

Representations of arrangement groups

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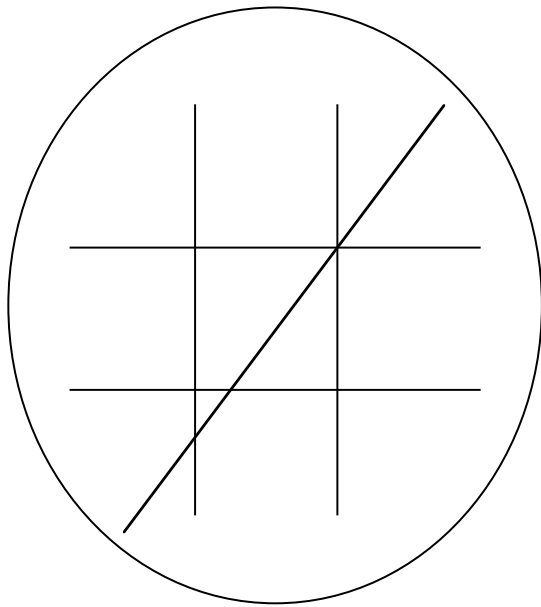
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(joint work w/ Dick Randell and Dan Cohen)

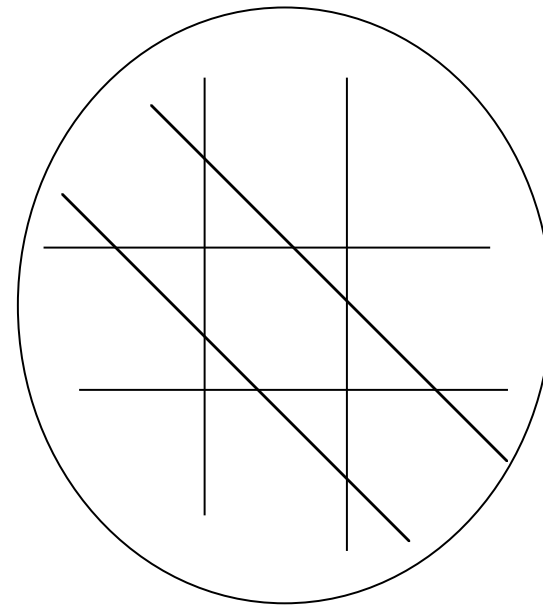
Lib60ber: Topology of algebraic varieties

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The motivating example



A_1



A_2

$\mathbb{P}^2 - \cup \mathcal{A}_1$ is an infinite cyclic cover of $\mathbb{P}^2 - \cup \mathcal{A}_2$.
(Matei-Suciu, 2004, private communication)

Notation

\mathcal{A} : an arrangement of lines in \mathbb{P}^2

$$\overline{M} := \mathbb{P}^2 - \bigcup \mathcal{A}$$

$$\overline{G} := \pi_1(\overline{M})$$

\mathcal{X} : the set of intersection points of multiplicity ≥ 3

For $X \in \mathcal{X}$:

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \in H\} \quad \text{Assume } \bigcup_{X \in \mathcal{X}} \mathcal{A}_X = \mathcal{A}$$

$$\overline{G}_X := \pi_1(\mathbb{P}^2 - \bigcup \mathcal{A}_X) \quad \overline{G}_X \text{ is free of rank } |\mathcal{A}_X| - 1$$

$$G_X := \pi_1(B(X, \epsilon) - \bigcup \mathcal{A}_X) \quad G_X \cong \overline{G}_X \times \mathbb{Z}$$

A split homomorphism

For $X \in \mathcal{X}$, we have an inclusion-induced homomorphism

$$\phi_X : \bar{G} \rightarrow \bar{G}_X,$$

which splits because \bar{G}_X is free.

Consider the product homomorphism

$$\phi : \bar{G} \rightarrow \prod_{X \in \mathcal{X}} \bar{G}_X$$

Theorem *The image of ϕ is normal, and the cokernel is free abelian of rank*

$$\sum_{X \in \mathcal{X}} (|\mathcal{A}_X| - 1) - |\mathcal{A}| + 1.$$

Can ϕ be injective?

Consequences of injectivity

If ϕ is injective:

- \overline{G} is linear, hence residually finite.
- \overline{G} is residually free, hence residually nilpotent.
- \overline{G} has solvable word and conjugacy problems.
- \overline{G} is determined up to isomorphism by the combinatorics (e.g., incidence graph) of \mathcal{A} .
- \overline{G} is of type F_{k-1} but not F_k^1 , hence \overline{M} is not aspherical.

¹Meier, Meinert, Van Wyk, Commentarii Math. Helvetici, 73:2244, 1998.

Decomposable arrangements

Let $\bar{G} = \bar{G}^1 \supseteq \bar{G}^2 \supseteq \dots \bar{G}^n \supseteq \dots$ be the lower central series² and $\bar{G}^\infty = \bigcap_{n \geq 1} \bar{G}^n$ the nilpotent residue of \bar{G} .

Definition \mathcal{A} is *decomposable* if

$$\phi_*: (\bar{G}^n / \bar{G}^{n+1}) \otimes \mathbb{Q} \rightarrow \bigoplus_{X \in \mathcal{X}} [(\bar{G}_X)^n / (\bar{G}_X)^{n+1}] \otimes \mathbb{Q}$$

is an isomorphism for every $n \geq 2$.

