On the Connectedness of the Branch Locus of Moduli Spaces of Riemann Surfaces

joint work with A. F. Costa
and G. Bartolini

Milagros Izquierdo

Jaca June 2009

Riemann Surfaces

$$\left(X, \text{complex atla}\right) \longleftrightarrow \left(\mathcal{C}_X, \text{birational structure}\right)$$

$$\updownarrow$$

$$X = \mathbb{H}^2/\Delta \quad , \Delta \text{ Fuchsian Group.}$$

Δ = cocompact discrete subgroup of $PSL(2, \mathbb{R})$

T_g = space of geometries on a surface of genus g

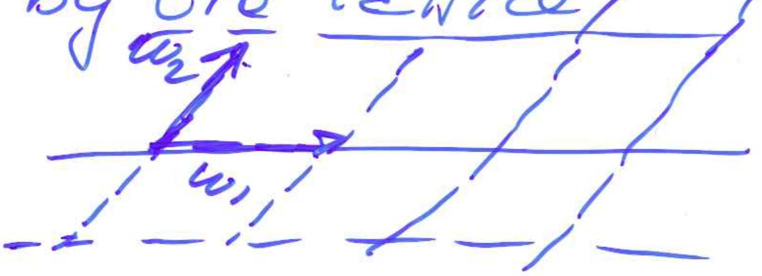
$\mathcal{M}_g$ = space (orbifold) of conformal classes of surfaces

$$\mathcal{M}_g = T_g/M_g \qquad M_g = \mathrm{Diff}(S_g)/\mathrm{Diff}_0(S_g)$$

Mapping Class Gr = Teichmüller Modular Group

Tori $T_1 = \mathbb{H}^2$

A torus X is given by the lattice

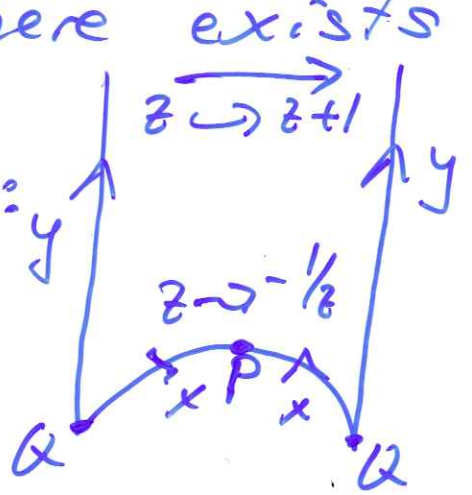


$$\{m\omega_1 + n\omega_2\}_{m,n\in\mathbb{Z}} = \Lambda$$

Marking $j(\Lambda) = \frac{\omega_2}{\omega_1}$

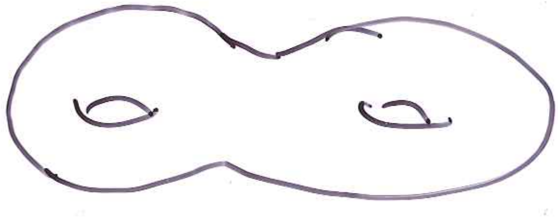
Two lattices are conformal if and only if there exists $f(z) = \frac{az+b}{cz+d} \in PSL(2,\mathbb{Z})$

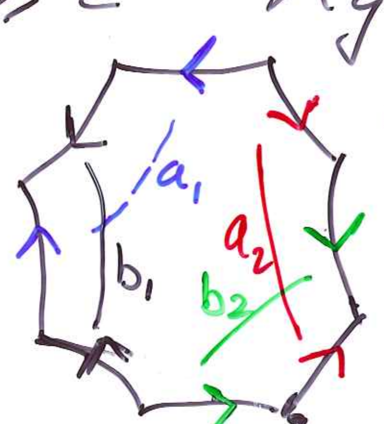
$\mathcal{M}_1$:



branching : $\{P, Q\}$

In general $g \geq 2$ $\quad X_g = \mathbb{H}^2/\Gamma_g$





$a_1 b_1 a_1^{-1} b_1^{-1} b_2 a_2 b_2^{-1} a_2^{-1}$

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \,/\, \prod_{i=1} [a_i, b_i] = 1 \rangle$$

