ACC for log canonical thresholds

Mircea Mustaţă
University of Michigan

joint work with

T. de Fernex and L. Ein

Jaca, June 26, 2009

Basics about log canonical thresholds

Work over k alg closed, char$(k) = 0$.

Let $f \in k[x_1, \ldots, x_n]$, $f(0) = 0$ defining $H \subset \mathbf{A}^n$

Recall: $\text{mult}_0(f) = \max\{r \geq 0 \mid f \in (x_1, \ldots, x_n)^r\}$

$$= \text{ord}_E(f)$$

where E is exceptional divisor on $\text{Bl}_0(\mathbf{A}^n)$.

Idea: • consider **all** divisors over $\mathbf{A}^n$

• normalize order of vanishing along divisors

• take an infimum over all such choices

Consider: proper, birational morphisms $\pi\colon Y \to X$, with Y smooth. E prime divisor on Y giving a valuation ord_E of the function field of X

$K_{Y/X} \geq 0$ div on Y locally def by $\det(\mathrm{Jac}(\pi))$

Note: $\mathrm{Supp}(K_{Y/X}) = \mathrm{Exc}(\pi)$

Definition. $\mathrm{lct}(f) := \inf_{E/X} \frac{\mathrm{ord}_E(K_{Y/X})+1}{\mathrm{ord}_E(f)}$

If only E with $0 \in \pi(E)$, get $\mathrm{lct}_0(f)$.

Principle: "bad singularities $\Leftrightarrow$ small lct"

Fundamental fact: enough to consider E on a log resolution of f, i.e. when $\pi^{-1}(H) + K_{Y/X}$ SNC divisor: in local coord def by

$$y_1^{a_1} \cdots y_n^{a_n}$$

Consequences: • inf in the definition is a min

• $\mathrm{lct}(f) \in \mathbf{Q}$

Examples: 1) H smooth $\Rightarrow \mathrm{lct}(f) = 1$

2) $f = x_1^{a_1} \cdots x_n^{a_n} \Rightarrow \mathrm{lct}(f) = \min_i \frac{1}{a_i}$

3)$f = x_1^{a_1} + \cdots + x_n^{a_n} \Rightarrow \mathrm{lct}(f) = \min\left\{1, \sum_{i=1}^n \frac{1}{a_i}\right\}$

4) If f defines a hyperplane arrangement in $\mathbf{A}^n$, then

$$\mathrm{lct}(f) = \min_{W \in L'(\mathcal{A})} \frac{\mathrm{codim}(W)}{\#\{H \in \mathcal{A} \mid H \supseteq W\}}$$

5) If f is nondegenerate with respect to its Newton polytope P, then $\mathrm{lct}(f) = \min\{1/\lambda, 1\}$, where

$$(\lambda, \ldots, \lambda) \in \partial(P + \mathbf{R}_+^n)$$

In general: $\mathrm{lct}_0(f)$ is a refined version of $\frac{1}{\mathrm{mult}_0(f)}$

$$\frac{1}{\mathrm{mult}_0(f)} \leq \mathrm{lct}_0(f) \leq \frac{n}{\mathrm{mult}_0(f)}$$

Another useful property: for all f and g

$$\mathrm{lct}_0(f+g) \leq \mathrm{lct}_0(f) + \mathrm{lct}_0(g)$$

Variants of the definition: replace f by ideal, allow X with mild singularities, "mixed" case: for $g \in k[x_1, \ldots, x_n]$ with $\mathrm{lct}(g) \leq 1$, let

$$\mathrm{lct}((\mathbf{A}^n, g), f) = \inf_E \frac{\mathrm{ord}_E(K_{Y/X}) + 1 - \mathrm{ord}_E(g)}{\mathrm{ord}_E(f)}$$

Important for us: case of formal power series

$$f \in k[\![x_1, \ldots, x_n]\!]$$

For f formal power series, can put

$$\mathrm{lct}(f) = \lim_{d\to\infty} \mathrm{lct}_0(t_d(f)),$$

with $t_d(f)$ the truncation of f up to degree d.

