Instituto de Ciências Matemáticas e de Computação

# The Euler Obstruction and The Chern Obstruction


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## 0.1. Introduction

(a) The Euler obstruction;

(b) The Euler obstruction of a function;

(c) The Euler obstruction of a 1-form;

(d) The Euler obstruction of a $k$-vector field;

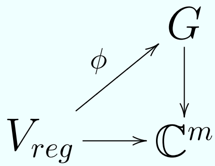
(e) The Euler obstruction of a map;

(f) The Chern obstruction.

### 0.1.1. Nash Modification

The Grassmannian of $d$-planes of $\mathbb{C}^m$ is denoted by $G(d,m)$. Let $(V,0) \subset (\mathbb{C}^m,0)$ be the germ of a complex analytic variety, equidimensional of complex dimension $d$. Let us consider the fiber bundle of Grassmannians of $d$-planes in $T\mathbb{C}^m$, denoted by $G$. The fiber $G_x$ on $x \in \mathbb{C}^m$ is the set of $d$-planes of $T_x\mathbb{C}^m$, isomorphic to $G(d,m)$. An element of $G$ is a pair $(x,P)$ where $x \in \mathbb{C}^m$ and $P \in G_x$. On the regular part of $V$, one can define the Gauss map $\phi : V_{reg} \to G$ as follows:

$$\phi(x) = (x, T_x V_{reg}).$$



**Definition 0.1.1.** The Nash modification of $V$ denoted by $\widetilde{V}$ is defined as the closure of the image of $\phi$ inside $G$.

Let $T$ be the tautological fiber bundle on $G$. We define the fiber bundle $\widetilde{T}$ with base $\widetilde{V}$ as the restriction of $T$ on $\widetilde{V}$, so we have the diagram:

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & T \\ \downarrow & & \downarrow \\ \widetilde{V} & \hookrightarrow & G \\ \nu\downarrow & & \downarrow\nu \\ V & \hookrightarrow & \mathbb{C}^m \end{array}$$

An element of $T$ is a triple $(x, P, v)$ where $x \in \mathbb{C}^m$, $P \in G_x$ and $v \in P$.

**Definition 0.1.2.** Let $v$ be a radial vector field on $V \cap \partial B_\varepsilon$ and $\tilde{v}$ the lifting up of $v$ on $\nu^{-1}(V \cap \partial B_\varepsilon)$. The vector field $\tilde{v}$ defines an obstruction cocycle $Obs(\tilde{v})$.
The Euler obstruction is evaluation of $Obs(\tilde{v})$ on the fundamental class of the pair $[\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap \partial B_\varepsilon)]$, it means:

$$Eu_V(0) := \langle Obs(\tilde{v}), [\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap \partial B_\varepsilon)] \rangle.$$

**Theorem 0.1.3** (BLS). *Let $(V,0) \subset (\mathbb{C}^m,0)$ be the germ of a complex analytic variety, and $\{V_\alpha\}$ a Whitney stratification of $V$. Let $l : U \to \mathbb{C}$ be a generic linear form, where $U$ is a open neighborhood of $0$ in $\mathbb{C}$. So:*

$$Eu_V(0) = \sum_\alpha \chi(V_\alpha \cap B_\varepsilon \cap l^{-1}(t_0)) \cdot Eu_V(V_\alpha)$$

*where $\varepsilon$ is sufficiently small, $t_0 \in \mathbb{C}\backslash\{0\}$ is near to the origin and $Eu_V(V_\alpha)$ is the obstruction of $V$ in $V_\alpha$.*

**Theorem 0.1.4** (BMPS). *Let $f : (V, 0) \to (\mathbb{C}, 0)$, be an analytic function with isolated singularity at the origin, and $\{V_\alpha\}$ a Whitney stratification of $V$. so:*

$$Eu_V(0) = \left( \sum_\alpha \chi(V_\alpha \cap B_\varepsilon \cap f^{-1}(t_0)) \cdot Eu_V(V_\alpha) \right) + Eu_{f,V}(0)$$

*where $\varepsilon$ is sufficiently small, $t_0 \in \mathbb{C} \backslash \{0\}$ near to the origin.*

### 0.1.2. The Euler obstruction of a $p$-frame

One says that a collection $v^{(p)} = \{v_1, \ldots, v_p\}$ of $p$ vector fields is a $p$-frame if the vector fields are $\mathbb{C}$-linearly independent. The point $x$ is a singular point for $v^{(p)}$ if the collection $v^{(p)}(x)$ is not linearly independent.

