

# **ACC for log canonical thresholds**

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joint work with

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## Basics about log canonical thresholds

Work over  $k$  alg closed,  $\text{char}(k) = 0$ .

Let  $f \in k[x_1, \dots, x_n]$ ,  $f(0) = 0$  defining  $H \subset \mathbf{A}^n$

$$\begin{aligned} \text{Recall: } \text{mult}_0(f) &= \max\{r \geq 0 \mid f \in (x_1, \dots, x_n)^r\} \\ &= \text{ord}_E(f) \end{aligned}$$

where  $E$  is exceptional divisor on  $\text{Bl}_0(\mathbf{A}^n)$ .

Idea: • consider **all** divisors over  $\mathbf{A}^n$

- normalize order of vanishing along divisors
- take an infimum over all such choices

Consider: proper, birational morphisms  $\pi: Y \rightarrow X$ , with  $Y$  smooth.  $E$  prime divisor on  $Y$  giving a valuation  $\text{ord}_E$  of the function field of  $X$

$K_{Y/X} \geq 0$  div on  $Y$  locally def by  $\det(\text{Jac}(\pi))$

Note:  $\text{Supp}(K_{Y/X}) = \text{Exc}(\pi)$

**Definition.**  $\text{lct}(f) := \inf_{E/X} \frac{\text{ord}_E(K_{Y/X})+1}{\text{ord}_E(f)}$

If only  $E$  with  $0 \in \pi(E)$ , get  $\text{lct}_0(f)$ .

Principle: “bad singularities  $\Leftrightarrow$  small  $\text{lct}$ ”

**Fundamental fact:** enough to consider  $E$  on a log resolution of  $f$ , i.e. when  $\pi^{-1}(H) + K_{Y/X}$  SNC divisor: in local coord def by

$$y_1^{a_1} \cdots y_n^{a_n}$$

Consequences: •  $\inf$  in the definition is a  $\min$

$$\bullet \text{lct}(f) \in \mathbb{Q}$$

**Examples:** 1)  $H$  smooth  $\Rightarrow \text{lct}(f) = 1$

$$2) f = x_1^{a_1} \cdots x_n^{a_n} \Rightarrow \text{lct}(f) = \min_i \frac{1}{a_i}$$

$$3) f = x_1^{a_1} + \cdots + x_n^{a_n} \Rightarrow \text{lct}(f) = \min \left\{ 1, \sum_{i=1}^n \frac{1}{a_i} \right\}$$

4) If  $f$  defines a hyperplane arrangement in  $\mathbf{A}^n$ , then

$$\text{lct}(f) = \min_{W \in L'(\mathcal{A})} \frac{\text{codim}(W)}{\#\{H \in \mathcal{A} \mid H \supseteq W\}}$$

5) If  $f$  is nondegenerate with respect to its Newton polytope  $P$ , then  $\text{lct}(f) = \min\{1/\lambda, 1\}$ , where

$$(\lambda, \dots, \lambda) \in \partial(P + \mathbf{R}_+^n)$$

In general:  $\text{lct}_0(f)$  is a refined version of  $\frac{1}{\text{mult}_0(f)}$

$$\frac{1}{\text{mult}_0(f)} \leq \text{lct}_0(f) \leq \frac{n}{\text{mult}_0(f)}$$

Another useful property: for all  $f$  and  $g$

$$\text{lct}_0(f + g) \leq \text{lct}_0(f) + \text{lct}_0(g)$$

**Variants of the definition:** replace  $f$  by ideal, allow  $X$  with mild singularities, “mixed” case: for  $g \in k[x_1, \dots, x_n]$  with  $\text{lct}(g) \leq 1$ , let

$$\text{lct}((\mathbf{A}^n, g), f) = \inf_E \frac{\text{ord}_E(K_{Y/X}) + 1 - \text{ord}_E(g)}{\text{ord}_E(f)}$$

Important for us: case of formal power series

$$f \in k[[x_1, \dots, x_n]]$$

For  $f$  formal power series, can put

$$\text{lct}(f) = \lim_{d \rightarrow \infty} \text{lct}_0(t_d(f)),$$

with  $t_d(f)$  the truncation of  $f$  up to degree  $d$ .