The above is convergent since

$$|\,\mathrm{lct}_0(t_{d+m}(f)) - \mathrm{lct}_0(t_d(f))|$$

$$\leq \mathrm{lct}_0(t_{d+m}(f) - t_d(f))| \leq \frac{n}{d+1}$$

Important: this can also be computed using a log resolution of f (Hironaka, Temkin)

History of log canonical thresholds: Varchenko, Shokurov,...

Why care about lct's: they show up in various contexts

1) **Birational geometry**: lct(f) is the largest $q > 0$ s.t. $(\mathbf{A}^n, qH)$ log canonical

2) **Complex singularity exponents**: over $\mathbf{C}$

$$\text{lct}(f) = \sup\left\{ s > 0 \mid \frac{1}{|f(z)|^{2s}} \text{ loc integrable} \right\}$$

3) p-**adic integration** (Igusa): for p prime, f over $\mathbf{Z}$, it controls asymptotic behavior of

$$\#\{u \in (\mathbf{Z}/p^m\mathbf{Z})^n \mid f(u) = 0\}$$

Other contexts: motivic integration, eigenvalues of monodromy action, Bernstein poly, invar of sing in char p defined via Frobenius

Theorem 1 (dF-E-M). For all n, the set

$$\mathcal{T}_n := \{\mathrm{lct}_0(f) \mid f \in k[x_1, \ldots, x_n], f(0) = 0\}$$

satisfies the Ascending Chain Condition (ACC).

Remarks: • This treats ambient smooth var.

• Conjectured more generally by Shokurov (for ambient log canonical varieties and "mixed" log canonical thresholds)

• Was known for $n = 2$ (Shokurov, Phong-Sturm, Favre-Jonsson), $n = 3$ (Alexeev)

• Could be extended to: varieties with quotient, or lci singularities

• General case in dim ≥ 4 ?

Theorem 2 (dF-M; Kollár). The set of accumulation points of $\mathcal{T}_n$ is $\mathcal{T}_{n-1} \smallsetminus \{1\}$.

This was conjectured by Kollár. The case of decreasing sequences treated by dF-M; Kollár then dealt with arbitrary sequences (now increasing sequences excluded by Thm. 1)

One inclusion is easy: if $g \in k[x_1, \ldots, x_{n-1}]$ has $\mathrm{lct}(g) < 1$, then

$$f_m(x_1, \ldots, x_n) = g(x_1, \ldots, x_{n-1}) + x_n^m$$

has $\mathrm{lct}(f_m) = \min\left\{1, \mathrm{lct}(g) + \frac{1}{m}\right\} \searrow \mathrm{lct}(g)$

An interpretation of Theorem 1. Suppose k uncountable. Then Theorem 1 is equivalent with the following statement:

For every $c > 0$, the set

$$\{f \in k[\![x_1, \ldots, x_n]\!] \mid \mathrm{lct}(f) \geq c\}$$

is a *cylinder* in $k[\![x_1, \ldots, x_n]\!]$, i.e. there is N such that $\mathrm{lct}(f) \geq c$ iff $\mathrm{lct}(t_N(f)) \geq c$.

Furthermore: the above set is a cylinder iff it is open in the projective limit topology. Hence Thm.1 can be interpreted as a semicontinuity theorem for the *infinite-dimensional* family of all formal power series.

Note: for finite-dimensional families, such a semicontinuity result was known: Varchenko, Siu, Demailly-Kollár,...

Key ideas in the proof of ACC

Suppose $f_m \in k[x_1, \ldots, x_n]$ are such that $c_m :=$ $\mathrm{lct}(f_m) \nearrow c$

1) Special case that can be treated geometrically: f_m converges to some f in $(x_1, \ldots, x_n)$-adic toology, i.e. $\mathrm{mult}_0(f_m - f) \to \infty$.

$$|\,\mathrm{lct}_0(f_m) - \mathrm{lct}_0(f)| \leq \mathrm{lct}_0(f_m - f) \leq \frac{n}{\mathrm{mult}_0(f_m - f)} \to 0$$