$a_1, b_1, \ldots, a_g, b_g$ = basis of homotopy

$$T_g = \left\{ \sigma : \Gamma_g \longrightarrow PSL(2,\mathbb{R}) \;\middle/\; \begin{array}{l} \sigma \text{ injective} \\ \sigma(\Gamma_g) \text{ discrete} \end{array} \right\} \Big/ PSL(2,\mathbb{R})$$

$$M_g = \mathrm{Out}(\Gamma_g) = \mathrm{Diff}(X_g) / \mathrm{Diff}_0(X_g)$$

universal covering

$$T_g \longrightarrow T_g / M_g = \mathcal{M}_g$$

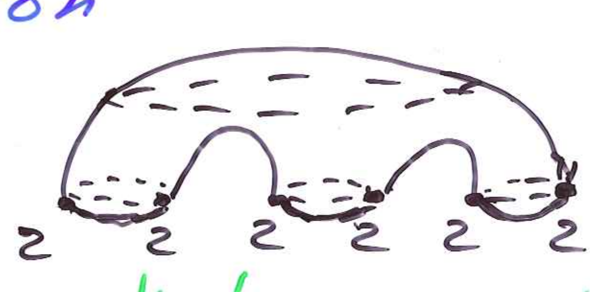
$\mathcal{B}_g$ = Branch locus

$$\mathcal{B}_g = \{ X \in \mathcal{M}_g \;/\; \mathrm{Aut}\, X \neq Id \} \quad g \geq 3$$

$$\mathcal{B}_2 = \{ X \in \mathcal{M}_2 \;/\; \mathrm{Aut}\, X \geq C_2 = \langle h \rangle \}$$

h: hyperelliptic involution

$\circlearrowleft h$ quotient



Two surfaces X, $\bar{X}$ are called <u>equisymmetric</u> if they have the same symmetry, i.e., $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\bar{X})$ conjugated in M_g

Spaces of Fuchsian Groups

Δ Fuchsian gr $\equiv$ (cocompact) discrete subgroup of $PSL(2,\mathbb{R})$

(Δ acts prop. discont. on $\mathbb{H}^2$)

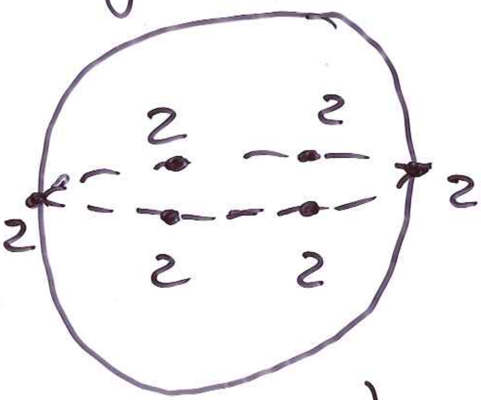
(compact) R. S. $X_g \cong \mathbb{H}^2/\Delta$, $g \geq 2$

$$\Delta = \left\langle x_1, \dots, x_r, a_1, b_1, \dots, a_h, b_h \;/\; x_i^{m_i} = 1 \ (1 \leq i \leq r);\ \prod x_i \prod [a_i, b_i] = 1 \right\rangle$$

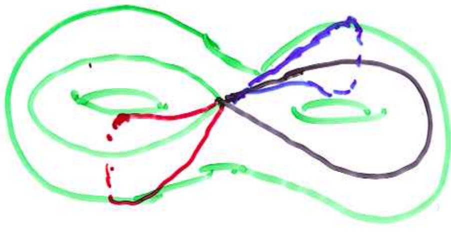
X_g = orbifold with underlying surface of genus h and r cone points of order m_i

(x_i : generator of maximal cyclic subgr.

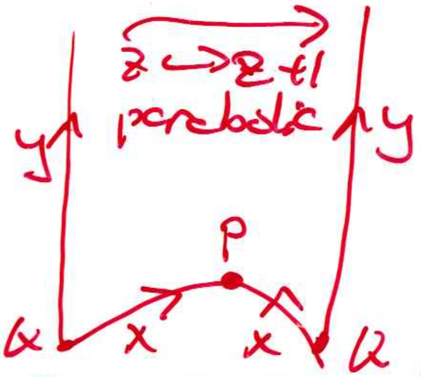
signature $s(\Delta) = (h; m_1, \dots, m_r)$



$(0; 2,2,2,2,2,2)$



$(2; -)$



$(0; 2, 3, \infty)$

$(0; 1; 2, 3)$

P : Fundamental region for Δ

$P/_{(\text{pairing})} \simeq X_g$: hyperbolic structure

Area $\mu(\Delta) = 2\pi\left(2h - 2 + \sum\left(1 - \frac{1}{m_i}\right)\right)$

$s(\Delta) = (h; -)$, Δ Surface Fuchsian Gr.