Let us denote by $\{V_\alpha\}$ a Whitney stratification of $\mathbb{C}^m$ compatible with $V$, *i.e.* $\mathbb{C}^m \setminus V$ is a stratum. Let $(K)$ be a triangulation of $\mathbb{C}^m$ subordinated to the stratification $\{V_\alpha\}$, and $(D)$ a cell decomposition of $\mathbb{C}^m$ dual to $(K)$. Let us denote by $\sigma$ a $(D)$-cell of (real) dimension $2(m-p+1)$. The cell $\sigma$ is transverse to all strata of $\{V_\alpha\}$. One says that the $p$-frame $v^{(p)} = \{v_1, \dots, v_p\}$ is stratified if each vector field $v_i$ is a stratified vector field.

Let us denote by

$$Obs(\tilde{v}^{(p)}, \sigma \cap V) \in H^{2(d-p+1)}(\nu^{-1}(\sigma \cap V), (\nu^{-1}(\partial\sigma \cap V)))$$

the class of the obstruction cocycle to extending $\tilde{v}^{(p)}$ as a set of $p$ linearly independent sections of $\widetilde{T}$ on $\nu^{-1}(\sigma \cap V)$.

**Definition 0.1.5.** The local Euler obstruction $Eu(v^{(p)}, V, \sigma)$ of a stratified $p$-frame $v^{(p)}$ defined on $\sigma \cap V$ with an isolated singularity at the barycenter $a$ of $\sigma$ is defined as the evaluation of the obstruction cocycle $Obs(\tilde{v}^{(p)}, \sigma \cap V)$ on the fundamental class of the pair $[\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]$. That is,

$$Eu(v^{(p)}, V, \sigma) = \langle Obs(\tilde{v}^{(p)}, \sigma \cap V), [\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]\rangle.$$

**Definition 0.1.6.** Let $\{\omega_j\}$ be a collection of $p$ 1-forms. The local Euler obstruction $Eu(\{\omega_j\}, V, \sigma)$ of the collection is defined in a similar way, but in this case we will take a section of the dual nash bundle $\widetilde{T}^*$.

### 0.1.3. Euler obstruction of a map

Let us fix an integer $p$, $1 \leq p \leq d$. Let us consider a germ of analytic map $f : (V, 0) \to (\mathbb{C}^p, 0)$, restriction of $F : (U, 0) \to (\mathbb{C}^p, 0)$, $f(z) = (f_1(x), f_2(x), ..., f_p(x))$ where $U$ is a neighborhood of 0 in of $\mathbb{C}^m$ and $F(x) = (F_1(x), F_2(x), ..., F_p(x))$.
We denote by $B_\varepsilon$ a closed ball centered at 0 with radius $\varepsilon$ and by $\Sigma f$ the singular set of $f$.

**Definition 0.1.7.** Let $(V,0) \subset (\mathbb{C}^m,0)$ the germ of an analytic variety and $f:(V,0) \to (\mathbb{C}^p,0)$ an analytic germ with singular set $\Sigma f$. One say that $f$ satisfies the $(\delta)$ condition if there exists one cell $\sigma$ of barycenter 0, of real dimension $2(m-p+1)$ of a cellular decomposition (D) of $\mathbb{C}^m$, such that :

$$\Sigma f \cap \partial\sigma = \emptyset. \tag{$\delta$}$$

If $f$ satisfies the $(\delta)$ condition for the cell $\sigma$, we can lift up the $p$-frame $\overline{\nabla}_V^{(p)} f$ as a set of $p$ linearly independent sections $\widetilde{\overline{\nabla}}_V^{(p)} f$ of $\widetilde{T}$ on $\nu^{-1}(V^\sigma)$ where $V^\sigma = V \cap \partial\sigma$. Let us denote by $\xi \in H^{2(d-p+1)}(\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma))$ the obstruction cocycle to extend $\widetilde{\overline{\nabla}}_V^{(p)} f$ as a set of $p$ linearly independent sections of $\widetilde{T}$ on $\nu^{-1}(V^\sigma)$.

