LIB60BER
Gevrey solutions of hypergeometric systems

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Example:

$X = \mathbb{C}$, $Y = \{0\}$:

$f = \sum_{m\geq 0} m! x^m$

$\rho_s(f) = \sum_{m\geq} \frac{1}{m!^{s-2}} x^m$ is convergent iff $s \geq 2$.

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Irregularity of $\mathcal{D}$-modules:

Let X be a complex manifold and $Y \subseteq X$ a smooth subvariety.

Denote $\mathcal{Q}_Y(s) := \mathcal{O}_{X|Y}(s)/\mathcal{O}_{X|Y}$, $s \geq 1$.

Definition (Mebkhout)

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module and Y a smooth subvariety of X. The irregularity complex of order s of $\mathcal{M}$, $1 \leq s \leq \infty$, with respect to Y is:

$$\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{Q}_Y(s))$$

If $s = +\infty$, it is called the irregularity complex of $\mathcal{M}$ with respect to Y, denoted by $\mathrm{Irr}_Y(\mathcal{M})$.

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$\mathcal{M}$ holonomic $\mathcal{D}$-module, Y smooth hypersurface

Theorem (Mebkhout)

$\mathrm{Irr}_Y^{(s)}(\mathcal{M})$ is a *perverse sheaf on* Y, for all $1 \leq s \leq +\infty$. Moreover, $\{\mathrm{Irr}_Y^{(s)}(\mathcal{M})\}_{s\geq 1}$ is an increasing filtration of $\mathrm{Irr}_Y(\mathcal{M})$.

Definition (Mebkhout)

$s > 1$ is an analytic slope of $\mathcal{M}$ along Y if s is a gap in the Gevrey filtration $\{\mathrm{Irr}_Y^{(s)}(\mathcal{M})\}_{s\geq 1}$.

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Y. Laurent defined algebraic slopes of a coherent $\mathcal{D}$-module along smooth subvarieties of X of any codimension.

Comparison Theorem of the slopes (Laurent-Mebkhout)

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Definition

- Gel'fand, Graev, Kapranov, Zelevinsky (1987-1994).

  Associated with a pair (A, β), where

- $A = (a_1 \cdots a_n)$, $a_i \in \mathbb{Z}^d$, $\text{rank}(A) = d \leq n$, $\beta \in \mathbb{C}^d$.

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Toric ideal: $I_A = \langle \partial^{u_+} - \partial^{u_-} : \; u \in \ker A \cap \mathbb{Z}^n \rangle \subseteq \mathbb{C}[\partial]$.
Euler operators: $E_i = \sum_{j=1}^n a_{ij} x_j \partial_j$, $i = 1, \ldots, d$.

Hypergeometric $\mathcal{D}$-module:

$$\mathcal{M}_A(\beta) := \frac{\mathcal{D}}{\mathcal{D} I_A + \mathcal{D}\langle E_1 - \beta_1, \ldots, E_d - \beta_d \rangle}$$

Example:
$A = (1, 2)$, $I_A = \langle \partial_1^2 - \partial_2 \rangle$, $E - \beta = x_1\partial_1 + 2x_2\partial_2 - \beta$
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Γ-series (Gel'fand-Zelevinsky-Kapranov; Saito-Sturmfels-Takayama)

- $v = (v_1, \dots, v_n) \in \mathbb{C}^n$:
- $[v]_u := \prod_{i=1}^{n} \prod_{j=1}^{u_i} (v_i - j + 1)$.

$$\phi_v := \sum_{u \in N_v} \frac{[v]_{u_-}}{[v+u]_{u_+}} x^{v+u} \in x^v \mathbb{C}[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$$

where $N_v \subseteq \ker A \cap \mathbb{Z}^n$.

Problem: to find v that guaranties that ϕ_v is a Gevrey solution of $\mathcal{M}_A(\beta)$ along a coordinate subspace Y.

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Hipergeometric $\mathcal{D}$-modules and some well-known results

Theorem (Gel'fand-Zelevinsky-Kapranov, Adolphson)

$\Delta_A = \text{convex-hull}(0, a_1, \ldots, a_n) \subseteq \mathbb{R}^d$
If β is very generic, the holonomic rank of $\mathcal{M}_A(\beta)$ is $\text{vol}_{\mathbb{Z}A}(\Delta_A) := \frac{d!\,\text{vol}(\Delta_A)}{[\mathbb{Z}^d : \mathbb{Z}A]}$.