A finite gr. G is a gr of automorphis.. of $X_g = \mathbb{H}^2/\Gamma$, Γ: Surf Fuch. Gr.
iff $\exists$ Δ Fuch. gr. $\theta: \Delta \xrightarrow{epi} G$
$\ker \theta = \Gamma$

Universal covering

$$\begin{tikzcd} \mathbb{H}^2 \arrow[r] \arrow[d] & \mathbb{H}^2/\Delta \arrow[d, leftrightarrow] \\ X_g = \mathbb{H}^2/\Gamma \arrow[ur] \arrow[r] & X_g/G \end{tikzcd}$$

θ: monodromy of cover $\mathbb{H}^2/\Gamma \to \mathbb{H}^2/\Delta$

An automorphism of X_g will fix the class of the uniformizing Surf. Fuch. Gr.

$|\mathrm{Aut}(X_g)| \leq 84(g-1)$

A morphism $f: X \to Y$, X, Y R.S
<u>covering</u>, holomorphic function

f is given by inclusion $i: \Lambda \to \Delta$
$X = \mathbb{H}^2/\Lambda$; $Y = \mathbb{H}^2/\Delta$

f is determined by $\theta: \Delta \xrightarrow{trans} \Sigma_N$ symmetric gr on the Λ-cosets
$\Lambda = \theta^{-1}(\mathrm{Stb}(1))$

Singerman (1970)

Λ (and i) determined by θ and Δ

iff $s(\Delta) = (h; m_1, \dots, m_r)$ then

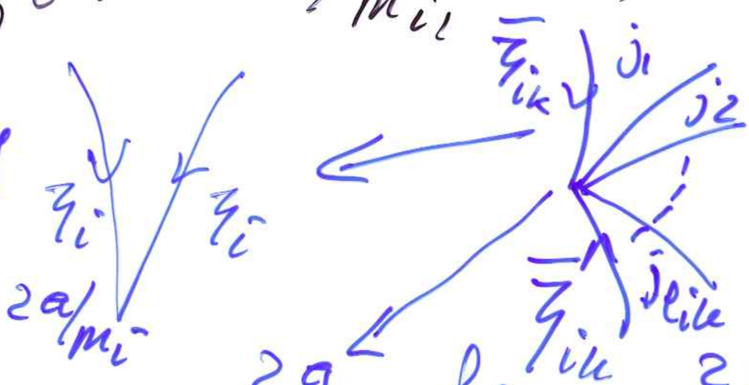
$s(\Lambda) = (h'; m'_{i1}, \dots, m'_{is_i}, \dots, m'_{r1}, \dots, m'_{rs_r})$

iff $\theta : \Delta \longrightarrow \Sigma_{|\Delta:\Lambda|}$ such that

i) Riemen-Hurwitz formula $\mu(\Lambda) = \mu(\Delta)|\Delta:\Lambda|$

ii) $\theta(x_i)$ = product of s_i cycles, each of length $m_i/m'_{i1}, \dots, m_i/m'_{is_i}$

Geometrically



$l_{ik} = m_i/m'_{ik}$

$$\frac{2a}{m_i} \cdot l_{ik} = \frac{2a}{m'_{ik}}$$

<u>Example</u>: Consider X_4 uniformized by $\Gamma = \ker\alpha$ $\quad \alpha : \Delta(0; 2, 2, 3, 6) \longrightarrow D_6 = \langle s, a \rangle$

$\alpha(x_1) = s$, $\alpha(x_2) = sa^3$; $\alpha(x_3) = a^2$; $\alpha(x_4) = a$

Consider $\Lambda = \alpha^{-1}\langle a^2 \rangle$ with $|\Delta:\Lambda| = 4$

The monodromy $\theta : \Delta \longrightarrow \Sigma_4(\bar{Id}, \bar{s}, \bar{a}, \bar{sa})$

