On Nash problem for surface singularities

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$\mathrm{char}(\mathbb{K}) = 0$

① X alg. var / $\mathbb{K}$. $\pi : \tilde{X} \longrightarrow X$ res of singularities.

E exep divisor $E = \cup E_i$ irred. comp.

Def: E_i essential $\Leftrightarrow$

$$\begin{tikzcd} \tilde{X} \arrow[rr, dashed, "\alpha"] \arrow[dr, "\pi"'] & & \tilde{X}' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

$\overline{\alpha(E_i)}$ irred comp of E', exep. div of π'.

(E_i appears in any resolution)

② Arc space $\mathcal{X}_\infty(X) = \{ \gamma : \mathrm{Spec}\,\mathbb{K}[[t]] \longrightarrow X \}$.

$\mathcal{X}_\infty(X) = \varprojlim \mathcal{X}_n(X) = \{ \gamma : \mathrm{Spec}\,\mathbb{K}[[t]]/(t^{n+1}) \longrightarrow X \}$

infinite dim alg. variety.

$\mathcal{X}_\infty(X)_{\mathrm{sing}} = \{ \gamma : \mathcal{X}_\infty(X) : \gamma(0) \in \mathrm{Sing}\,X \}$

Nash construction

$$\begin{tikzcd} & \tilde{X} \arrow[d, "\pi"] \\ \operatorname{Spec}(\mathbb{K}[[t]]) \arrow[r] \arrow[ur, "\tilde{\gamma}"] & X \end{tikzcd}$$ lifting.

$\gamma \in \mathcal{X}_\infty(X)_{sing} \Leftrightarrow \tilde{\gamma}(0) \in E.$

Thus $\mathcal{X}_\infty(X)_{sing} = \bigcup \overline{\pi_* \mathcal{X}_\infty(\tilde{X})_{E_i}}$ ← irreducible.

$\rightsquigarrow$ $\mathcal{X}(X)_{sing}$ has finitely many irred. components.
$= \bigcup \mathcal{X}_i$

Nash mapping $\{\mathcal{X}_i\} \longrightarrow \{E_i \text{ essential}\}$

$$\mathcal{X}_i \longmapsto E_{\alpha(i)}$$
$$\text{generic} \leftarrow \gamma \in \mathcal{X}_i \qquad \gamma(0) \in E_{\alpha(i)}$$

- $\mathcal{N}$ es injective.

Nash problem: Is $\mathcal{N}$ bijective?

True for many classes of singularities:

- Toric (any dim)
- Quasi-ordinary (any dim)
- Minimal surface sing.
- Sandwiched surface sing.
- Some infinite families of higher dim. non toric. sing.

Results by: M. Lejeune, A. Reguera, S. Ishii, C. Plenat, P. Popescu-Pampu, P. Gonzalez-Perez, M. Morales, J. Kollar.

However:

$\boxed{\text{Counterexample in dim} = 4 \quad (\text{Ishii \& Kollar})}$

→ Surface case of different nature: easier birational geometry

we concentrate in this case.

$(X,0)$ germ of normal surface singularity.

$\pi : \tilde{X} \longrightarrow X$ minimal resolution $\quad E = \pi^{-1}(0) = \bigcup_i E_i$

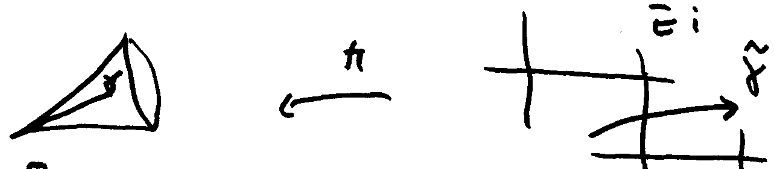
$\{E_i\}$ are the essential components.

$X_\infty(X)_{sing}$: arcs centered at the origin

$\| $

$\bigcup_i \overline{N_{E_i}} \qquad N_{E_i} := \{\text{arcs with lifting centered at } E_i\}$

$N_{E_i} \supset \mathring{N}_{E_i} := \{\text{arcs with lifting transverse to } E_i \text{ at smooth point of } E\}$



- $\mathring{N}_{E_i} \subset N_{E_i}$ is Zariski dense in N_{E_i}.
- $\mathcal{N}$ bijective $\Leftrightarrow$ $N_{E_i} \not\subset \overline{N_{E_j}}$ for $i \neq j$.