Theorem (Schulze-Walther)

Combinatorial description of the (algebraic) slopes of $\mathcal{M}_A(\beta)$ along coordinate subspaces Y at $0 \in Y$ under the assumptions:

1. $\mathbb{Z}A = \mathbb{Z}^d$.
2. A is *pointed*, i.e. $a_1, \ldots, a_n$ lie in a single open half-space (equivalently, $\mathcal{V}(I_A)$ passes through the origin).

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Results

Let Y be a coordinate hyperplane. Reordering the variables: $Y = \{x_n = 0\}$. $A' = (a_1 \cdots a_{n-1})$.

For η a facet of $\Delta_{A'}$, denote H_η the unique hyperplane with $a_i \in H_\eta, ; \forall i \in \eta$.

Theorem 1

The slopes of $\mathcal{M}_A(\beta)$ along Y at any point $p \in Y$ are:

$$\{s > 1 : \ a_n/s \in H_\eta, \ \text{ for certain facet } \eta \text{ of } \Delta_{A'}\}$$

Moreover, all the slopes occur as the Gevrey index of certain formal solutions along Y that we will explicitly construct.

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$\tau \subseteq \{1, \ldots, n\}$, $Y_\tau := \{x_i = 0 : \ i \notin \tau\}$.
$\Delta_\tau :=$ convex-hull$(0, a_i : \ i \in \tau)$.

Theorem 2

For all $\beta \in \mathbb{C}^d$ and for generic points $p \in Y_\tau$:

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$$

More precisely, for all $s \in \mathbb{R}$,

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{X|Y_\tau}(s))_p \geq \sum_{\eta \subseteq \tau} \mathrm{vol}_{\mathbb{Z}A}(\Delta_\eta)$$

where η varies between the facets of Δ_τ such that $a_i/s \in H_\eta \cup H_\eta^-, \forall i \notin \tau$.
Moreover, equality holds for very generic parameters $\beta \in \mathbb{C}^d$.

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$\Delta_\tau :=$ convex-hull$(0, a_i : \; i \in \tau)$.

Theorem 2

For all $\beta \in \mathbb{C}^d$ and for generic points $p \in Y_\tau$:

$$\dim_{\mathbb{C}} \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y_\tau}})_p \geq \mathrm{vol}_{\mathbb{Z}A}(\Delta_\tau)$$

More precisely, for all $s \in \mathbb{R}$,

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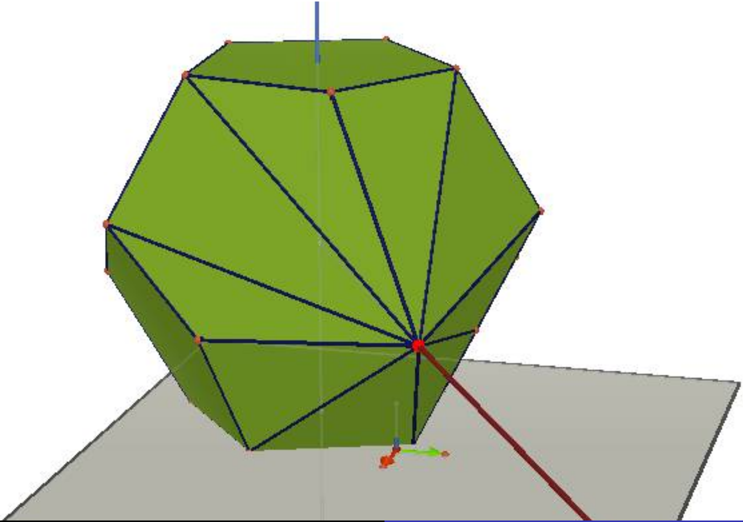
The moving point is a_n/s where s varies in $[1,+\infty)$



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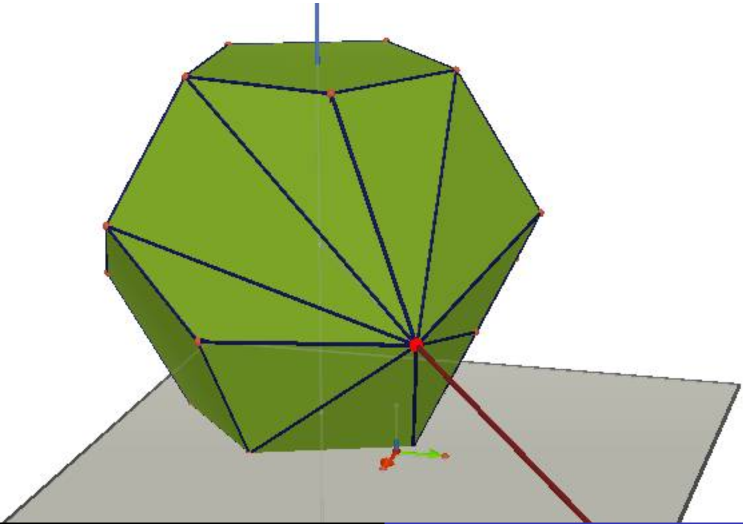

