Variation on the skein relation

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- $I(\text{trivial knot}) = 1$;
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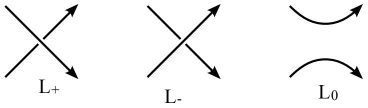

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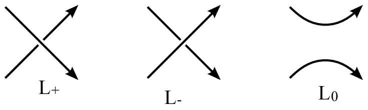

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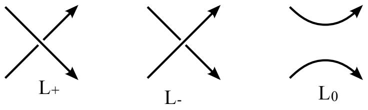

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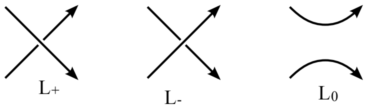

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Corollary $\widetilde{\mathrm{Skein}}(\mathcal{L})$ is of dimension 1.

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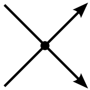
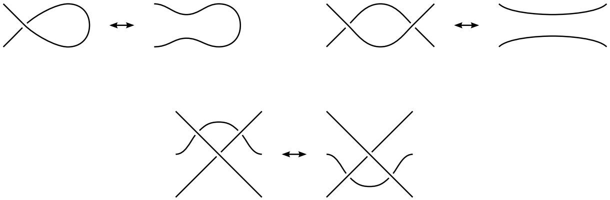
Example 2: Singular links.
Links that admit singular crossings



Reidemeister moves



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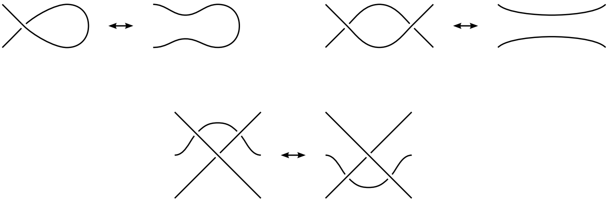
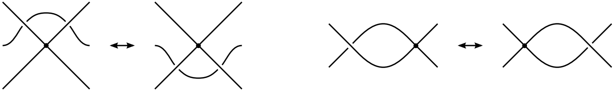


Singular Reidemeister moves (Kauffman)



Example 3: Virtual links (Kauffman)

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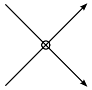

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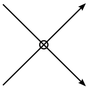
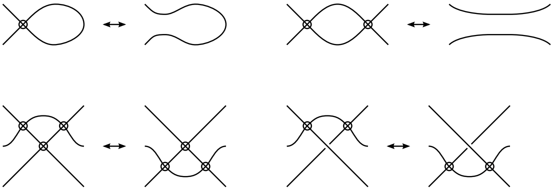
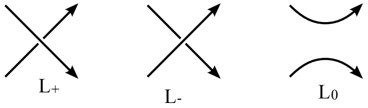


Virtual Reidemeister moves (Kauffman)



Definition Let $\mathcal{L}$ be a knot-like category.
The Skein module of $\mathcal{L}$, denoted by $\mathrm{Skein}(\mathcal{L})$, is the quotient of $\mathbb{C}(x,t)[\mathcal{L}]$ by the relations

$$t^{-1} \cdot L_+ - t \cdot L_- = x \cdot L_0$$



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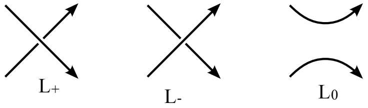

Note: The space $\widetilde{\mathrm{Skein}}(\mathcal{L})$ of invariants $I : \mathcal{L} \to \mathbb{C}(x,t)$ which satisfies the skein relation is

$$L(\mathrm{Skein}(\mathcal{L}), \mathbb{C}(x,t))$$

Skein relation:

$$t^{-1} \cdot I(L_+) - t \cdot I(L_-) = x \cdot I(L_0)$$

Theorem **(Przytycki)** Let F be a surface and let $\mathcal{L}$ the category of links in $F \times I$.
$\mathrm{Skein}(\mathcal{L})$ is an algebra.
It is isomorphic to the symmetric tensor algebra $S\mathbb{C}(x,t)[\hat{\pi}^0]$.
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Theorem (P, Rabenda) Let $\mathcal{L}$ be the category of singular links in the sphere.
$\mathrm{Skein}(\mathcal{L})$ is an algebra.
It is isomorphic to $\mathbb{C}(x, t)[X, Y]$.

