On a relation between algebraic and geometric properties of braids

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Introduction

Braids form a group: $B_n$
Braids can also be seen as automorphisms of the punctured disc.

Center of $B_n : \langle \Delta^2 \rangle$

Modulo this center, braids can be seen as automorphisms of the **punctured sphere**.

Hence one can apply **Nielsen-Thurston** theory to braids.
Geometric classification of braids:

**Periodic braids:** Finite order elements in $B_n / \langle \Delta^2 \rangle$

That is, roots of $\Delta^m$. 

Rotations
Geometric classification of braids:

**Periodic braids:** Rotations

**Pseudo-Anosov braids:** Preserve two transverse measured foliations…

…scaling the measure of $\mathcal{F}^u$ by $\lambda > 1$

and the measure of $\mathcal{F}^s$ by $\lambda^{-1}$
Geometric classification of braids:

**Periodic braids:** Rotations

**Pseudo-Anosov braids:** Preserve two transverse measured foliations…

**Reducible braids:** Preserve a family of disjoint simple curves

Reduction system
Every braid has a **canonical reduction system.**  (Birman-Lubotzky-McCarthy)

In general, the canonical reduction system can be quite complicated.

But it can always be simplified by an automorphism

\[ \alpha \rightarrow \eta \alpha \eta^{-1} \]

This corresponds to a conjugation
We can then decompose the disc $D$ along the reduction curves:
Example:

Every reducible braid can be decomposed into simpler braids.
Example:

Every reducible braid can be decomposed into simpler braids.
The geometric side

Thurston decomposition

**Theorem:** (Thurston) (applied to braids)

Every braid is either **periodic**, or **pseudo-Anosov**, or **reducible**.

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**Theorem:** (Thurston) (applied to braids)

The canonical reduction system decomposes a braid into braids which are either **periodic** or **pseudo-Anosov**.
The geometric side

Thurston decomposition allows to show theoretical results.

First solving the periodic and pseudo-Anosov case…

… and then deducing the reducible case from the above ones.

Examples:

(GM, 2003) The n-th root of a braid is unique up to conjugacy.

In practice, knowing the canonical reduction system improves algorithms.

Example: (Birman-Gebhardt-GM, 2003)
A project to solve the conjugacy problem in braid groups.

But... how can we compute the canonical reduction system?

Remark: There is a well-known algorithm by Bestvina-Handel, using train tracks.

We will use a different approach.
If the reduction curves are **round**, they are very easy to detect.

Benardete-Gutiérrez-Nitecki, 1991
The geometric side

Finding reduction curves

If the reduction curves are **almost round**, they are very easy to detect...

... if the interior braid is trivial.

**Theorem**: (GM-Wiest, 2009) There is a polynomial algorithm which decides whether a braid admits an almost-round reduction curve, with trivial interior braid.

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But... what happens if the reduction curves are very tangled?

Recall that they can be simplified by a conjugation.

But... what conjugation?
The algebraic side

Generators and relations

E. Artin (1925)

\[ B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i - j| \geq 2) \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (1 \leq i \leq n - 2) \right\rangle \]
Positive elements: Braids in which every crossing is positive

Simple elements: Positive elements in which every pair of strands cross at most once.

Simple elements of $B_n$  Bij.  Permutations of $\Sigma_n$

$\Delta =$ Biggest simple element = Half twist
The algebraic side

Left normal form


Left normal form: \[ x = \Delta^p x_1 \cdots x_r \]

Simple elements

Every product \( x_i x_{i+1} \) must be **left-weighted**.

Biggest possible simple element in any such writing of this product.

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In general: Given simple elements $a, b$, $a \rightarrow b$ is not left-weighted.
In general: Given simple elements $a, b,$

\[ a \quad b \]

Not left-weighted

\[ a \quad st \]
In general: Given simple elements $a$, $b$, $s$, and $t$.