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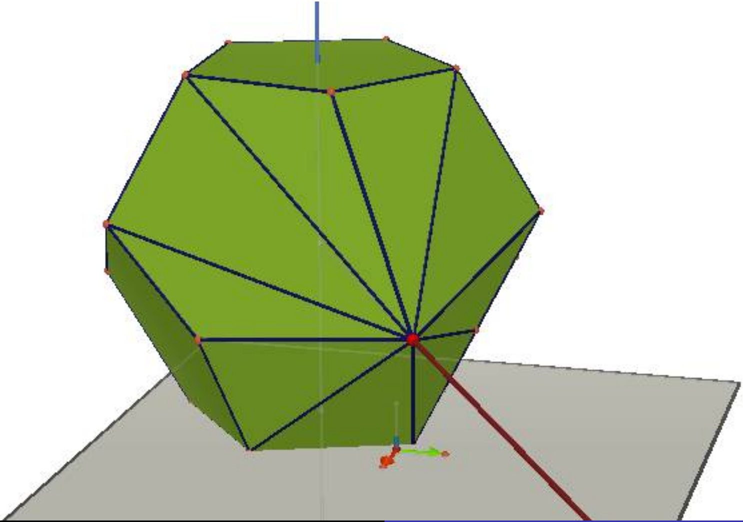

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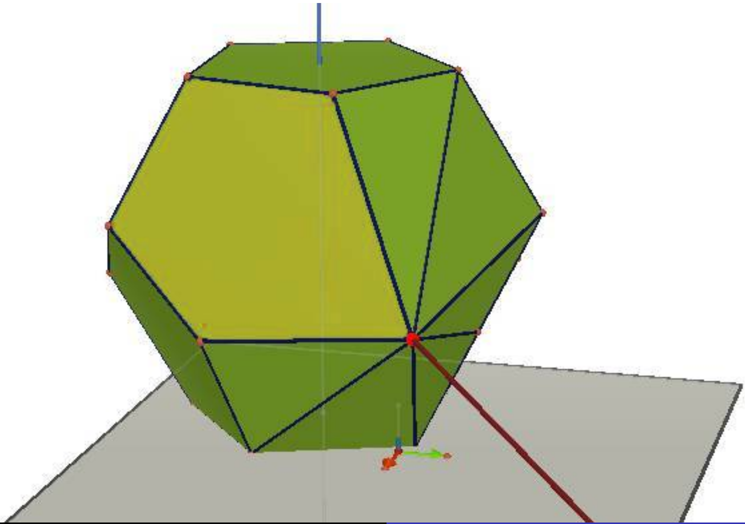

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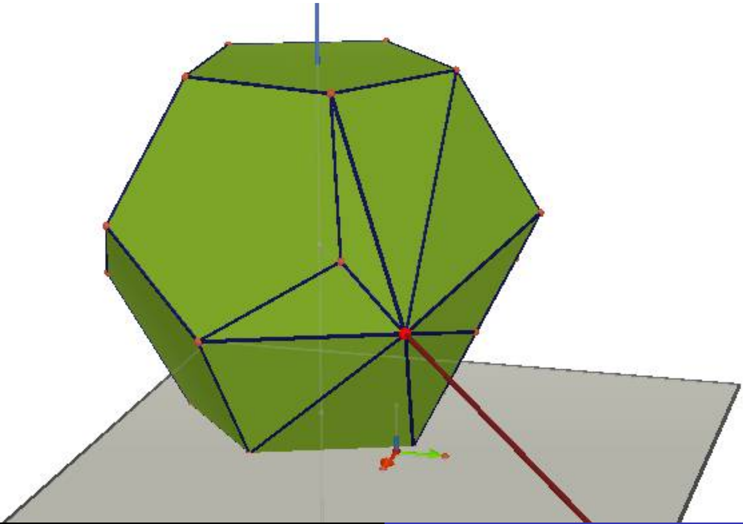