Question Let $\mathcal{L}$ be the category of virtual links in the sphere. Determine $\mathrm{Skein}(\mathcal{L})$.

Goal now: To present an approach to the study of skein modules by means of generalizations of Hecke algebras.
(Ocneanu, Jones, Lambropoulou, Przytycki, P, Rabenda).

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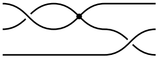
Treat the case of singular links.
BUT the tools exist for many other categories (links in 3-manifolds, virtual links, so on).

Definition A singular braid is a n-tuple $\beta = (b_1, \ldots, b_n)$, $b_i : [0,1] \to \mathbb{R}^2 \times [0,1]$, such that

- $b_i(0) = (i, 0, 0)$, $b_i(1) = (\chi(i), 0, 0)$, where $\chi \in \mathrm{Sym}_n$;
- $b_i(t) = (*, *, t)$;
- $b_1 \cup \cdots \cup b_n$ has finitely many singularities (ordinary double points).



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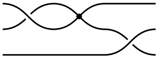

The set of singular braids form a monoid, SB_n.

Observation SB_n is generated by $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$, $\tau_1, \ldots, \tau_{n-1}$,

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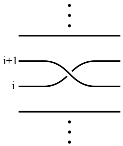

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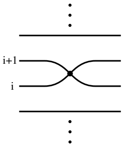
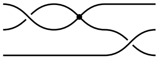


Definition From a singular braid β one can construct a closed braid, $\hat{\beta}$.

β =



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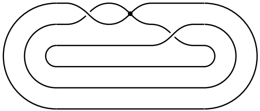

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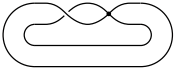

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Definition Set $\sqcup SB = \sqcup_{n=1}^{+\infty} SB_n$.
$(\alpha, n), (\beta, m) \in \sqcup SB$ are connected by a Markov move if either

- $n = m$, $\alpha = \gamma_1\gamma_2$, $\beta = \gamma_2\gamma_1$, where $\gamma_1, \gamma_2 \in SB_n$; or
- $n = m + 1$ and $\alpha = \beta\sigma_n^{\pm 1}$; or
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Theorem **(Markov, Gemein)** Let $(\alpha, n), (\beta, m) \in \sqcup SB$. We have $\hat{\alpha} = \hat{\beta}$ if and only if (α, n) and (β, m) are connected by finitely many Markov moves.

Definition The singular Hecke algebra, denoted by $\mathcal{H}(SB_n)$, is the quotient of $\mathbb{C}(q)[SB_n]$ by the relations

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Observe that the inclusion $SB_n \subset SB_{n+1}$ gives rise to a homomorphism $\iota_n : \mathcal{H}(SB_n) \to \mathcal{H}(SB_{n+1})$.

Definition The Markov module (of singular braids), denoted by $\mathrm{Markov}(\sqcup SB)$, is the quotient of

$$\oplus_{n=1}^{\infty}(\mathbb{C}(z) \otimes \mathcal{H}(SB_n))$$

by the relations

- $ab = ba$ for all $a, b \in \mathcal{H}(SB_n)$ and all $n \geq 1$;
- $a = \iota_n(a)$ for all $a \in \mathcal{H}(SB_n)$ and all $n \geq 1$;
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Theorem (Folklore, P, Rabenda) Let $\mathcal{L}$ be the category of singular links. Then $\mathrm{Skein}(\mathcal{L})$ is isomorphic to $\mathrm{Markov}(\sqcup SB)$.