The above is convergent since

$$\begin{aligned} & |\text{lct}_0(t_{d+m}(f)) - \text{lct}_0(t_d(f))| \\ & \leq |\text{lct}_0(t_{d+m}(f) - t_d(f))| \leq \frac{n}{d+1} \end{aligned}$$

Important: this can also be computed using a log resolution of  $f$  (Hironaka, Temkin)

**History of log canonical thresholds:** Varchenko, Shokurov,...

Why care about lct's: they show up in various contexts

1) **Birational geometry**:  $\text{lct}(f)$  is the largest  $q > 0$  s.t.  $(\mathbf{A}^n, qH)$  log canonical

2) **Complex singularity exponents**: over  $\mathbf{C}$

$$\text{lct}(f) = \sup \left\{ s > 0 \mid \frac{1}{|f(z)|^{2s}} \text{loc integrable} \right\}$$

3)  **$p$ -adic integration** (Igusa): for  $p$  prime,  $f$  over  $\mathbf{Z}$ , it controls asymptotic behavior of

$$\#\{u \in (\mathbf{Z}/p^m\mathbf{Z})^n \mid f(u) = 0\}$$

Other contexts: motivic integration, eigenvalues of monodromy action, Bernstein poly, invar of sing in char  $p$  defined via Frobenius

**Theorem 1** (dF-E-M). For all  $n$ , the set

$$\mathcal{T}_n := \{\text{lct}_0(f) \mid f \in k[x_1, \dots, x_n], f(0) = 0\}$$

satisfies the Ascending Chain Condition (ACC).

**Remarks:** • This treats ambient smooth var.

- Conjectured more generally by Shokurov (for ambient log canonical varieties and “mixed” log canonical thresholds)
- Was known for  $n = 2$  (Shokurov, Phong-Sturm, Favre-Jonsson),  $n = 3$  (Alexeev)
- Could be extended to: varieties with quotient, or lci singularities
- General case in  $\dim \geq 4$  ?

**Theorem 2** (dF-M; Kollár). The set of accumulation points of  $\mathcal{I}_n$  is  $\mathcal{I}_{n-1} \setminus \{1\}$ .

This was conjectured by Kollár. The case of decreasing sequences treated by dF-M; Kollár then dealt with arbitrary sequences (now increasing sequences excluded by Thm. 1)

One inclusion is easy: if  $g \in k[x_1, \dots, x_{n-1}]$  has  $\text{lct}(g) < 1$ , then

$$f_m(x_1, \dots, x_n) = g(x_1, \dots, x_{n-1}) + x_n^m$$

has  $\text{lct}(f_m) = \min \left\{ 1, \text{lct}(g) + \frac{1}{m} \right\} \searrow \text{lct}(g)$

**An interpretation of Theorem 1.** Suppose  $k$  uncountable. Then Theorem 1 is equivalent with the following statement:

For every  $c > 0$ , the set

$$\{f \in k[[x_1, \dots, x_n]] \mid \text{lct}(f) \geq c\}$$

is a *cylinder* in  $k[[x_1, \dots, x_n]]$ , i.e. there is  $N$  such that  $\text{lct}(f) \geq c$  iff  $\text{lct}(t_N(f)) \geq c$ .

Furthermore: the above set is a cylinder iff it is open in the projective limit topology. Hence Thm.1 can be interpreted as a semicontinuity theorem for the *infinite-dimensional* family of all formal power series.

Note: for finite-dimensional families, such a semicontinuity result was known: Varchenko, Siu, Demailly-Kollár,...

## Key ideas in the proof of ACC

Suppose  $f_m \in k[x_1, \dots, x_n]$  are such that  $c_m := \text{lct}(f_m) \nearrow c$

1) Special case that can be treated geometrically:  $f_m$  converges to some  $f$  in  $(x_1, \dots, x_n)$ -adic topology, i.e.  $\text{mult}_0(f_m - f) \rightarrow \infty$ .

$$|\text{lct}_0(f_m) - \text{lct}_0(f)| \leq \text{lct}_0(f_m - f) \leq \frac{n}{\text{mult}_0(f_m - f)} \rightarrow 0$$