**Definition 0.1.8.** In the above situation one can define the Euler obstruction of $f$ relatively to $\sigma$, denoted by $Eu_{f,V}(\sigma)$, as the evaluation of the cocycle $\xi$ on the fundamental class of the pair $[\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma)]$. That means

$$Eu_{f,V}(\sigma) = \langle \xi, [\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma)] \rangle.$$

### 0.1.4. Local Chern obstruction of collections of 1-forms and special points.

The notion of local Chern obstruction extends the notion of local Euler obstruction in the case of collections of germs of 1-forms, This number is well defined for any germ of a reduced equidimensional complex analytic space. The Chern obstruction can be characterized as a intersection number. More precisely, W. Ebeling and S. M. Gusein-Zade perfom the following construction.

Let $(V^d, 0) \subset (\mathbb{C}^m, 0)$ be the germ of a purely $d$-dimensional reduced complex analytic variety at the origin. Let $\mathbf{k} = \{k_i\}, (i = 1, \cdots, s; j = 1, \cdots, d - k_i + 1)$, $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(\mathbb{C}^m, 0)$. Let $\varepsilon > 0$ be small enough so that there is a representative $V$ of the germ $(V, 0)$ and representatives $\{\omega_j^{(i)}\}$ of the germs of 1-forms inside the ball $B_\varepsilon(0) \subset \mathbb{C}^m$.

**Definition 0.1.9.** A point $x \in V$ is called a special point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety $V$ if there exists a sequence $x_n$ of points from the non-singular part $V_{reg}$ of the variety $V$ such that the sequence $T_{x_n}V_{reg}$ of the tangent spaces at the points $x_n$ has a limit $L$ (in $G(d,m)$) and the restriction of the 1-forms $\omega_1^{(i)}, \cdots, \omega_{d-k_i+1}^{(i)}$ to the subspace $L \subset T_x\mathbb{C}^m$ are linearly dependent for each $i = 1, \cdots, s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an isolated special point on $(V, 0)$ if it has no special point on $V$ in a punctured neighborhood of the origin.

Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(V, 0)$ with an isolated special point at the origin. Let $\nu : \widetilde{V} \to V$ be the Nash transformation of the variety $V$ and $\widetilde{T}$ the Nash bundle. The collection of 1-forms $\{\omega_j^{(i)}\}$ gives rise to a section $\Gamma(\omega)$ of the bundle

$$\widetilde{\mathbb{T}} = \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{d-k_i+1} \widetilde{T}^*_{i,j}$$

where $\widetilde{T}^*_{i,j}$ are copies of the dual Nash bundle $\widetilde{T}^*$ over the Nash transform $\widetilde{V}$ numbered by indices $i$ and $j$.

Let $\mathbb{D} \subset \widetilde{\mathbb{T}}$ be the set of pairs $(x, \{\alpha_j^{(i)}\})$ where $x \in \widetilde{V}$ and the collection $\{\alpha_j^{(i)}\}$ is such that $\alpha_1^{(i)}, \cdots, \alpha_{n-k_i+1}^{(i)}$ are linearly dependent for each $i = 1, \cdots ., s$.