$x_1 \mapsto (\bar{Id}, \bar{s})(\bar{a}, \bar{sa})$ } regular pts

$x_2 \mapsto (\bar{Id}, \bar{sa})(\bar{s}, \bar{a})$

$x_3 \mapsto (\bar{Id})(\bar{s})(\bar{a})(\bar{sa})$ 4 cone points of order 3

$x_4 \mapsto (\bar{Id}, \bar{a})(\bar{s}, \bar{sa})$ 2 cone points of order 3

Genus $4(-1 + 2/3 + 5/6) = 2h' - 2 + 6 \cdot 2/3$, $\underline{h' = 0}$

$$s(\Lambda) = (0; \underset{31}{3}, \underset{32}{3}, \underset{33}{3}, \underset{34}{3}, \underset{41}{3}, \underset{42}{3})$$

Teichmüller Space

Δ abstract Fuch. gr $s(\Delta) = (h; m_1, \dots m_r)$

$$T(\Delta) = \left\{ \sigma : \Delta \to PSL(2,\mathbb{R}) \;\middle|\; \begin{array}{l}\sigma \text{ inject.} \\ \sigma(\Delta) \text{ discret}\end{array} \right\} \Big/ PSL(2,\mathbb{R})$$

$g \in PSL(2,\mathbb{R})$ acts by $\sigma \to \sigma^g$

$\sigma^g(\delta) = g^{-1}\sigma(\delta)g \quad , \delta \in \Delta$

$$\begin{tikzcd} \mathbb{H}^2/\Delta \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_2"'] & \mathbb{H}^2/\sigma_1(\Delta) \arrow[d, "\bar{g}\ \text{(conformal)}"] \\ & \mathbb{H}^2/\sigma_2(\Delta) \end{tikzcd}$$

$\sigma_2^{-1}\bar{g}\,\sigma_1$: homotopic to Id.

If Λ subgroup of Δ : $i : \Lambda \to \Delta$

$$i_* : T(\Delta) \to T(\Lambda)$$
$$[\sigma] \mapsto [\sigma \circ i]$$

Γ Surf. Fuch gr with $\Gamma \leq \Delta$

$$T(\Delta) \leq T_g$$

G finite gr. $T_g^G = \{[\sigma] \in T_g \mid g[\sigma] = [\sigma] \;\forall g \in G\}$

$T_g^G \neq \emptyset$; T_g^G : surfaces with G as a group of automorphisms

Teichmüller Modular Gr. $M(\Delta) = \mathrm{Out}(\Delta)$

$$M(\Delta) = \mathrm{Diff}(\mathbb{H}^2/\Delta) \big/ \mathrm{Diff}_0(\mathbb{H}^2/\Delta)$$

since $\Delta = \pi_1(\mathbb{H}^2/\Delta)$ as orbifold

$M(\Delta)$ acts on $T(\Delta)$ by $\beta[\sigma] = [\sigma\beta]$

Moduli Spaces

Quotient Space $T(\Delta)/M(\Delta) = \mathcal{M}(\Delta)$

$$\begin{tikzcd} \mathbb{H}^2/\Delta \arrow[r, "{[\sigma_1]}"] \arrow[d, "\beta"'] & \mathbb{H}^2/\sigma_1(\Delta) \arrow[d, "\sigma\ \text{conformal}"] \\ \mathbb{H}^2/\Delta \arrow[r, "{[\sigma_2]}"] & \mathbb{H}^2/\sigma(\Delta) \end{tikzcd}$$

Surfaces $\mathbb{H}^2/\sigma_1(\Delta) = X_1$ and $\mathbb{H}^2/\sigma_2(\Delta) = X_2$ conformal (the same complex structure) iff $[\sigma_2 \sigma_1^{-1}]$ is conformal

M_g acts propert. discont on T_g

Branch Locus: surfaces with automorphs

$$\begin{tikzcd} \mathbb{H}^2/\Gamma_g = X \arrow[r, "\sigma"] \arrow[d] & \sigma X \arrow[d, "\text{conformal}"] \\ \beta_* X \arrow[r, "\sigma\beta"] & \sigma\beta X \end{tikzcd}$$