<u>Def</u>: There is adjacency from E_j to E_i $\Leftrightarrow$ $N_{E_i} \subset \overline{N_{E_j}}$.

The problem of wedges (M. Lejeune-Jalabert)

Imagine: $N_{E_i} \subset \overline{N_{E_i}}$ & Curve selection lemma true in $\mathcal{X}_\infty(X)$

$\longrightarrow$ $\cancel{\exists}$ $\forall \gamma \in \mathring{N}_{E_i}$ $\exists$ $\alpha : \operatorname{Spec} \mathbb{K}[[s]] \longrightarrow \mathcal{X}_\infty(X)$ such that

(1) $\alpha(0) = \gamma$

(2) The image of the generic point of $\operatorname{Spec} \mathbb{K}[[s]]$ by α is a $\mathbb{K}((s))$-arc through N_{E_i} (K-arc $\leftrightarrow$ K-point of $\mathcal{X}_\infty(X)$ $\leftrightarrow$ power series with coefs in K)

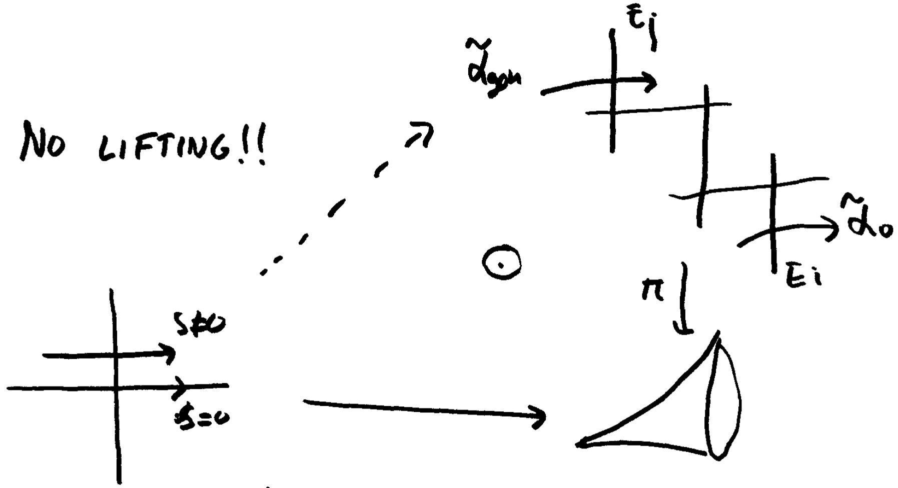
In other words:

$\alpha : \mathbb{K}[[t,s]] \longrightarrow (X,0)$.

(1) $\alpha_0 := \alpha(t,0) = \gamma$

(2) $\alpha_{gen} := \alpha(t,s) \in N_{E_i}$ (viewed as arc in N_{E_i})

In that situation:



① <u>Problem of wedges</u>: Let E_i be essential.

$\alpha : \mathrm{Spec}\, \mathbb{K}[[s,t]] \longrightarrow (X,0)$ such that.

(1) $\alpha_0 \in \mathring{N}_{E_i}$

(2) $\alpha(s,0) = 0$.

| → wedge at E_i

Does α lifts to $\tilde{X}$?

If E_i is not essential the answer is <u>NO</u>.

② <u>Equivalence of Nash and wedge problem</u>:

Does the lifting wedge property charactorise the image of the Nash map?

Affirmative solutions to ① & ② imply $\mathcal{N}$ bijective.

② True + ① False $\Rightarrow$ $\mathcal{N}$ non-bijective.

Def: A wedge α realises an adyacency from E_j to E_i if
$\alpha_0 \in \mathring{N}_{E_i}$ y $\alpha_{gen} \in N_{E_j}$

Remark: It could happen that a wedge does not lift without realising an adyacency.

We propose

(1') Let E_i be essential.
There exist wedges realising adyacencies from E_j to E_i for $j \neq i$?

(2') Does the NOT existence of wedges realising adyacencies characterise the image of the Nash map?