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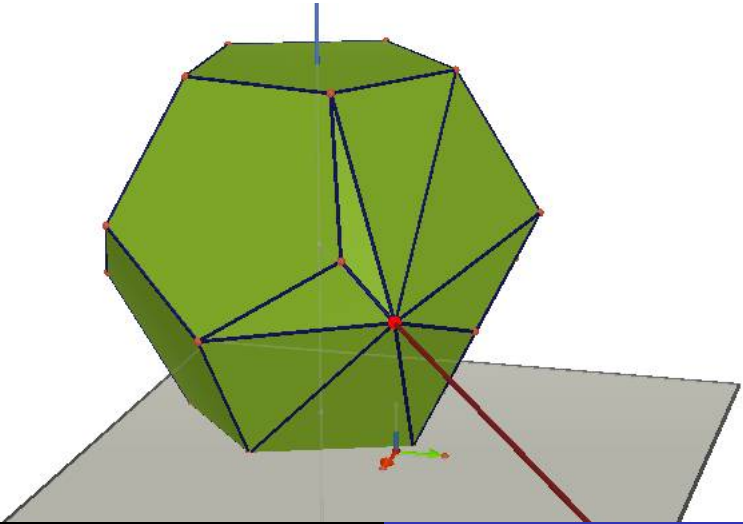

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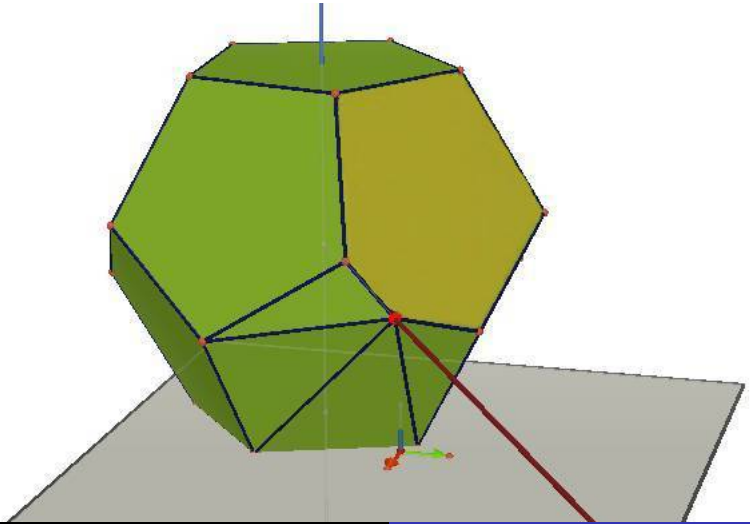

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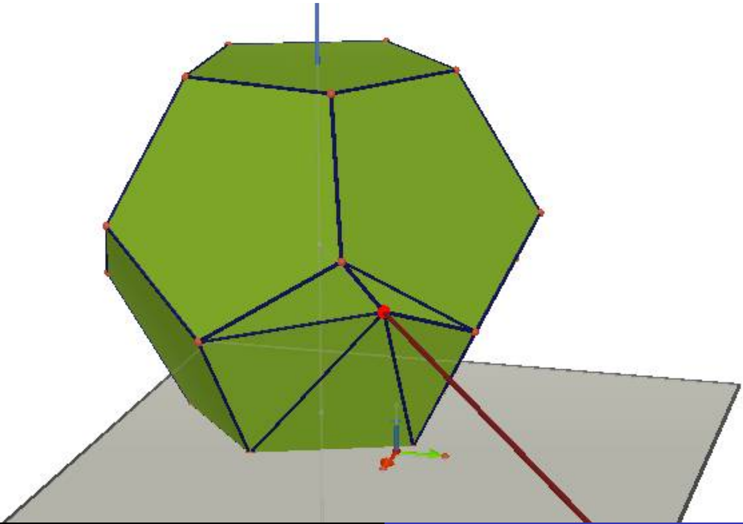

