Nash modification and polar curves on normal surfaces

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Nash modification

Consider a germ of complex analytic space $(X, 0) \subset (\mathbb{C}^N, 0)$ of pure dimension d.
Consider the map:

$$\begin{array}{rcl} \lambda : X \setminus Sing(X) & \to & \mathbf{G}(N, d) \\ x & \mapsto & T_x X \end{array}$$

The closure of the graph of λ in $\mathbb{C}^N \times \mathbf{G}(N, d)$ with the restriction of the first projection:
$\nu : \tilde{X} \to X$ is the Nash modification of X.

$\tilde{X}$ is analytic.

ν is proper and is an isomorphism outside the singular locus of X.

If x is singular, its fiber is :

$$\nu^{-1}(x) = \{(x, \lim_{x_n \to x} T_{x_n}X)\ , (x_n) \subset X \setminus Sing(X)\}$$

The Nash modification has a desingularisation vertue:

Theorem (M. Spivakovsky)

A finite sequence of normalised Nash modifications resolves the singularities of a complex surface.

To be compared with:

Theorem (O. Zariski)

A normal surface singularity is resolved by a finite sequence of normalised blow-ups of points.

Let $(S, 0)$ be a germ of normal surface singularity.

Theorem [S]

The normalised Nash modification of S factors through the blow-up of the origin if and only if the tangent cone of S at the origin does not have any plane as irreducible component.

The normalised blow-up of the origin of S factors through the normalised Nash modification if and only if they coincide, and in this case they both resolve the singularity of S.

Corollary Let $(S, 0)$ be a germ of normal surface singularity, for which the minimal resolution factors through the blow-up of the origin.
To any 2-dimensional plane in the tangent cone corresponds at least a singular point in the normalised Nash modification.

Consider a linear projection $\pi_L : \mathbb{C}^N \to \mathbb{C}^2$ whose kernel is an $(N-2)$-plane L.
For a general axis L, the restriction of π_L to S is a finite map.
The closure of the critical locus of the restriction of π_L to the non-singular part of S, is the polar curve P_L defined by L on S.

When S is normal, and whenever $L \cap S = \{0\}$, the polar curve P_L is always a non empty curve.

When $(S, 0)$ is an isolated hypersurface singularity, defined by $f = 0$, the family of polar curves on S corresponds to the linear system:

$$\{\alpha \frac{\partial f}{\partial X} + \beta \frac{\partial f}{\partial Y} + \gamma \frac{\partial f}{\partial Z} = 0\},\ (\alpha : \beta : \gamma) \in \mathbb{P}^2.$$

In the case of isolated singularities of hypersurfaces in $\mathbb{C}^3$ we can obtain:

A lower bound for the number of branches of the general polar curve in terms of the degree of the tangent cone, multiplicity and Milnor number of its singular points and the number of exceptional tangents.