Hirzebruch Invariants of Symmetric Products

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1. Symmetric products

2. Generating series
   - History and Results
   - Extensions to the singular setting

Symmetric products

Definition

The *n-th symmetric product* of a space X is defined by

$$X^{(n)} := \overbrace{X \times \cdots \times X}^{n \ times} / \Sigma_n$$

the quotient of the product of n copies of X by the natural action of the symmetric group on n elements, Σ_n.

What are symmetric products good for?

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- If X is a smooth complex algebraic surface, $X^{(n)}$ is used to understand the topology of the n-th Hilbert scheme $X^{[n]}$ (Cheah, Göttsche-Soergel). Higher-dimensional generalizations: Gusein-Zade, Luengo, Melle-Hernández.

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  - Then $\mathcal{I}(X^{(n)})$ is equal to the coefficient of t^n in the resulting expression in invariants of X.

Euler-Poincaré characteristic and Chern classes

- Macdonald ('62): X- compact triangulated space

$$\sum_{n\geq 0} \chi(X^{(n)}) \cdot t^n = (1-t)^{-\chi(X)} = \exp\left(\sum_{r\geq 1} \chi(X) \cdot \frac{t^r}{r}\right)$$

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- Ohmoto ('08): Chern class version of Macdonald's result for the Chern-MacPherson classes of complex quasi-projective varieties.

Signature and L-classes

- Hirzebruch-Zagier ('70): if X is a closed oriented manifold,

$$\sum_{n\geq 0} \sigma(X^{(n)}) \cdot t^n = \frac{(1+t)^{\frac{\sigma(X)-\chi(X)}{2}}}{(1-t)^{\frac{\sigma(X)+\chi(X)}{2}}},$$

for $\mathcal{I} = \sigma$ the signature of a compact rational homology manifold.

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- Hirzebruch-Zagier ('70): class version for the Thom-Milnor L-classes.

Arithmetic genus and Todd classes

- Moonen ('78): if X is a complex projective variety, then

$$\sum_{n\geq 0} \chi_a(X^{(n)}) \cdot t^n = (1-t)^{-\chi_a(X)} = \exp\left(\sum_{r\geq 1} \chi_a(X) \cdot \frac{t^r}{r}\right),$$

for $\mathcal{I}(X) = \chi_a(X) := \sum_{k\geq 0} (-1)^k \cdot \dim H^k(X, \mathcal{O}_X)$ the arithmetic genus.

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- Moonen ('78): class version for the Baum-Fulton-MacPherson Todd classes of symmetric products.

Hirzebruch χ_y-genus

- If X is smooth and compact, $H^k(X;\mathbb{Q})$ carries a natural weight k pure Hodge structure, i.e.,

$$H^k(X;\mathbb{C}) = \oplus_{p+q=k} H^{p,q},$$

with $H^{p,q} = \overline{H^{q,p}}$. In fact, $H^{p,q} = H^q(X, \Omega_X^p)$ (Deligne).

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- The Hirzebruch χ_y-genus of X is:

$$\chi_y(X) = \sum_{p,q} (-1)^q h^{p,q}(X) \cdot y^p,$$

with $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$ the Hodge numbers of X.

- Borisov-Libgober, Zhou ('00): X compact smooth complex algebraic variety:

$$\sum_{n\geq 0} \chi_{-y}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r\geq 1} \chi_{-y^r}(X) \cdot \frac{t^r}{r} \right).$$

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- Borisov-Libgober ('00): generating series for the 2-variables elliptic genus of compact complex algebraic manifolds.
- For X a complex projective manifold:

$$\chi_{-1} = \chi, \ \ \chi_0 = \chi_a, \ \ \chi_1 = \sigma$$

so get back all previous results for genera in the smooth complex algebraic context.