(1') NO $\forall i$ (2') YES $\Rightarrow$ Nash bijective.

(1') YES FOR SOME i + (2') YES $\Rightarrow$ Nash non bijective.

A. Reguera's results:

- Curve selection Lemma false in general in arc spaces.

BUT

Theorem (A. Reguera) Let $X \subset \overline{Y} \subset \mathcal{X}_\infty(X)$, $X, \overline{Y}$ stable (kind of finite codimensionality). Let X be irreducible, $K(X)$ its function field. There is a finite extension $K(X) \subset K$ and a K-~~wedge~~ arc wedge

$$\alpha : \operatorname{Spec} K[[s,t]] \longrightarrow X \quad \text{such that:}$$

1) $\alpha(0,t)$ is $\boxed{\text{THE generic point}}$ of X (in the sense of alg. geom.)

2) $\alpha_{gen} \in \overline{Y} \setminus X$.

Theorem (A. Reguera) N_{E_i} is stable.

Theorem (A. Reguera) Equivalent are:

(1) E_i is not in the image of the Nash map.

(2) There is a K-wedge α with α_0 $\boxed{\text{THE GENERIC point}}$ of N_{E_i} and $\alpha_{gen} \in N_{E_j}$.

- The main difficulty in working with such wedges is that K is huge: infinite trascendence degree, and that is hard to find wedges with α_0 the generic point of N_{E_i}.

__Theorem__ (A. Reguera, M. Lejeune-Jalabert). $\mathbb{K}$ uncountable base field. If there is a __very dense__ subset $A \subset E_i$ such that for any arc γ satisfying $\tilde{\gamma}(0) \in A$, $\tilde{\gamma} \in N_{E_i}$, any $\mathbb{K}$-wedge α having $\alpha_0 = \gamma$ lifts, then E_i __is__ in the image of the Nash map.

→ One has to check the lifting of MANY wedges.

The work I present here is an attempt to improve this situation.

$\mathbb{K}$ alg. closed, non-necesarily uncountable.

<u>Theorem A</u>: Let E_i, E_j different essential components. Equivalent are:

(1) There is an adyacency from E_j to E_i

(2) There is a $\mathbb{K}$-wedge α with $\alpha_0 \in \mathring{N}_{E_i}$ y $\alpha_{gen} \in N_{E_j}$.

If $\mathbb{K} = \mathbb{C}$. the following is also eq.

(3) $\forall\, \gamma \in \mathring{N}_{E_i}$ convergente $\exists$ $\mathbb{C}$-wedge α convergente with $\alpha_0 = \gamma$, $\alpha_{gen} \in N_{E_j}$

<u>Corollary</u>: E_i essential div. Eq. are:

(1) E_i is in the image of the Nash map.

(2) $\nexists$ E_j and a $\mathbb{K}$-wedge α with $\alpha_0 \in \mathring{N}_{E_i}$. $\alpha_{gen} \in N_{E_j}$

If $K = \mathbb{C}$ the following is also eq.

(3) There is $\gamma \in \mathring{N}_{E_i}$ convergent such that: $\nexists$ convergent $\mathbb{C}$-wedge α with $\alpha_0 = \gamma$ and $\alpha_{gen} \in N_{E_j}$ for $j \neq i$.

In terms of lifting wedges:

<u>Theorem B</u> $(\mathbb{K} = \mathbb{C})$ E_i essential component.

If $\exists \gamma \in \mathring{N}_{E_i}$ convergent / any ~~$\mathbb{K}$~~ $\mathbb{C}$-wedge α with $\alpha_0 = \gamma$ lifts to $\tilde{X}$ then E_i is in the image of the Nash map.

The technique used to prove this gives also:

<u>Theorem C</u>: The set of adyacencies between essential components is topological/combinatorial property of the singularity: only depends on the plumbing graph.

<u>Theorem D</u>: If the Nash mapping is bijective for surface sing. having rational homology sphere link, then it is bijective in general.

(Improvement of a Theorem by M. Lejeune-Jalabert and A. Reguera)

Other nice consequences:

<u>Theorem</u>:

(1) If the plumbing graph of $(X,0)$ has an automorphisms interchanging two vertices, then there is no adyacency between the associated divisors.