$\exists \gamma \in PSL(2,\mathbb{R})$ $\quad \sigma(\Gamma_g) = \gamma^{-1}\sigma\beta(\Gamma_g)\gamma$

γ induces an automorphism of $[\sigma X]$

$\mathrm{stb}_{\mathcal{M}_g}[\sigma] = \{\beta \in M_g : \beta[\sigma] = [\sigma]\} = \mathrm{Aut}(X)$

$G \leq \mathrm{Aut}(X)$ finite

G determines a conjugacy class of finite subgroups of M_g: <u>the symmetry of X</u>

X_g and Y_g equisymmetric if $\mathrm{Aut}(X_g)$ conjugated to $\mathrm{Aut}(Y_g)$

An important consequence was proved by Greenberg:

Theorem 1.5.2. *(See [10], [15]) The following conditions are equivalent:*

1. *$Mod(\Gamma)$ fails to act faithfully on $T(\Gamma)$.*
2. *There exists a Fuchsian group Γ' and a group monomorphism $\alpha : \Gamma \to \Gamma'$ such that $d(\Gamma) = d(\Gamma')$ and $\alpha(\Gamma)$ is a normal subgroup of Γ'.*

The full list such of pairs $(s(\Gamma), s(\Gamma'))$ of signatures of Fuchsian groups such was obtained by Singerman in [22].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(2; [-])$	$(0; [2,2,2,2,2,2])$	2
$(1; [t,t])$	$(0; [2,2,2,2,t])$	2
$(1; [t])$	$(0; [2,2,2,2t])$	2
$(0; [t,t,u,u])$	$(0; [2,2,t,u])$	2
$(0; [t,t,u])$	$(0; [2,t,2u])$	2
$(0; [t,t,t,t])$	$(0; [2,2,2,t])$	4
$(0; [t,t,t])$	$(0; [3,3,t])$	3
$(0; [t,t,t])$	$(0; [2,3,2t])$	6

Normal pairs of Fuchsian groups

To decide whether a given finite group can be the full group of automorphism of some compact Riemann surface we will need all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for some Fuchsian groups Γ and Γ' such that $\Gamma \leq \Gamma'$ and $d(\Gamma) = d(\Gamma')$. The full list in the non-normal case was also obtained by Singerman in [22].

$s(\Gamma)$	$s(\Gamma')$	$[\Gamma' : \alpha(\Gamma)]$
$(0; [7,7,7])$	$(0; [2,3,7])$	24
$(0; [2,7,7])$	$(0; [2,3,7])$	9
$(0; [3,3,7])$	$(0; [2,3,7])$	8
$(0; [4,8,8])$	$(0; [2,3,8])$	12
$(0; [3,8,8])$	$(0; [2,3,8])$	10
$(0; [9,9,9])$	$(0; [2,3,9])$	12
$(0; [4,4,5])$	$(0; [2,4,5])$	6
$(0; [n,4n,4n])$	$(0; [2,3,4n])$	6
$(0; [n,2n,2n])$	$(0; [2,4,2n])$	4
$(0; [n,3,3n])$	$(0; [2,3,3n])$	4
$(0; [n,2,2n])$	$(0; [2,3,2n])$	3

Non-normal pairs of Fuchsian signatures

Broughton 1990

$\mathcal{M}_g^G = \{X \in \mathcal{M}_g ;$ symmetry type of X is $G\}$

$\overline{\mathcal{M}}_g^G = \{X \in \mathcal{M}_g ;$ symmetry type of X contains $G\}$

$(G \leq Aut(X))$

i) $\mathcal{M}_g^G$ is a smooth, connected, locally closed alg subvariety of $\mathcal{M}_g$, dense in $\overline{\mathcal{M}}_g^G$

ii) $\overline{\mathcal{M}}_g^G$ is a closed, irreducible, alg. subv. of $\mathcal{M}_g$

We obtain a stratification of $\mathcal{B}_g$

$$\mathcal{B}_g = \bigcup \mathcal{M}_g^{G,\theta}$$

Costa-I (2008) As every finite group G contains a cyclic subgroup of prime order $\Longrightarrow \mathcal{B}_g = \bigcup \overline{\mathcal{M}}_g^{C_p,\theta}$