**Definition 0.1.10.** Let 0 be a special point of the collection $\{\omega_j^{(i)}\}$. The local Chern obstruction $Ch_{V,0}\{\omega_j^{(i)}\}$ of the collection of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(V,0)$ at the origin is the obstruction to extend the section $\Gamma(\omega)$ of the fibre bundle $\widetilde{\mathbb{T}} \setminus \mathbb{D} \to \widetilde{X}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to $\widetilde{V}$. More precisely its value (as an element of the cohomology group

$$H^{2d}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon), \mathbb{Z}))$$

on the fundamental class of the pair

$$(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon)).$$

Let $V$ be a complex analytic equidimensional reduced variety in $\mathbb{C}^m$, $\dim V = d$, and $f : (V,0) \to (\mathbb{C}^p, 0)$, $1 \leq p \leq d$, a map-germ defined on $V$.
In what follows we adapt the definition of the Euler obstruction of a map (Definition 0.1.8)in the context of collections of 1-forms.

Let us denote by $df_i$ the 1-form dual to the vector field $\overline{\nabla}_V^{(i)} f$. We denote by $\widetilde{T}^*$ the dual bundle of $\widetilde{T}$. In the same way as above, if $f$ satisfies the condition $(\delta)$ for the cell $\sigma$, the 1-forms $df_i$ can be lifted as linearly independent sections $\widetilde{df_i}$ of $\widetilde{T}^*$ over $\nu^{-1}(\partial V^\sigma)$. Let $\xi^* \in H^{2(d-p+1)}(\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma))$ the obstruction cocycle for the extension of the $\widetilde{df_i}$ as a set of $k$ linearly independent sections of $\widetilde{T}^*$ over $\nu^{-1}(V^\sigma)$.

**Definition 0.1.11.** One denotes by $Eu^*_{f,V}(\sigma)$, the evaluation of the cocycle $\xi^*$ over the fundamental class of $[\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma)]$. That is,

$$Eu^*_{f,V}(\sigma) = \langle \xi^*, [\nu^{-1}(V^\sigma), \nu^{-1}(\partial V^\sigma)] \rangle.$$

**Theorem 0.1.12.** *Let* $(V^d, 0) \subset (\mathbb{C}^m, 0)$ *be the germ of a purely* $d$*-dimensional reduced complex analytic variety at the origin. Let* $\boldsymbol{k} = \{k_i\}, (i = 1, 2; j = 1, \cdots, d - k_i + 1)$, $\{\omega_j^{(i)}\}$ *a collection of germs of* 1*-forms on* $(\mathbb{C}^m, 0)$. *Let* $\sigma$ *be a* $2k_1$*-cell from a dual decomposition* $(D)$ *as above and* $\tau$ *the* $2k_2$*-simplex dual to* $\sigma$ *, so* $\tau$ *is transverse to* $\sigma$ *and* $\sigma \cap \tau = \{0\}$*. In this case we have the product formula,*

$$Ch_{V,0}\{\omega_j^{(i)}\} = Eu(\omega^{(1)}, V, \sigma) \times Ind_{PH}(\omega^{(2)}, \tau, 0).$$

**Corollary 0.1.13.** *The Euler obstruction* $Eu^*_{f,V}(0)$ *can be characterized as the intersection number* $\Gamma(\omega) \circ \mathbb{D}^p_V$.

**Proposition 0.1.14.** *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a finitely determined map germ, let us take $\omega_j^{(i)} = \{\omega^1, \omega^2\}$ where we have $\omega^1 = \{df_1, df_2\}$ and $\omega^2 = \{df_1, d\Delta\}$, where $d\Delta$ is the determinant of the jacobian matrix of $f$. In this case, where $V = \mathbb{C}^2$, we have that*

$$Ch_{V,0}\{\omega_j^{(i)}\} = c(f),$$

*where $c(f)$ is the number of cusps of $f$ and $V = \mathbb{C}^2$.*

*Proof.* Let us denote by $M$ the $3 \times 2$-matrix with columns $df_1, df_2$ and $d\Delta$. If we denote by $I$ the ideal generate by the determinants of de $2 \times 2$ minors of the matrix $M$, we know by [GM] that

$$c(f) = dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2}/I.$$

By the other hand, from [EG] and using that $V = \mathbb{C}^2$, we also have that

$$Ch_{V,0} = dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2}/I,$$

in this case, we have $Ch_{V,0} = c(f)$.

□

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