ACC predicts: $\mathrm{lct}(f_m) \geq \mathrm{lct}(f)$ for $m \gg 0$

Suppose E computes $\mathrm{lct}(f)$ has image $\{0\}$ on $\mathbf{A}^n$. If $\mathrm{ord}_E(f_m - f) \geq \mathrm{ord}_E(f)$, then

$$\mathrm{lct}_0(f_m) \leq \frac{\mathrm{ord}_E(K_{Y/X}) + 1}{\mathrm{ord}_E(f_m)} \leq \frac{\mathrm{ord}_E(K_{Y/X}) + 1}{\mathrm{ord}_E(f)}$$

$$= \mathrm{lct}_0(f)$$

Theorem 3 (Kollár; dF-E-M) Let f, $g \in k[x_1, \dots, x_n]$. If E is a divisor computing $\mathrm{lct}(f)$ such that $\mathrm{ord}_E(f-g) > \mathrm{ord}_E(f)$, then $\mathrm{lct}(g) = \mathrm{lct}(f)$ around the image of E.

Kollár's proof: uses the results on MMP of Birkar-Cascini-Hacon-M$^{\mathrm{c}}$Kernan

dF-E-M: uses the Connectedness Theorem of Shokurov and Kollár (easy consequence of Kawamata-Viehweg vanishing)

2) Second point in the proof of ACC: given a sequence $\{f_m\}_m$, $f_m \in k[x_1, \ldots, x_n]$ with $c_m :=$ $\mathrm{lct}_0(f_m) \to c$, construct $F \in K[\![x_1, \ldots, x_n]\!]$, $K \supset$ k field extension, such that $\mathrm{lct}(F) = c$.

In fact, we will have the following property: for every $d \geq 1$, have infinitely many m such that

$$\mathrm{lct}_0(t_d(F)) = \mathrm{lct}_0(t_d(f_m))$$

For every such m, have

$$|\,\mathrm{lct}(F) - \mathrm{lct}_0(f_m)| \leq |\,\mathrm{lct}(F) - \mathrm{lct}_0(t_d(F))| +$$

$$|\,\mathrm{lct}_0(t_d(f_m)) - \mathrm{lct}_0(f_m)| \leq \frac{2n}{d+1}$$

Since $\mathrm{lct}_0(f_m) \to c$, it follows $\mathrm{lct}(F) = c$.

Two ways of constructing such $F \in K[\![x_1, \ldots, x_n]\!]$:

- dF-M: using ultrafilter constructions

- Kollár: using a sequence of generic points

The nonstandard construction: if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbf{N}$, then the sequence (f_m) defines an *internal polynomial* in ${}^*(k[x_1, \ldots, x_n])$. Truncating to keep just the monomials with standard exponents gives $F \in {}^*k[\![x_1, \ldots, x_n]\!]$.

One can show: for all d

$$\mathrm{lct}(t_d(F)) = \mathrm{lct}_0(t_d(f_m))$$

whenever $m \in \mathcal{U}$.

Kollár's construction: consider the truncation maps

$$P = k[\![x_1, \ldots, x_n]\!] \xrightarrow{t_d} P_d = k[\![x_1, \ldots, x_n]\!]/(x_1, \ldots, x_n)^{d+1}$$
$$\xrightarrow{\varphi_d} P_{d-1} = k[\![x_1, \ldots, x_n]\!]/(x_1, \ldots, x_n)^d$$

Each P_d is an affine space over k.

Construct by induction on $d \geq 1$ a sequence of *irreducible, closed* subsets $Z_d \subseteq P_d$ such that

i) Each Z_d is minimal with the property that there are infinitely many m such that $t_d(f_m) \in Z_d$.

ii) Each φ_d induces a *dominant* map $Z_d \to Z_{d-1}$.

Get a sequence of field extensions $k(Z_d) \subseteq k(Z_{d+1}) \subseteq \cdots$. Let $K = \bigcup_d k(Z_d)$.

The sequence of compatible maps

$$\mathrm{Spec}(K) \to P_d$$

defines a formal power series $F \in K[\![x_1, \ldots, x_n]\!]$.