(2) If the plumbing graph G_1 of $(X_1,0)$ is included in G_2, plumbing graph of $(X_2,0)$, v_1, v_2 vertices of G_1 and E_{v_1}, E_{v_2} are not adyacent in $(X_2,0)$ then E_{v_1}, E_{v_2} are not adyacent in $(X_1,0)$.

(3) $(X_1,0) \rightsquigarrow G_1$ plumbing graph.

$(X_2,0) \rightsquigarrow G_2$ " "

G_1 obtained from G_2 decreasing self intersections.

$v_1, v_2 \in G_1$.

E_{v_1} y E_{v_2} not adyacent en $(X_2,0) \Rightarrow E_{v_1}$ & E_{v_2} not adyacent in $(X_1,0)$.

<u>Cor</u>: The class of rat. hom. spheres can be restricted.

MAIN IDEAS OF PROOFS: (work / $\mathbb{C}$)

Eq. are

(1) Adjacency $E_j \longrightarrow E_i$

(2) $\exists$ $\mathbb{C}$-wedge α / $\alpha_0 \in \mathring{N}_{E_i}$ $\quad \alpha_{gen} \in N_{E_j}$

(3) $\forall\ \gamma \in \mathring{N}_{E_i}$ convergent $\exists$ $\mathbb{C}$-wedge α convergent with $\alpha_0 = \gamma$ $\quad \alpha_{gen} \in N_{E_j}$.

$\boxed{(3) \Rightarrow (2)}$ Obvious

$\boxed{(3) \Rightarrow (1)}$ $\{\gamma \in \mathring{N}_{E_i} : \gamma \text{ convergent}\}$ is dense in N_{E_i} (Artin's approximation theorem).

(3) Means that an $\gamma \in \mathring{N}_{E_i}$ is in the closure of E_j

$\boxed{(1) \Rightarrow (2)}$ $N_{E_i} \subset \overline{N_{E_j}} \xRightarrow{\text{A. Reguera}} \exists\ K \supset k_i$ $(k_i = K(N_{E_i}))$ finite extension and a K-wedge α such that

- α_0 is the generic arc of N_{E_i}
- $\alpha_{gen} \in N_{E_j}$.

- $K_i \subset K$ finite $\Rightarrow \exists$ alg. variety (infinite dimensional) Z and <u>finite</u> morphism $Z \xrightarrow{\rho} N_{E_i}$ such that $K(Z) = K$ and $\rho : K_i \longrightarrow K$ induces the given inclussion of fields.

- Take $\gamma \in N_{E_i}$ and $z \in Z$ / $\rho(z) = \gamma$

If $X \subset \mathbb{C}^n$ write $\alpha(s,t) = \left(\sum_{i,j} a_{ij}^1 s^i t^j, \dots, \sum_{i,j} a_{ij}^N s^i t^j \right)$.

with $a_{ij}^k \in K$ i.e. functions of Z.

Choosing z and γ very general we <u>can evaluate</u> $a_{ij}^k(z)$ $\forall i,j,k$.
and get $\mathbb{C}$-wedge $\alpha(z)$.

- Since α_0 is the generic point of N_{E_i} we have

$\rightarrow \gamma = \left(\sum_j a_{0j}^1(z) t^j, \dots, \sum_j a_{0j}^N(z) t^j \right) = \alpha(z)_0$

$\rightarrow \alpha(z)_{gen} \in N_{E_j}$ ∎

$\boxed{(2) \Rightarrow (3)}$ is the most difficult step.

→ Involves a technique to move wedges which take us outside the algebraic category.

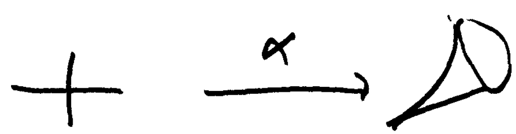
STEP 1 <u>Approximation of Wedges</u>:

Popescu's APPROXIMATION THEOREM $\Rightarrow$ can replace α for a $\mathbb{C}$-wedge with the same properties, but such that the defining series α^i $\left(\alpha = (\alpha^1, \dots, \alpha^N)\right)$ are algebraic functions.

This means: $\exists \; P^i \in \mathbb{C}[s,t,z_i] \;/\; P^i(s,t,\alpha^i(s,t)) \equiv 0$.