ACC predicts:  $\text{lct}(f_m) \geq \text{lct}(f)$  for  $m \gg 0$

Suppose  $E$  computes  $\text{lct}(f)$  has image  $\{0\}$  on  $\mathbf{A}^n$ . If  $\text{ord}_E(f_m - f) \geq \text{ord}_E(f)$ , then

$$\begin{aligned} \text{lct}_0(f_m) &\leq \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(f_m)} \leq \frac{\text{ord}_E(K_{Y/X}) + 1}{\text{ord}_E(f)} \\ &= \text{lct}_0(f) \end{aligned}$$

**Theorem 3** (Kollár; dF-E-M) Let  $f, g \in k[x_1, \dots, x_n]$ . If  $E$  is a divisor computing  $\text{lct}(f)$  such that  $\text{ord}_E(f - g) > \text{ord}_E(f)$ , then  $\text{lct}(g) = \text{lct}(f)$  around the image of  $E$ .

Kollár's proof: uses the results on MMP of Birkar-Cascini-Hacon-M<sup>c</sup>Kernan

dF-E-M: uses the Connectedness Theorem of Shokurov and Kollár (easy consequence of Kawamata-Viehweg vanishing)

2) Second point in the proof of ACC: given a sequence  $\{f_m\}_m$ ,  $f_m \in k[x_1, \dots, x_n]$  with  $c_m := \text{lct}_0(f_m) \rightarrow c$ , construct  $F \in K[[x_1, \dots, x_n]]$ ,  $K \supset k$  field extension, such that  $\text{lct}(F) = c$ .

In fact, we will have the following property: for every  $d \geq 1$ , have infinitely many  $m$  such that

$$\text{lct}_0(t_d(F)) = \text{lct}_0(t_d(f_m))$$

For every such  $m$ , have

$$\begin{aligned} |\text{lct}(F) - \text{lct}_0(f_m)| &\leq |\text{lct}(F) - \text{lct}_0(t_d(F))| + \\ &|\text{lct}_0(t_d(f_m)) - \text{lct}_0(f_m)| \leq \frac{2n}{d+1} \end{aligned}$$

Since  $\text{lct}_0(f_m) \rightarrow c$ , it follows  $\text{lct}(F) = c$ .

Two ways of constructing such  $F \in K[[x_1, \dots, x_n]]$ :

- dF-M: using ultrafilter constructions
- Kollár: using a sequence of generic points

The nonstandard construction: if  $\mathcal{U}$  is a non-principal ultrafilter on  $\mathbb{N}$ , then the sequence  $(f_m)$  defines an *internal polynomial* in  ${}^*(k[x_1, \dots, x_n])$ . Truncating to keep just the monomials with standard exponents gives  $F \in {}^*k[[x_1, \dots, x_n]]$ .

One can show: for all  $d$

$$\text{lct}(t_d(F)) = \text{lct}_0(t_d(f_m))$$

whenever  $m \in \mathcal{U}$ .

Kollár's construction: consider the truncation maps

$$P = k[[x_1, \dots, x_n]] \xrightarrow{t_d} P_d = k[[x_1, \dots, x_n]] / (x_1, \dots, x_n)^{d+1}$$

$$\xrightarrow{\varphi_d} P_{d-1} = k[[x_1, \dots, x_n]] / (x_1, \dots, x_n)^d$$

Each  $P_d$  is an affine space over  $k$ .

Construct by induction on  $d \geq 1$  a sequence of *irreducible, closed* subsets  $Z_d \subseteq P_d$  such that

- i) Each  $Z_d$  is minimal with the property that there are infinitely many  $m$  such that  $t_d(f_m) \in Z_d$ .
- ii) Each  $\varphi_d$  induces a *dominant* map  $Z_d \rightarrow Z_{d-1}$ .

Get a sequence of field extensions  $k(Z_d) \subseteq k(Z_{d+1}) \subseteq \dots$ . Let  $K = \bigcup_d k(Z_d)$ .

The sequence of compatible maps

$$\mathrm{Spec}(K) \rightarrow P_d$$

defines a formal power series  $F \in K[[x_1, \dots, x_n]]$ .