Immediate Corollaries

- If X_g is a smooth projective curve of genus g, then

$$\sum_n \chi_{-y}(X_g^{(n)}) \cdot t^n = [(1-t)(1-yt)]^{g-1} .$$

In particular,

$$h^{p,q}(X_g^{(n)}) = \sum_{0 \leq k \leq p} \binom{g}{p-k}\binom{g}{q-k}, \quad 0 \leq p \leq q, \; p+q \leq n.$$

Immediate Corollaries

- If X is a smooth projective *surface* and $X^{[n]}$ is the n-th Hilbert scheme, then $X^{[n]} \to X^{(n)}$ is birational (crepant resolution). So,

$$h^{p,0}(X^{[n]}) = h^{p,0}(X^{(n)}).$$

The generating series formula yields Göttsche's formula:

$$\sum_{n,p} h^{p,0}(X^{[n]}) y^p t^n = \prod_{p \geq 0} (1 - (-1)^p y^p t)^{(-1)^{p+1} h^{p,0}(X)}$$

- Aim: Unify and extend these results to the singular setting, e.g., find generating series for (intersection homology) Hodge polynomials of (possibly singular) quasi-projective varieties, and in particular, for the intersection homology Euler characteristic and the Goresky-MacPherson signature.

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- Approach: Allow coefficients in mixed Hodge modules, i.e., consider twisted Hodge polynomials, twisted signatures etc.

Extensions of Hirzebruch's genus to the singular setting

Hirzebruch's χ_y-genus of a complex projective manifold, $\chi_y(X)$, admits several generalizations to the singular setting.

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Definition

The χ_y-genus transformation is the ring homomorphism

$$\chi_y : K_0(\mathrm{mHs}) \to \mathbb{Z}[y, y^{-1}]$$

$$[(V, F^\bullet, W_\bullet)] \mapsto \sum_p \dim_{\mathbb{C}}(gr_F^p(V \otimes_{\mathbb{Q}} \mathbb{C})) \cdot (-y)^p,$$

where $K_0(\mathrm{mHs})$ is the Grothendieck ring of the category of $\mathbb{Q}$-mHs.

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- If X is a compact algebraic manifold, then $\chi_y^{(c)}(X) = I\chi_y^{(c)}(X)$ is the Hirzebruch χ_y-genus.
- If X is projective (but possibly singular) then $I\chi_1(X) = \sigma(X)$ is the Goresky-MacPherson signature defined via Poincaré duality in intersection cohomology.

Crash course on Mixed Hodge Modules

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- All six Grothendieck operations on $D^b_c(X)$ "lift" to $D^b\mathrm{MHM}(X)$.

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- Then $\chi^{(c)}_y(X) = \chi^{(c)}_y(X,\mathbb{Q}^H_X)$, $I\chi^{(c)}_y(X) = \chi^{(c)}_y(X, IC^H_X)$.

Symmetric powers of mixed Hodge modules

Definition

Let $p_n : X^n \to X^{(n)}$ be the projection to the symmetric product $X^{(n)} = X^n/\Sigma_n$. The n-th symmetric power of $\mathcal{M} \in D^b\mathrm{MHM}(X)$ is defined as:

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- $\mathcal{M}^{\boxtimes n} \in D^b\mathrm{MHM}(X^n)$ is the n-th external product of $\mathcal{M}$ with the induced Σ_n-action.
- $(-)^{\Sigma_n} := \frac{1}{n!}\sum_{\sigma \in \Sigma_n} \psi_\sigma$ is the projector on the Σ_n-invariant sub-object.

Important special cases

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Important special cases

- if $\mathcal{M} = \mathbb{Q}_X^H$ then: $(\mathbb{Q}_X^H)^{(n)} = \mathbb{Q}_{X^{(n)}}^H$
- if $\mathcal{M} = IC_X'^H := IC_X^H[-\dim X]$ then: $\left(IC'^H_X\right)^{(n)} = IC'^H_{X^{(n)}}$
- if $\mathcal{L}$ is a "nice" variation of mHs on $U \subset X$, then $p_n : U^n \to U^{(n)}$ is a finite ramified covering branched along the "fat diagonal", i.e. the induced map of the configuration spaces on n (un)ordered points in U:

$$F(U,n) \xrightarrow{p_n} B(U,n) := F(U,n)/\Sigma_n,$$

with

$$F(U,n) := \{(x_1, x_2, \ldots, x_n) \in U^n \mid x_i \neq x_j \;\; \text{for} \;\; i \neq j\},$$

is a finite unramified covering. So $\mathcal{L}^{(n)}|B(U,n)$ is a "nice" variation on $B(U,n)$. Then $\left(IC'^H_X(\mathcal{L})\right)^{(n)} = IC'^H_{X^{(n)}}(\mathcal{L}^{(n)})$