By construction, for every $d \geq 1$, we have an infinite subset $I_d \subseteq \mathbf{N}$ such that

$$t_d(F) \text{ corresponds to the generic point of}$$

$$\{t_d(f_m) \mid m \in I_d\}$$

But there is $U_d \subset Z_d$ open such that $\mathrm{lct}_0(t_d(F)) = \mathrm{lct}_0(g)$ for every $g \in U_d$. The set

$$\{m \in I_d \mid t_d(f_m) \in U_d\}$$

is infinite, hence F satisfies our requirement.

Rough outline of the proof of Thm. 1:

Step 1. Given $f_m \in k[x_1, \ldots, x_n]$ with $\text{lct}_0(f_m) \nearrow c$, construct $F \in K[\![x_1, \ldots, x_n]\!]$ as above.

Step 2. Reduce to the case when $\text{lct}(F)$ is computed by a divisor E with image $\{0\}$. This is done by replacing f_m by $f_m^r g^s$, for suitable $r, s \geq 1$, and a general polynomial g of suitable degree. Can do this such that

$$\text{lct}(F^r) = \text{lct}(F^r g^s) > \text{lct}(f_m^r) \geq \text{lct}(f_m^r g^s)$$

Step 3. Use Thm. 3 to show that if $\text{lct}(F)$ is computed by a divisor E with image $\{0\}$, then $\text{lct}(f_m) = c$ for some c (shown by Kollár).

For Thm. 2: Step 3 in the previous proof shows that if $\mathrm{lct}_0(f_m)$ is a (strictly) decreasing sequence with limit c, then $\mathrm{lct}(F)$ can not be computed by a divisor with image $\{0\}$.

Let E be a divisor computing $\mathrm{lct}(F)$. Localize at the generic point of the image of E, and complete, to get to a "variety" of dimension $\leq n-1$. It is then standard to get $f \in k[x_1, \ldots, x_n]$ with $\mathrm{lct}_0(f) = c$.

Why care about ACC: related to termination of flips (Shokurov, Birkar)

Let $(X_1, \Delta_1) \overset{\varphi_1}{\dashrightarrow} (X_2, \Delta_2) \overset{\varphi_2}{\dashrightarrow} \cdots (X_m, \Delta_m) \overset{\varphi_m}{\dashrightarrow}$

sequence of flips

φ_i rational map, isomorphism in codim one

$$\Delta_{i+1} = (\varphi_i)_*(\Delta_i),\ \Delta_i = \mathbf{Q}\text{-divisor}$$

Idea: φ_i replaces some $(K_{X_i} + \Delta_i)$-negative curves by $(K_{X_{i+1}} + \Delta_{i+1})$-positive curves

Consequence: if $\Gamma \geq 0$, $\Gamma \sim_{\mathbf{Q}} (K_{X_i} + \Delta_i)$, then

$$\mathrm{lct}((X_i, \Delta_i), \Gamma) \leq \mathrm{lct}((X_{i+1}, \Delta_{i+1}), (\varphi_i)_*(\Gamma)))$$

Furthermore: strict inequality if lct's computed by divisors with center inside "flipping locus"

Theorem (Birkar). Suppose that

- MMP holds in dim $\leq (n-1)$.

- ACC holds (in a general form) in dim. n

Then there is no infinite sequence of flips as above **if** there is $\Gamma \geq 0$ with $\Gamma \sim_{\mathbf{Q}} (K_{X_1} + \Delta_1)$

Note: this is the case when one expects a minimal model at the end of MMP

Idea of proof:

Let Γ_i the direct image of Γ on X_i. Consider the weakly increasing sequence consisting of

$$c_i = \mathrm{lct}((X_i, \Delta_i), \Gamma_i)$$

Each φ_i is also a flip w.r.t. $(K_{X_i} + \Delta_i + c_i\Gamma_i)$

Key point: if $\Delta_i + c_i\Gamma_i$ contains F with coeff 1: can restrict to F and use adjunction More generally, have Shokurov's Special Termination: given MMP dim $\leq (n-1)$

for $i \gg 0$, if E computes $\mathrm{lct}((X_i, \Delta_i), \Gamma_i)$, then its image does not intersect "flipping locus"

Birkar uses this to produce an increasing sequence of lct's