By construction, for every  $d \geq 1$ , we have an infinite subset  $I_d \subseteq \mathbb{N}$  such that

$t_d(F)$  corresponds to the generic point of

$$\{t_d(f_m) \mid m \in I_d\}$$

But there is  $U_d \subset Z_d$  open such that  $\mathrm{lct}_0(t_d(F)) = \mathrm{lct}_0(g)$  for every  $g \in U_d$ . The set

$$\{m \in I_d \mid t_d(f_m) \in U_d\}$$

is infinite, hence  $F$  satisfies our requirement.

## Rough outline of the proof of Thm. 1:

Step 1. Given  $f_m \in k[x_1, \dots, x_n]$  with  $\text{lct}_0(f_m) \nearrow c$ , construct  $F \in K[[x_1, \dots, x_n]]$  as above.

Step 2. Reduce to the case when  $\text{lct}(F)$  is computed by a divisor  $E$  with image  $\{0\}$ . This is done by replacing  $f_m$  by  $f_m^r g^s$ , for suitable  $r, s \geq 1$ , and a general polynomial  $g$  of suitable degree. Can do this such that

$$\text{lct}(F^r) = \text{lct}(F^r g^s) > \text{lct}(f_m^r) \geq \text{lct}(f_m^r g^s)$$

Step 3. Use Thm. 3 to show that if  $\text{lct}(F)$  is computed by a divisor  $E$  with image  $\{0\}$ , then  $\text{lct}(f_m) = c$  for some  $c$  (shown by Kollár).

**For Thm. 2:** Step 3 in the previous proof shows that if  $\text{lct}_0(f_m)$  is a (strictly) decreasing sequence with limit  $c$ , then  $\text{lct}(F)$  can not be computed by a divisor with image  $\{0\}$ .

Let  $E$  be a divisor computing  $\text{lct}(F)$ . Localize at the generic point of the image of  $E$ , and complete, to get to a “variety” of dimension  $\leq n-1$ . It is then standard to get  $f \in k[x_1, \dots, x_n]$  with  $\text{lct}_0(f) = c$ .

Why care about ACC: related to termination of flips (Shokurov, Birkar)

Let  $(X_1, \Delta_1) \xrightarrow{\varphi_1} (X_2, \Delta_2) \xrightarrow{\varphi_2} \cdots (X_m, \Delta_m) \xrightarrow{\varphi_m}$

sequence of flips

$\varphi_i$  rational map, isomorphism in codim one

$$\Delta_{i+1} = (\varphi_i)_*(\Delta_i), \Delta_i = \mathbb{Q}\text{-divisor}$$

Idea:  $\varphi_i$  replaces some  $(K_{X_i} + \Delta_i)$ -negative curves by  $(K_{X_{i+1}} + \Delta_{i+1})$ -positive curves

Consequence: if  $\Gamma \geq 0$ ,  $\Gamma \sim_{\mathbb{Q}} (K_{X_i} + \Delta_i)$ , then

$$\text{lct}((X_i, \Delta_i), \Gamma) \leq \text{lct}((X_{i+1}, \Delta_{i+1}), (\varphi_i)_*(\Gamma))$$

Furthermore: strict inequality if lct's computed by divisors with center inside "flipping locus"

**Theorem** (Birkar). Suppose that

- MMP holds in  $\dim \leq (n - 1)$ .
- ACC holds (in a general form) in  $\dim. n$

Then there is no infinite sequence of flips as above **if** there is  $\Gamma \geq 0$  with  $\Gamma \sim_{\mathbb{Q}} (K_{X_1} + \Delta_1)$

Note: this is the case when one expects a minimal model at the end of MMP

## Idea of proof:

Let  $\Gamma_i$  the direct image of  $\Gamma$  on  $X_i$ . Consider the weakly increasing sequence consisting of

$$c_i = \text{lct}((X_i, \Delta_i), \Gamma_i)$$

Each  $\varphi_i$  is also a flip w.r.t.  $(K_{X_i} + \Delta_i + c_i\Gamma_i)$

Key point: if  $\Delta_i + c_i\Gamma_i$  contains  $F$  with coeff 1: can restrict to  $F$  and use adjunction  
More generally, have Shokurov's Special Termination: given MMP  $\dim \leq (n - 1)$

for  $i \gg 0$ , if  $E$  computes  $\text{lct}((X_i, \Delta_i), \Gamma_i)$ , then its image does not intersect "flipping locus"

Birkar uses this to produce an increasing sequence of lct's