In particular α^i is <u>convergent</u>.

We have now $\alpha : (\mathbb{C}^2, 0) \longrightarrow (X, 0)$.



- α non proper.
- Easier to work with proper maps.

(STEP 2) <u>Compactification of α:</u> Want to add something to the source of α in order to make it proper.

→ May assume X algebraic (isolated singularity)

$$Z := \{(s,t,z_1,\dots,z_n) : p^i(s,t,z_i)=0 \ \forall i\} \subset \mathbb{C}^2\times\mathbb{C}^N$$

$$\begin{tikzcd}
 & \mathbb{C}^2\times\mathbb{C}^N \arrow[ddl] \arrow[ddr] & \\
 & Z \arrow[u, phantom, "\cup"] \arrow[dl, shift left, "\text{Projection.}"] \arrow[dr, "\text{Projection}"] & \\
\mathbb{C}^2 \arrow[ur, shift left, "\Gamma_\alpha = (s,t,\alpha(s,t))"] \arrow[rr, "\alpha"] & & X\subset\mathbb{C}^N
\end{tikzcd}$$

→ Taking the irred. component Z' of Z containing $\Gamma_\alpha(\mathbb{C}^2,0)$, suitable projective compactifications and a desingularisation $W \longrightarrow Z'$ we arrive to:

$$\begin{tikzcd}
 & W \arrow[d, "\text{proper.}"] \\
(\mathbb{C}^2,0) \arrow[ur] \arrow[r, "\alpha"] \arrow[ur, phantom, shift right=3, "\circ"] & (X,0)
\end{tikzcd}$$

$\boxed{\text{STEP 3}}$ Reduction to branched covers

$$\begin{tikzcd}
 & W \arrow[d, "\text{proper}"] \arrow[dr, "\rho"] & & \\
(\mathbb{C}^2,0) \arrow[ur, hook, "\beta"] \arrow[r, "\alpha"] \arrow[ur, phantom, "\odot"] & (X,0) \arrow[r, phantom, "\odot"] & (Y,0) \arrow[l, "\sigma"] & (\mathbb{C},0) \arrow[l, "\varphi"]
\end{tikzcd}$$

→ smooth.

birational morphism.

finite. (branched cover)

↓

STEIN FACTORISATION

Define: $\varphi(t) = \rho\beta(0,t)$.

Obs $\boxed{\cancel{\sigma\circ}\gamma = \sigma\circ\varphi}$

→ From $\alpha:(\mathbb{C}^2,0)\longrightarrow(X,0)$ we get branched cover:

$$\begin{tikzcd}
(Y,0) \arrow[rr, "\sigma"] & & (X,0) \\
 & (\mathbb{C},0) \arrow[ul, "\varphi"] \arrow[ur, "\gamma"'] \arrow[u, phantom, "\odot"] &
\end{tikzcd}$$

What do they have in common?

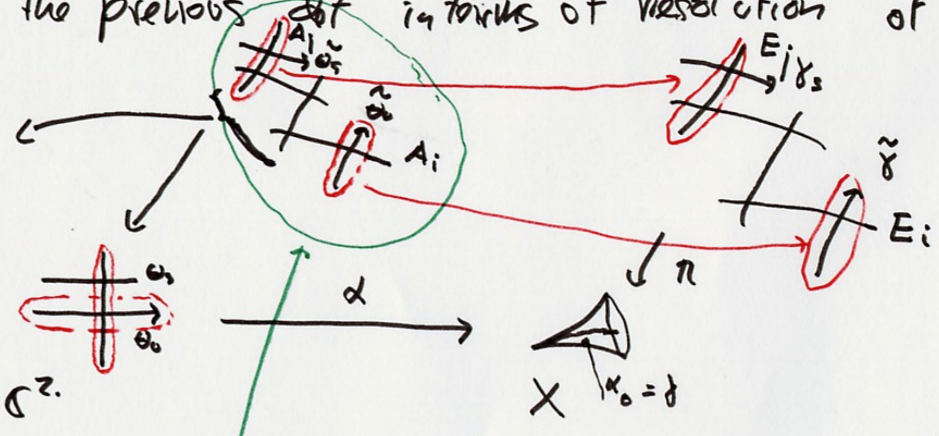
Definition: $\alpha:(\mathbb{C}^2,0)\longrightarrow(X,0)$ is a wedge realising an adjacency from E_j to $\gamma\in \mathring{N}_{E_i}$ $\Leftrightarrow$

(1) $\alpha_0=\gamma$

(2) $\gamma_s\in N_{E_j}$ $\forall s\neq 0$.