θ : conjugacy class of G in M_g

(it depends on $\theta : \Delta \longrightarrow G$)

Each action of G is given by Δ Fuch gr. and $\theta : \Delta \longrightarrow G$ $\ker \theta = \Gamma_g$

$Aut(X)$ conjugate $Aut(Y)$ iff $\exists\, \omega \in Aut(G)$, $h \in Diff(X_0)$ s.t

$\varepsilon, \varepsilon' : G \longrightarrow Diff(X_0)$ $\varepsilon'(g) = h\,\varepsilon(g)\,h^{-1}$

Two (surface) monodromies $\theta_1, \theta_2 : \Delta \longrightarrow G$ define equiv actions of G on X iff

$$\begin{tikzcd} \Delta \arrow[r, "\theta_1"] \arrow[d, "\beta \in Aut(\Delta)"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[d, "\omega \in Aut(G)"] \\ \Delta \arrow[r, "\theta_2"] & G \end{tikzcd}$$

[θ_1, θ_2 equiv under the action of $\mathcal{B}^{(\Delta)} \times Aut(G)$
$\mathcal{B}(\Delta)$ subgroup of $Aut(\Delta)$ induced by or. homeomorphisms of $\mathbb{H}^2/\Delta$

We are interested in $Y \in \overline{\mathcal{M}}_g^{G_1,\theta} \cap \overline{\mathcal{M}}_g^{G_2,\bar{\theta}}$

$G = Aut(Y)$ contains G_1 and G_2 and

$\varphi : \overline{\Delta} \longrightarrow G$ extends $\Gamma = \ker(\varphi)$

both $\theta : \Delta_1 \longrightarrow G_1$ and

$\bar{\theta} : \Delta_2 \longrightarrow G_2$

$$\begin{tikzcd}
\Gamma \arrow[rrr, "\varphi, \theta, \bar{\theta}"] \arrow[d, hook] \arrow[dr, hook] & & & Id \arrow[d, hook] \arrow[dr, hook] & \\
\Delta_1 \arrow[rrr, dashed, bend left, "\theta = \varphi/\Delta_1"] \arrow[dr, hook] & \Delta_2 \arrow[rr, dashed, "\bar{\theta} = \varphi/\Delta_2"'] \arrow[d, hook] & & G_1 \arrow[r, dashed] \arrow[d, hook] & G_2 \arrow[dl, hook] \\
& \bar{\Delta} \arrow[rr, "\varphi"] & & Aut(Y) &
\end{tikzcd}$$

Geometrically

$$\begin{tikzcd}
& Y_g \arrow[dl, "f_\theta"'] \arrow[dr, "f_{\bar{\theta}}"] \arrow[dd, "f_\varphi"] & \\
\mathbb{H}^2/\Delta_1 \arrow[dr] & & \mathbb{H}^2/\Delta_2 \arrow[dl] \\
& \mathbb{H}^2/\bar{\Delta} &
\end{tikzcd}$$

I-Ying (2008): Equisymmetric stratification of the trigonal R-S of genus 4

Costa-I (2008): Equisymmetric stratification of $\mathcal{B}_4$ (41 strata)

This gives a structure of cw-complex to M_4

Bartolini-I (2009): Equisym. Strat. of $\mathcal{B}_5$

Example There are three strata in B_4 associated to actions of C_5.

They correspond to

$\theta_1 : \Delta(0; 5,5,5,5) \longrightarrow C_5 = \langle a \mid a^5 = 1 \rangle$

$x_1 \mapsto a$, $x_2 \mapsto a$, $x_3 \mapsto a$ $(x_4 \mapsto a^2)$

$\theta_2 : \Delta \longrightarrow C_5$

$x_1 \mapsto a$, $x_2 \mapsto a$, $x_3 \mapsto a^4$, $(x_4 \mapsto a^4)$

$\theta_3 : \Delta \longrightarrow C_5$

$x_1 \mapsto a$, $x_2 \mapsto a^2$, $x_3 \mapsto a^3$, $(x_4 \mapsto a^4)$

(θ_1, Δ) gives maximal groups : symmetry is C_5

$(\theta_2 : \Delta)$ extends to $\bar{\theta}_2 : \Delta(0; 2,2,2,5) \longrightarrow D_{10}$

the symmetry is D_{10}

$(\theta_3 : \Delta)$ extend to $\bar{\theta}_3 : \Delta(0; 2,2,5,5) \longrightarrow D_5$