- $a$ and $b$ not left-weighted.
- $s$ and $t$ left-weighted.
In general: Given simple elements $a, b,$

\[
\begin{array}{c}
\begin{array}{c}
\text{simple} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{simple} \\
\end{array}
\end{array}
\] $a b$

Not left-weighted

\[
\begin{array}{c}
\begin{array}{c}
\text{simple} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{simple} \\
\end{array}
\end{array}
\] $a s t$

left-weighted

We call this procedure a \textbf{local sliding} applied to $a b$.

\textbf{Remark:} Possibly $as = \Delta$, or $t = 1$. 

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The algebraic side

Computation of a left normal form, given a product of simple elements:

\[ x = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} \]

Apply all possible local slidings, until all consecutive factors are left weighted

\[ x = \Delta \Delta \Delta x_1 x_2 x_3 x_4 x_5 1 1 \]

Left normal form:

\[ x = \Delta^3 x_1 x_2 x_3 x_4 x_5. \]

Maximal power of \( \Delta \).

Minimal number of factors.

(canonical length)

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Conjugacy problem

Garside (1969)
Birman-Ko-Lee (1998)
Gebhardt (2005)
Gebhardt-GM (2009)

Charney: (1992) Artin-Tits groups of spherical type are biautomatic.
Conjugacy problem

Given an element $x$, compute the set of simplest conjugates of $x$.

(in some sense)

$x$ and $y$ are conjugate $\iff$ their corresponding sets coincide.
Could \( x \) be simplified by a conjugation?

For simplicity, we will assume that there is no power of \( \Delta \):

\[
x = \Delta^3 x_1 x_2 x_3 x_4 x_5.
\]

Consecutive factors are left-weighted. What about \( x_5 \) and \( x_1 \)?

Up to conjugacy, we can consider that \( x_5 \) and \( x_1 \) are consecutive.
Conjugacy problem

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Cyclic sliding:
Conjugacy problem

Cyclic sliding:

The resulting braid is not in left normal form.

We compute its left normal form, and it may become simpler.

Iterate…
Conjugacy problem

Sliding circuits

\[ x \rightarrow \text{sliding} \rightarrow x_1 \rightarrow \text{sliding} \rightarrow x_2 \rightarrow \text{sliding} \rightarrow x_3 \cdots \]

Set of sliding circuits in the Conjugacy class of \( x \).

\[ SC(x) \]

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Two braids $x$ and $y$ are conjugate $\iff SC(x) = SC(y)$

We solve the conjugacy problem by computing these sets.

**Remark:** Cyclic sliding simplifies braids algebraically…

…but also geometrically!
Suppose that \( x \) is reducible.

The reducing curves are very tangled

The reducing curves are simpler

In general, either round or almost-round!
Theorem: (GM-Wiest, 2009) Suppose that $y \in B_n$ belongs to a sliding circuit and so do $y^2$, $y^3$, ..., $y^m$, where $m = \left(\frac{n(n-1)}{2}\right)^3$.

Then $y$ admits a reduction curve which is either round or almost-round.

Ideas of the proof:

1) Reduction curves of a braid are preserved by powers.

2) Some small power of $y$ has the right property, provided it belongs to a sliding circuit.
Suppose that \( y \) is **rigid**.

This means that it is in left normal form, even if considered *around a circle*.

Then, the braid inside an innermost tube must be either **trivial** or **pseudo-Anosov**.
Proposition: (GM-Wiest, 2009) If a **rigid** braid has a **pseudo-Anosov** interior braid, the corresponding reduction curve is **round**.

Proposition: (GM-Wiest, 2009) If a braid has a **trivial** interior braid, the corresponding reduction curve is either **round** or **almost-round**.

The rigid case is solved!
We investigate the conjugating element along the circuit.

Up to replacing $y$ by a small power, $P(y)$ is rigid.

The reduction curves of $P(y)$ are reduction curves of $y$.

We are done!?

Unless $P(y)$ has no reduction curves. But then it is trivial. $\Rightarrow y$ is rigid.

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Cyclic sliding simplifies left normal forms, and also reduction curves.

This provides an algorithm to determine the geometric type of the braid, and to find the reducing curves.

This algorithm has polynomial complexity if this distance has a polynomial bound.

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