Theorem A. (M.-Schürmann)

Let X be a complex quasi-projective variety and $\mathcal{M} \in D^bMHM(X)$. For $p, q, k \in \mathbb{Z}$, denote by

$$h^{p,q,k}_{(c)}(X, \mathcal{M}) := h^{p,q}(H^k_{(c)}(X; \mathcal{M})) := dim(Gr^p_F Gr^W_{p+q} H^k_{(c)}(X; \mathcal{M}))$$

the corresponding Hodge numbers. Then:

$$\sum_{n \geq 0} \left(\sum_{p,q,k} h^{p,q,k}_{(c)}(X^{(n)}, \mathcal{M}^{(n)}) \cdot y^p x^q (-z)^k \right) \cdot t^n$$

$$= \prod_{p,q,k} \left(\frac{1}{1 - y^p x^q z^k t} \right)^{(-1)^k \cdot h^{p,q,k}_{(c)}(X,\mathcal{M})}$$

Idea of proof (for the experts)

- Let $\bar{K}_0(D^b\mathrm{MHM}(pt))$ be the Grothendieck ring associated to the abelian monoid of isomorphism classes of objects with the direct sum; the product is induced by $\otimes$, and the unit is $[\mathbb{Q}^H_{pt}]$.

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- Let $h : \bar{K}_0(D^b\mathrm{MHM}(pt)) \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$ be given by

$$[\mathcal{V}] \mapsto \sum_{p,q,k} h^{p,q}(H^k(\mathcal{V})) \cdot y^p x^q (-z)^k$$

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- Then h is a homomorphism of pre-lambda rings, with pre-lambda structure on $\bar{K}_0(D^b\mathrm{MHM}(pt))$ given by

$$\sigma_t([\mathcal{V}]) := 1 + \sum_{n \geq 1} [(\mathcal{V}^{\otimes n})^{\Sigma_n}] \cdot t^n$$

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- apply this to $\mathcal{V} = k_{*(!)}\mathcal{M}$, with $(\mathcal{V}^{\otimes n})^{\Sigma_n} \simeq k_{*(!)}(\mathcal{M}^{(n)})$.

Alternating objects and Configuration spaces

We can work with the opposite pre-lambda structure $\lambda_t = \sigma_{-t}^{-1}$ on $\bar{K}_0(D^b\mathrm{MHM}(pt))$ given by

$$\lambda_t([\mathcal{V}]) := 1 + \sum_{n\geq 1} [(\mathcal{V}^{\otimes n})^{sign-\Sigma_n}] \cdot t^n,$$

for

$$(-)^{sign-\Sigma_n} := \frac{1}{n!} \sum_{\sigma\in\Sigma_n} (-1)^{sign(\sigma)} \cdot \psi_\sigma$$

the projector onto the alternating Σ_n-equivariant sub-object.

Theorem B. (M.-Schürmann)

Let $X^{\{n\}} := B(X,n)$ the configuration space of all unordered n-tuples of different points in X, and

$$\mathcal{M}^{\{n\}} := (p_{n*}\mathcal{M}^{\boxtimes n})^{sign-\Sigma_n} \in D^b MHM(X^{(n)}).$$

Then:

$$\sum_{n\geq 0}\left(\sum_{p,q,k} h_c^{p,q,k}(X^{\{n\}},\mathcal{M}^{\{n\}})\cdot y^p x^q (-z)^k\right)\cdot t^n$$

$$= \prod_{p,q,k}\left(1+y^p x^q z^k t\right)^{(-1)^k\cdot h_c^{p,q,k}(X,\mathcal{M})}.$$

Corollary of Theorem A.