² $\bar{G}^{n+1} = [\bar{G}, \bar{G}^n]$ for $n \geq 1$

Since \overline{G} and \overline{G}_X are 1-formal, decomposability of \mathcal{A} is determined by the cohomology ring of \overline{M} . A result of [S. Papadima and A. Suciu](#)³ makes it easy to check for decomposability: it is enough to check the $n = 3$ stage. This amounts to a straightforward matrix calculation. (Note: If \mathcal{A} is decomposable then “all resonance is local,” but not conversely.)

Theorem *If \mathcal{A} is decomposable, then $\ker(\phi) = \overline{G}^\infty$.*

Corollary *If \mathcal{A} is decomposable and ϕ is not injective, then \overline{G} is not residually nilpotent.*

³Commentarii Math. Helvetici, 81(4):859875, 2006

Brunnian elements

\overline{G} is generated by meridians g_H , $H \in \mathcal{A}$.

\overline{G}_X is isomorphic to the quotient

$$\overline{G} / \langle\langle g_H \mid H \notin \mathcal{A}_X \rangle\rangle$$

and ϕ_X is the canonical projection.

Ted Stanford gave a description of Brunnian⁴ braids in terms of the standard generators of the braid group⁵. His argument applies also to our situation.

⁴deleting any strand yields the trivial braid

⁵arxiv.org/math.GT/9907072

Definition An element $g \in \overline{G}$ is a *monic commutator* if $g = g_H$ for some $H \in \mathcal{A}$, or g can be expressed as a commutator with entries from $\{g_H \mid H \in \mathcal{A}\}$. The *support* of a monic commutator g is

$\{H \in \mathcal{A} \mid g \text{ can be expressed as a commutator involving } g_H\}$.

Theorem The kernel of ϕ is generated by monic commutators whose support meets $\mathcal{A} - \mathcal{A}_X$ for every $X \in \mathcal{X}$.

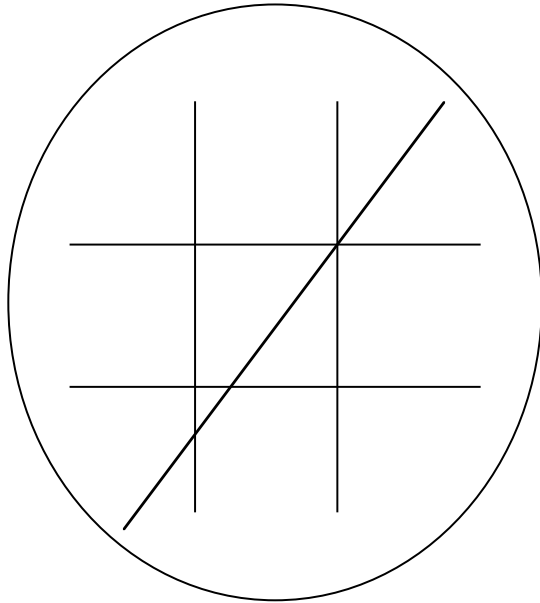
A simple criterion for injectivity

Theorem ϕ is injective if and only if, for every $X \in \mathcal{X}$, for every $H \notin \mathcal{A}_X$, $[g_H, (G_X)^2] = 1$.

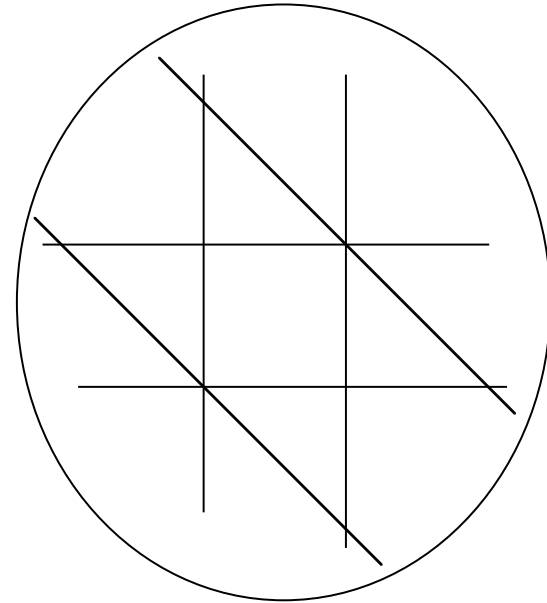
Since $G_X \cong \overline{G}_X \times \mathbb{Z}$, $(G_X)^2$ is generated by commutators of $|\mathcal{A}_X| - 1$ of the standard generators g_H , $H \in \mathcal{A}_X$.

Corollary Suppose, for every $X \in \mathcal{X}$, for every $H \notin \mathcal{A}_X$, $H \cap K$ is a double point for all but one $K \in \mathcal{A}_X$. Then ϕ is injective.

Examples



A_1



A_3

The corollary applies to the family of examples (generalizing A_1) constructed by [E. Artal-Bartolo, J. Cogolludo-Augustin, and D. Matei](#).⁶ A_3 is a “new” example.

⁶in preparation