Monodromy and spectrum of quasi-ordinary surface singularities

Mirel Caibar, Gary Kennedy, Lee McEwan

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- Thus we can find local coordinates so that the surface is the zero locus of

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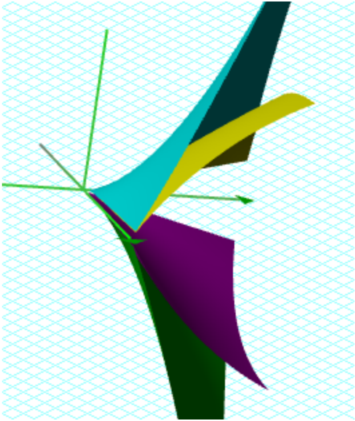
- $z = x^{1/2}y + xy^{3/2}$



- We want to calculate certain invariants of a quasi-ordinary surface, and to understand the relationships among them. The invariants include:
  1. The monodromy of the Milnor fibration, as recorded in the graded characteristic function

$$\frac{\det(tI - m_0)\det(tI - m_2)}{\det(tI - m_1)}$$

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- Consider a transverse slice of the surface by $x = C$, where C is sufficiently small. This is a plane curve germ, with its own Milnor fibration, which we call the horizontal fibration.
- Continuing our list of invariants:
  - (3) The monodromy of the horizontal fibration:

$$\mathbf{H}(t) = \frac{\det(tI - h_0)}{\det(tI - h_1)}.$$

  - (4) The spectrum of the same fibration.
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- For the horizontal fibration we fix the value of x while varying the parameter in $f(x, y, z) = \epsilon$. Alternatively we can fix ϵ and let x move around a small circle, obtaining the vertical fibration.
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Calculations

- Without changing the local topology, we may find local coordinates so that each branch is parametrized by

$$\zeta = \sum_{i=1}^{e} x^{\lambda_i} y^{\mu_i}$$

with $\lambda_i \geq \lambda_{i-1}$, $\mu_i \geq \mu_{i-1}$, and (λ_i, μ_i) not contained in the group generated by the previous pairs. (Abhyankar-Lipman) Each (λ_i, μ_i) is called a characteristic pair.

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- An example: a recursion for vertical monodromy. Given

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- …its truncation is

$$\zeta_1 = x^{\lambda_1} y^{\mu_1} = x^{\frac{a}{mb}} y^{\frac{n}{m}},$$

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- Let r and s be smallest nonnegative integers so that

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$$\mathbf{V}(t) = \frac{(\mathbf{V}_1(t))^{d'}\mathbf{V}'(t^b)}{(t^b - 1)^{d'}}.$$

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Relationships among invariants

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- McEwan's pairing (writing eigenvalues as elements of $\mathbf{Q}/\mathbf{Z}$):

$$\frac{i}{b} + \frac{j}{n} \leftrightarrow -\frac{ai}{b} \qquad \text{(for } 1 \le i \le b-1 \text{ and } 1 \le j \le n-1\text{).}$$

- The horizontal spectral numbers are $h_{ij} = \frac{i}{b} + \frac{j}{n} - 1$.
  Let v_{ij} be the fractional part of $-\frac{ai}{b}$ (but use 1 if it's an integer).
- By Steenbrink and Saito,

$$\mathrm{Sp}(f+L^k) - \mathrm{Sp}(f) = \frac{1-t}{1-t^{1/k}} \cdot \left[\sum t^{h_{ij}+v_{ij}/k} + \text{same for other slice} \right].$$

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- The Steenbrink-Saito formula applies to all quasi-ordinary surfaces. Can we find compatible recursions for computing all the ingredients?

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