Let $f^p_{(c)} := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^i_{(c)}(X, \mathcal{M})$, *so that* $\chi^{(c)}_{-y}(X, \mathcal{M}) = \sum_p f^p_{(c)}(X, \mathcal{M}) \cdot y^p$. *Then:*

$$\sum_{n \geq 0} \chi^{(c)}_{-y}(X^{(n)}, \mathcal{M}^{(n)}) \cdot t^n = \prod_p \left(\frac{1}{1 - y^p t} \right)^{f^p_{(c)}(X,\mathcal{M})}$$

$$= \exp \left(\sum_{r \geq 1} \chi^{(c)}_{-y^r}(X, \mathcal{M}) \cdot \frac{t^r}{r} \right) .$$

A different proof based on equivariant genera and traces

- Main ingredient: The Künneth isomorphism holds in mHs:

$$H^*_{(c)}(X^{(n)}; \mathcal{M}^{(n)}) \simeq (H^*_{(c)}(X^n; \mathcal{M}^{\boxtimes n}))^{\Sigma_n} \simeq ((H^*_{(c)}(X; \mathcal{M}))^{\otimes n})^{\Sigma_n}$$

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- Σ_n acts graded anti-symmetrically on $H^*_{(c)}(X^n,\mathcal{M}^{\boxtimes n})$, so can take traces of the action. Define equivariant Hodge genera by:

$$\chi^{(c)}_{-y}(X^n,\mathcal{M}^{\boxtimes n};\sigma) := \sum_{i,p}(-1)^i \mathrm{trace}\left(\sigma|\ \mathrm{Gr}^p_F\ H^i_{(c)}(X^n,\mathcal{M}^{\boxtimes n})\right)\cdot y^p.$$

- Step 1: For any $n \geq 0$,

$$\chi_{-y}^{(c)}(X^{(n)}, \mathcal{M}^{(n)}) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \chi_{-y}^{(c)}(X^n, \mathcal{M}^{\boxtimes n}; \sigma)$$

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- Step 2: If $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \cdots, k_n)$, i.e., $k_r = \#$ of length r cycles in σ, $\sum_{r=1}^{n} k_r \cdot r = n$, then

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- Step 3: For any r-cycle σ_r:

$$\chi_{-y}^{(c)}(X^r, \mathcal{M}^{\boxtimes r}; \sigma_r) = \chi_{-y^r}^{(c)}(X, \mathcal{M}) = \Psi_r \left(\chi_{-y}^{(c)}(X, \mathcal{M}) \right),$$

for Ψ_r the r-th Adams operation on $\mathbb{Z}[y^{\pm 1}]$.

Characteristic class version

- For X a complex projective variety,

$$\chi_y(X) = \int_X T_{y*}(X)$$

for $T_{y*}(X)$ the (homology) Hirzebruch class of Brasselet-Schürmann-Yokura.

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- The 3 steps above admit class versions and yield generating series for the Hirzebruch classes of symmetric products (extending a calculation by Moonen for the case when X is smooth and projective).

Theorem (Cappell-Schürmann-Shaneson-M.-Yokura)

Let X be a complex quasi-projective variety and $X^{(n)} := X^n/\Sigma_n$. Then the following identity holds in $\sum_n H_{2*}^{BM}(X^{(n)}; \mathbb{Q}[y]) \cdot t^n$:

$$\sum_{n \geq 0} T_{-y_*}(X^{(n)}) \cdot t^n = \exp\left(\sum_{r \geq 1} \Psi_r d_*^r T_{-y^r *}(X) \cdot \frac{t^r}{r} \right),$$

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- *The multiplication on the right-hand side is with respect to the Pontrjagin product induced by*

$$X^{(m)} \times X^{(n)} \to X^{(m+n)}, \quad m, n \in \mathbb{N}.$$

Happy Birthday, ANATOLY !!!