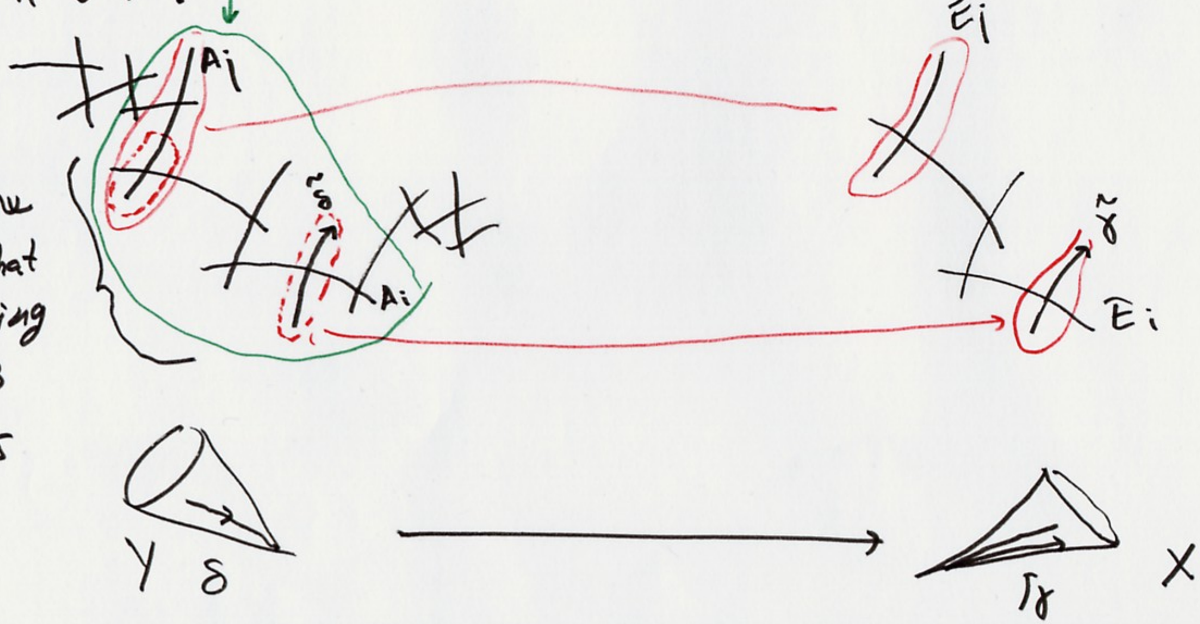
Express the previous of in terms of resolution of indeterminacy of α:

Chain of blow ups such that Θ_s y Θ_0 are normal crossings



By construction this picture <u>embedds</u> in the resolution of indeterminacy of $\pi^{-1} \circ \sigma$:

Chain of blow ups such that after collapsing A_j becomes n. crossings with $\tilde{\delta}$.



Modelling on this combinatorial picture it is possible to define

Def
$$\begin{tikzcd} (Y,0) \arrow[rr, "\sigma"] \arrow[dr, phantom, "\odot"] & & (X,0) \\ & (\mathbb{C},0) \arrow[ul, "\varphi"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$
branched cover. realising adjacency from E_j to γ

and prove:

