ is defined in the natural way. Pulling $\mathfrak{a}$ back via γ yields an ideal (t^e) in $\mathbb{C}[t]/(t^{m+1})$ or $\mathbb{C}[[t]]$, and one sets

$$\mathrm{ord}_\gamma(\mathfrak{a}) \;=\; \mathrm{ord}_\gamma(Y) \;=\; e.$$

For a fixed integer $p \geq 0$, we define the *contact locus*

$$\mathrm{Cont}^p(Y) \;=\; \mathrm{Cont}^p(\mathfrak{a}) \;=_{\text{def}}\; \{\, \gamma \in X_\infty \,|\, \mathrm{ord}_\gamma(\mathfrak{a}) = p \,\}$$

Note that this is a locally closed cylinder. A subset of X_∞ is called an *irreducible closed contact subvariety* if it is the closure of an irreducible component of $\mathrm{Cont}^p(Y)$ for some p and Y.

Suppose now that W is an arbitrary irreducible closed cylinder in X_∞. We can naturally associate a valuation of the function field of X to W as follows. If f is a nonzero rational function of X, we put

$$\mathrm{val}_W(f) = \mathrm{ord}_\gamma(f) \text{ for a general } \gamma \in W.$$

This valuation is not identically zero if and only if W does not dominate X.

If $\mu : X' \longrightarrow X$ is a proper birational morphism, with X' smooth, and if E is an irreducible divisor on X', then we define a valuation by

$$\mathrm{val}_E(f) = \text{ the vanishing order of } f \circ \mu \text{ along } E.$$

A valuation on the function field of X is called a *divisorial valuation* (with center on X) if it is of the form $m \cdot \mathrm{val}_E$ for some positive integer m and some divisor E as above.

A key invariant associated to a divisorial valuation v is its *log discrepancy*. If E is a divisor as above, Suppose that k_E is the coefficient of E in the relative canonical divisor $K_{X'/X}$. Note that k_E depends only on val_E (it does not depend on the model X'). Given an arbitrary divisorial valuation $m \cdot \mathrm{val}_E$, we define its log discrepancy as $m(k_E + 1)$.

Consider a divisor E on X' as above. If $C_m(E)$ is the closure of $\mu_\infty(\mathrm{Cont}^m(E))$, then it is not hard to see that $C_m(E)$ is an irreducible closed contact subvariety of X_∞ such that $\mathrm{val}_{C_m(E)} = m \cdot \mathrm{val}_E$. The following result of Ein, Lazarsfeld and Mustaţă [18] describes in general the connection between cylinders and divisorial valuations.

Theorem 5.6. *Let X be a smooth variety.*

(i) *If W is an irreducible, closed cylinder in X_∞ that does not dominate X, then the valuation val_W is divisorial.*
(ii) *For every divisorial valuation $m \cdot \mathrm{val}_E$, there is a unique maximal irreducible closed cylinder W such that $\mathrm{val}_W = m \cdot \mathrm{val}_E$: this is $W = C_m(E)$.*

(iii) *The map that sends* $m \cdot \mathrm{val}_E$ *to* $C_m(E)$ *gives a bijection between divisorial valuations of* $\mathbb{C}(X)$ *with center on* X *and the set of irreducible closed contact subvarieties of* X_∞.

The applicability of this result to the study of singularities is due the following description of log discrepancy of a divisorial valuation in terms of the codimension of a certain set of arcs.

Theorem 5.7. *Given a divisorial valuation* $v = m \cdot \mathrm{val}_E$ *with center on* X, *if* $C_m(E)$ *is its associated irreducible closed contact subvariety in* X_∞, *then the log discrepancy of* v *is equal to* $\mathrm{codim}(C_m(E), X_\infty)$.

Combining the statements of the above theorems, we deduce a lower bound for the codimension of an arbitrary cylinder in terms of the discrepancy of the corresponding divisor.

Corollary 5.8. *If W is a closed, irreducible cylinder in X_∞ that does not dominate X, then* $\operatorname{codim}(W, X_\infty)$ *is bounded below by the log discrepancy of* val_W.

Remark 5.9. The above two theorems also hold for singular varieties after some minor modifications using Nash's blow-up and Mather's canonical class.

As an application of Theorems 5.6 and 5.7, one gives in [18] a simple proof of the following result of Mustaţă [37] describing the log canonical threshold in terms of the geometry of the space of jets.

Theorem 5.10. *Let X be a smooth complex variety and Y be a proper closed subscheme of X. Let X_m and Y_m be the spaces of m^{th} order jets of X and Y, respectively. If $c = \mathrm{lc}(X, Y)$, then*

(i) *For every m we have* $\operatorname{codim}(Y_m, X_m) \geq c \cdot (m+1)$.

(ii) *If* $m+1$ *is sufficiently divisible, then* $\operatorname{codim}(Y_m, X_m) = c\cdot(m+1)$.

The above results relating divisorial valuations with the space of arcs can be used to study more subtle invariants of singularities of pairs. Let Y be a closed subscheme of the smooth variety X, and let λ be a positive real number. We associate a numerical invariant to the pair $(X, \lambda\cdot Y)$ and to an arbitrary nonempty closed subset $B\subseteq X$, as follows.

We end with a result that translates properties of the minimal log discrepancy over the singular locus of a locally complete intersection variety into geometric properties of its spaces of jets.

Theorem 5.11. *Let* X *be a normal locally complete intersection variety of dimension* n*.*

(i) X_m *is irreducible for every* m *(and in this case it is also reduced) if and only* *if* $\mathrm{mld}(X_{\mathrm{sing}}; X, \emptyset) \geq 1$ (*this says that* X *canonical singularities*).

(ii) X_m *is normal for every* m *if and only if* $\mathrm{mld}(X_{\mathrm{sing}}; X, \emptyset) > 1$ (*this says that* X *has* terminal singularities).

(iii) *In general, we have* $\mathrm{codim}((X_m)_{\mathrm{sing}}, X_m) \geq \mathrm{mld}(X_{\mathrm{sing}}; X, \emptyset)$ *for every* m.

Remark 5.12. The description in (ii) above was first proved in [38]. Note that since X is in particular Gorenstein, it is known that X has canonical singularities if and only if it has rational singularities. All the statements in the above theorem were obtained in [22] and [21] combining the description of minimal log discrepancies in terms of spaces of arcs and Inversion of Adjunction.

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