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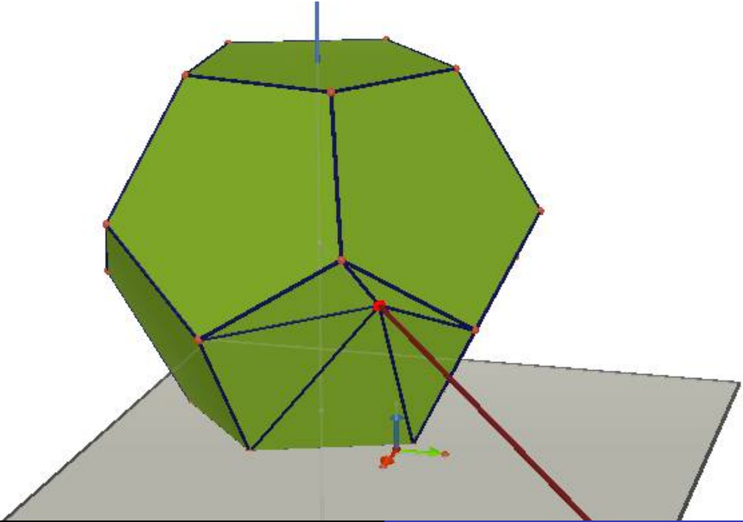

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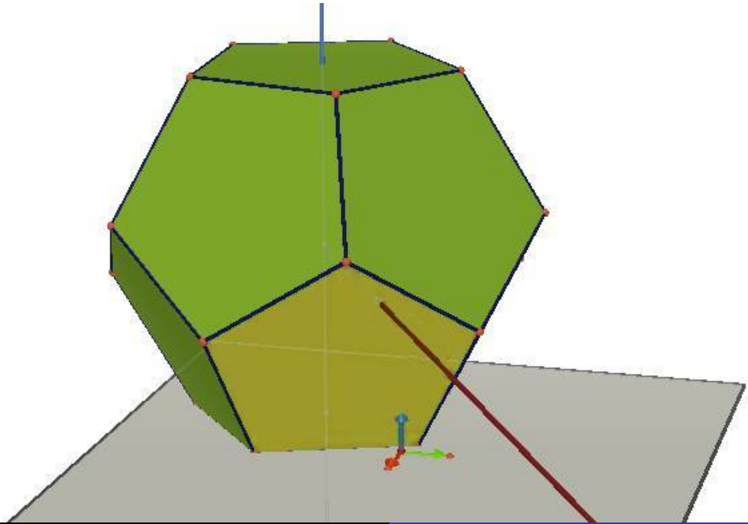

Gevrey solutions of $\mathcal{M}_A(\beta)$ associated with a simplex

A simplex of A is: $\sigma \subseteq \{1, \ldots, n\}$, wit cardinal d and $\det(A_\sigma) \neq 0$.

Reordering the variables, $\sigma = \{1, \ldots, d\}$.

Consider $v^k = (A_\sigma^{-1}(\beta - \sum_{i \notin \sigma} k_i a_i), k)$ with $k = (k_i)_{i \notin \sigma} \in \mathbb{Z}^{n-d}$.
It is clear that $Av^k = \beta$.

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The series $\varphi_{v^{\mathbf{k}}}$:

$$\sum_{\mathbf{k}+\mathbf{m}\in\Lambda_{\mathbf{k}}} \frac{[A_\sigma^{-1}(\beta - \sum_{i\notin\sigma} k_i a_i)]_{(A_\sigma^{-1}(\sum_{i\notin\sigma} m_i a_i))_+} x_\sigma^{A_\sigma^{-1}(\beta - \sum_{i\notin\sigma}(k_i+m_i)a_i)} x_{\overline{\sigma}}^{\mathbf{k}+\mathbf{m}}}{[A_\sigma^{-1}(\beta - \sum_{i\notin\sigma}(k_i+m_i)a_i)]_{(A_\sigma^{-1}(\sum_{i\notin\sigma} m_i a_i))_-} (\mathbf{k}+\mathbf{m})!}$$

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Example

In general, if Y is a coordinate subspace with codim $Y > 1$, not every slope appear as the index of a Gevrey solution:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$I_A = \langle \partial_1^3 - \partial_2 \partial_3 \rangle$.
$Y = \{x_2 = x_3 = 0\}$.
$s = 3/2$ is an algebraic slope of $\mathcal{M}_A(\beta)$ along Y (see Schulze-Walther). But s is the index of a Gevrey solution of $\mathcal{M}_A(\beta)$ (modulo convergent series) along Y if and only if $\beta_2 \in \mathbb{Z}$.

More details and results in the preprint:

Fernández-Fernández, M.C. *Irregular hypergeometric $\mathcal{D}$-modules*. arXiv:0906.3478v1 [math.AG]

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