<u>Theorem</u>: There exists a wedge realising an adjacency from E_j to γ

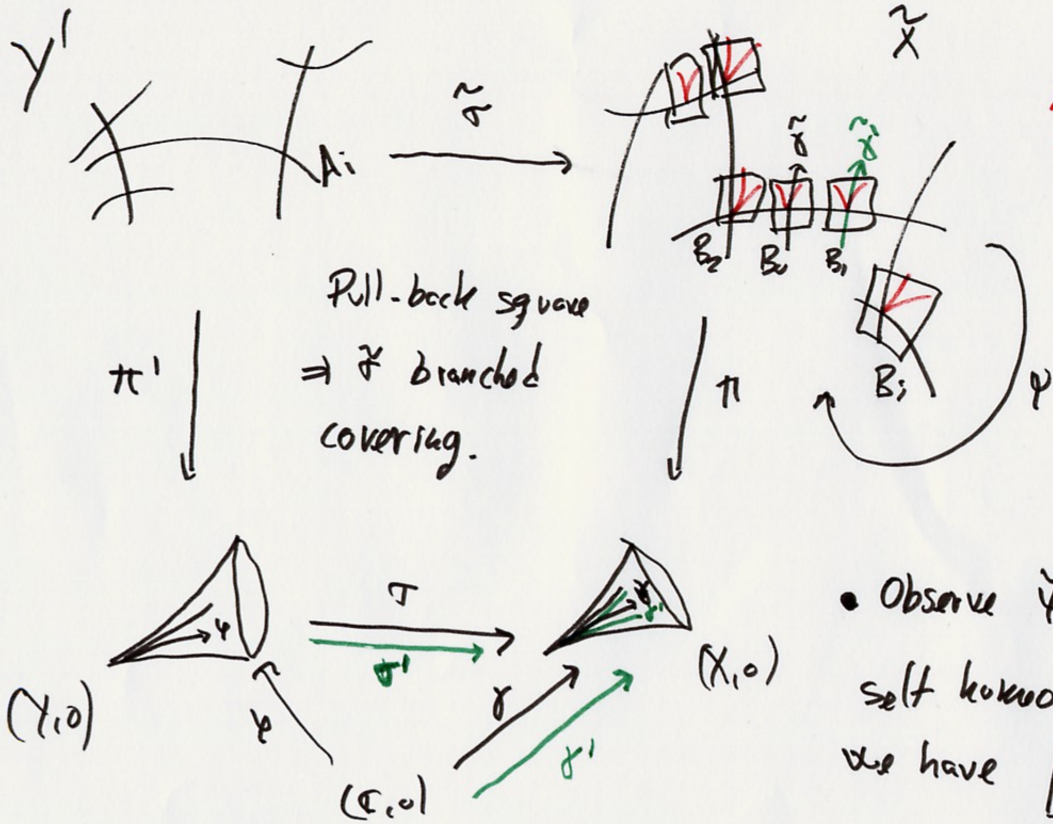
$\Leftrightarrow$ " " " branched cover " " " " " " " .

From now on we study the adjacency among divisors via branched covers instead of wedges.

Step 4 Moving ~~wedges~~/branched covers.

Since $\{\gamma \in \mathring{N}_{\bar{E}_i} : \gamma \text{ convergent}\} \subset N_{\bar{E}_i}$ is dense it only remains to prove:

<u>Proposition</u>: Let $\gamma, \gamma' \in \mathring{N}_{\bar{E}_i}$, convergent. If there exists a branched covering realising an adjacency from $\bar{e}_j$ to γ then there exists a branched covering realising an adjacency from $\bar{E}_j$ to γ'.



Pull-back square $\Rightarrow \tilde{\sigma}$ branched covering.

Δ part of discriminant of $\tilde{\sigma}$ not contained in E

$\tilde{\psi}$ = self diffeo $\tilde{X} \circlearrowleft$,

- holomorphic in boxes
- $\tilde{\psi}|_{B_1} : B_1 \longrightarrow B_2$ such that $\boxed{\psi \circ \tilde{\gamma} = \tilde{\gamma}'}$

- Observe $\tilde{\psi}$ descends to $\psi \circlearrowright (X,0)$, self homeo such that if $\sigma' = \psi \circ \sigma$ we have $\boxed{\sigma' \circ \varphi = \gamma'}$

<u>BUT</u>: σ' is not holomorphic!!

<u>Solution</u>: Since $\tilde{\psi}(\Delta)$ is alg. curve Grauert and Remmert Thm $\Rightarrow$
$\exists$ unique complex structure in Y' making $\tilde{\psi} \circ \tilde{\sigma}$ holomorphic.
It is possible to push the complex structure under π' $\leadsto$ with the new complex structure σ' is holomorphic, and a wedge realising adjacency from E_i to γ', since the combinatoric's did not change ∎

Happy birthday, Anatoly