$\bar{\theta}_3(y_1) = s$; $\bar{\theta}_3(y_2) = sa$; $\bar{\theta}_3(y_3) = a^3$

Kulkarni (1991) showed the existence of isolated points in $\mathcal{B}_g$ iff
$2g+1$ prime ≥ 11 (and $g=2, p=5$)

Isolated points are given by triangel gr. Δ, $s(\Delta)=(0; p, p, p)$

$g=3$, ($p=7$) there are two NON-MAXIMAL actions of C_7 extending to

$\bar{\theta}_1 : \Delta(0; 2, 7, 14) \longrightarrow C_{14}$

$\bar{\theta}_2 : \Delta(2, 3, 7) \longrightarrow PSL(2, 7)$ Klein Quartic

(2009) Bartolini - I

For all g $\quad \bigcup_{\theta_i} \mathcal{M}_g^{C_2, \theta_i} \cup \bigcup_{\alpha_j} \mathcal{M}_g^{C_3, \alpha_j}$

lies in the connected component of $\mathcal{B}_g$ of dimension $2g+2$

the largest dimension of $\mathcal{B}_g$ is attained for hyperelliptic surfaces

(2009) Costa – I B_g contains isolated strata of dim 1 iff $g+1=p$, p prime ≥ 11

Isolated stratum of dim 1

$\Downarrow$

Action of C_p, p prime $\theta: \Delta(0; p,p,p,p) \longrightarrow C_p$

$$p\left(-2+\frac{4(p-1)}{p}\right)=2(g-1) \; ; \; p=g+1$$

Classes of actions $\theta_i: \Delta(0; p,p,p,p) \longrightarrow C_p$

i) $\theta_1: x_1, x_2 \mapsto a, \quad x_3, x_4 \mapsto a^{-1}$ Non-Maximal

$$\mathcal{M}_{p-1}^{C_p,\theta_1} \subseteq \overline{\mathcal{M}}_{p-1}^{C_2, g/2} \cap \overline{\mathcal{M}}_{p-1}^{C_2, 0}$$

ii) $\theta_{2i}: x_1, x_2 \mapsto a, \; x_3 \mapsto a^i \quad 2 \leq i \leq p-3$

R.S $X_{p-1}^{\psi_i}$ $\psi_i: \Delta_2(0; p, 2p, 2p) \longrightarrow C_{2p}$

$$X_{p-1}^{\psi_i} \in \overline{\mathcal{M}}_g^{C_p, 2i} \cap \overline{\mathcal{M}}_g^{C_2, g/2}$$

iii) $\theta_3: x_1, x_2, x_3 \mapsto a, \; X_{p-1}^{\phi} \in \overline{\mathcal{M}}_{p-1}^{C_p,3} \cap \underbrace{\overline{\mathcal{M}}_{p-1}^{3,0}}_{\text{trigonal curves}}$

iv) $\theta_{4i}: x_1 \mapsto a, x_2 \mapsto a^{-1}, x_3 \mapsto a^i, x_4 \mapsto a^{-i}$

$$\mathcal{M}^{C_p, 4i} \subseteq \overline{\mathcal{M}}_g^{C_2, g/2}$$

v) $\theta_{5i}: x_1 \mapsto a, \; x_2 \mapsto a^i, \; x_3 \mapsto a^j$

$i \neq 1, p-1 \qquad j \notin \{1, p-1, i, p-i\}$

$1+i+j \not\equiv 0 \mod p$

Maximal action and the stratum $\mathcal{M}^{C_p, 5i}$ contains no curve with more symmetry

(2008)

Case - I : B_4 is connected

i) $B_4 = \bar{\mathcal{M}}_4^{C_2,0} \cup \bar{\mathcal{M}}_4^{C_2,1} \cup \bar{\mathcal{M}}_4^{C_2,2} \cup \bar{\mathcal{M}}_4^{C_3,01} \cup$
$\cup \bar{\mathcal{M}}_4^{C_3,02} \cup \bar{\mathcal{M}}_4^{C_5,1} \cup \bar{\mathcal{M}}_4^{C_3,1} \left[\cup \bar{\mathcal{M}}_4^{C_3,2} \cup \bar{\mathcal{M}}_4^{C_5,2} \cup \bar{\mathcal{M}}_4 \right.$

$\mathcal{M}_4^{C_2,0} = \{\theta : \Delta(0; \overbrace{2, \cdots, 2}^{10}) \longrightarrow C_2\}$

$\mathcal{M}_4^{C_2,1} = \{\theta : \Delta(1; \overbrace{2, \cdots, 2}^{6}) \longrightarrow C_2\}$

$\mathcal{M}_4^{C_2 2} = \{\theta : \Delta(2; 2, 2) \longrightarrow C_2\}$

$\mathcal{M}_4^{C_3,01} = \{\theta : \Delta(0; \overbrace{3, \cdots, 3}^{6}) \to C_3 ; \theta(x_{2i-1}) = a, \theta(x_{2i}) = \bar{a}$

$\mathcal{M}_4^{C_3,02} = \{\theta : \Delta(0; \overbrace{3, \cdots, 3}^{6}) \to C_3 ; \theta(x_i) = a\}$

$\mathcal{M}_4^{C_3,1} = \{\theta : \Delta(1; 3, 3, 3) \longrightarrow C_3\}$

$\mathcal{M}_4^{C_5,1} = \{\theta : \Delta(0; 5, 5, 5, 5) \longrightarrow C_5 / \theta(x_1) = \theta(x_2) = \theta(x_3) = a\}$

$\mathcal{M}_4^{C_3,2} = \{\theta : \Delta(2; -) \longrightarrow C_3\}$

$\mathcal{M}_4^{C_3,2} = \mathcal{M}_4^{D_3,\theta} \quad \theta : (0; \overbrace{2, -- 2}^{6}) \longrightarrow D_3$

$x_1 \mapsto s ; x_2 \mapsto sa, x_3, x_5 \mapsto s ; x_4, x_6 \mapsto sa$

$(\theta^{-1} \langle s \rangle) = (1; \overbrace{2, \cdots, 2}^{6})$

$\mathcal{M}_4^{C_3,2} \subseteq \bar{\mathcal{M}}_4^{C_2,1}$

In the same way

$$\mathcal{M}_4^{C_5,2} \cong \mathcal{M}_4^{D_{10}}$$

$$\mathcal{M}_4^{C_5,2} \subseteq \underset{(a^5)}{\overline{\mathcal{M}}_4^{C_2,0}} \cap \underset{(s)}{\overline{\mathcal{M}}_4^{C_2,2}}$$

and $\mathcal{M}_4^{C_5,3} = \mathcal{M}_4^{D_5,\theta_2}$, $\mathcal{M}_4^{C_5,5} \subseteq \overline{\mathcal{M}}_4^{C_2,2}$

From I-Ying (2008, Ying's PhD thesis)

i) $\mathcal{M}_4^{D_3 \times D_3} \subseteq \overline{\mathcal{M}}_4^{C_3,01} \cap \overline{\mathcal{M}}_4^{C_3,2} \subseteq \overline{\mathcal{M}}_4^{C_3,01} \cap \overline{\mathcal{M}}_4^{C_2,1}$

ii) $\mathcal{M}_4^{C_3 \times D_3} \subseteq \overline{\mathcal{M}}_4^{C_2,1} \cap \overline{\mathcal{M}}_4^{C_3,1} \cap \overline{\mathcal{M}}_4^{C_3,02}$

iii) $T_4 = \mathcal{M}_4^{C_1 5} \in \overline{\mathcal{M}}_4^{C_3,02} \cap \mathcal{M}_4^{C_5,1}$

iv) $\mathcal{M}^{C_6 \times C_2} \subseteq \overline{\mathcal{M}}^{C_3,02} \cap \overline{\mathcal{M}}_4^{C_2,2} \cap \overline{\mathcal{M}}^{C_2,1}$

(2009) Bartolini-I $\mathcal{B}_5$, $\mathcal{B}_6$, and $\mathcal{B}_8$ are connected with the exception of one, one, resp. two isolated points. $\mathcal{B}_7$ is connected.