The kite of a constellation

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(S, O) : germ of smooth complex surface.

Definition 0.1. — If $(\Sigma, E) \xrightarrow{\pi} (S, O)$ is a morphism obtained by composing point blowing-ups, then a point of the reduced exceptional divisor $E := \pi^{-1}(O)$ is called **an infinitely near point** of O.

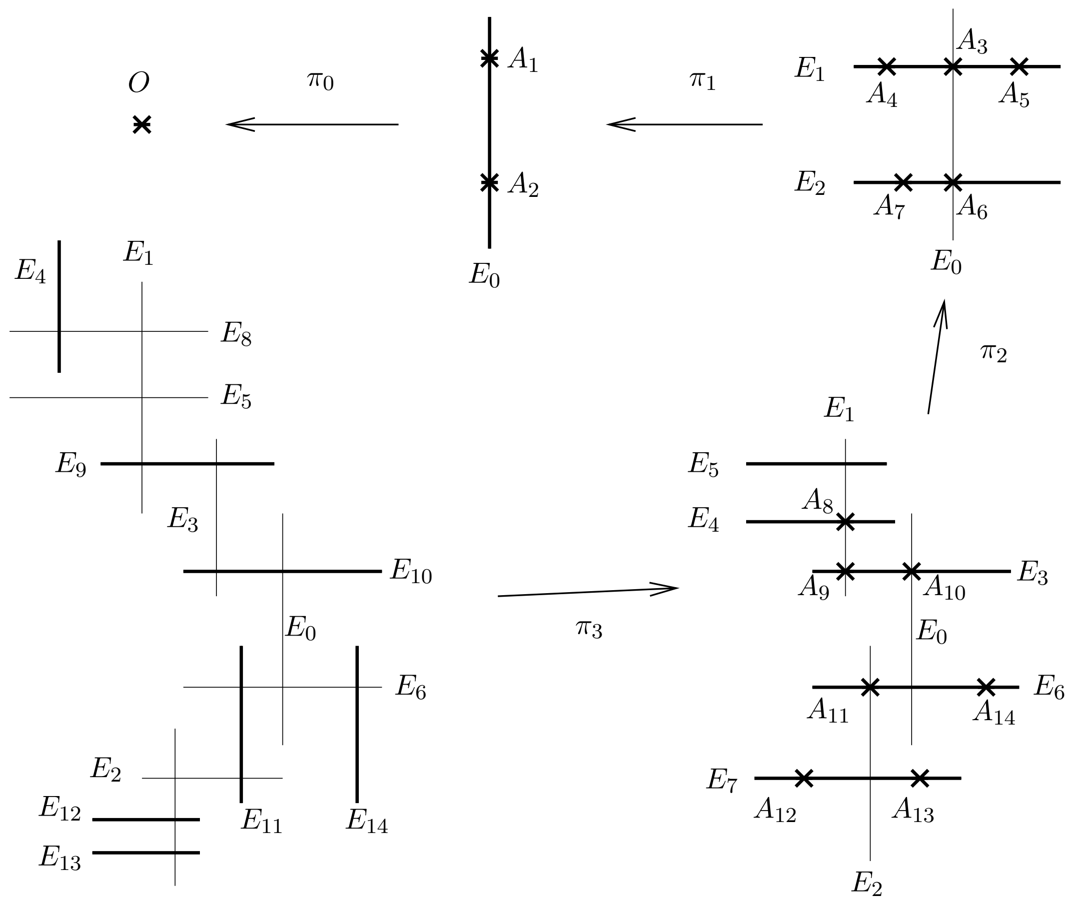
Definition 0.2. — Denote by $\mathcal{C}_O$ the set of points infinitely near O, including O itself. We call its elements **stars**, $\mathcal{C}_O$ being the **firmament** of O.

• *If A lies on a unique irreducible component E_i of E*, it is called a **free star**. Denote $p_D(A) = A_i$.
• *If A lies on two components E_i and E_j of E*, then it is called a **satellite star**. If A_i appeared *after* A_j, denote $p_D(A) := A_i$.

One gets the *predecessor map* :

$$p_D : \mathcal{C}_O \longrightarrow \mathcal{C}_O.$$

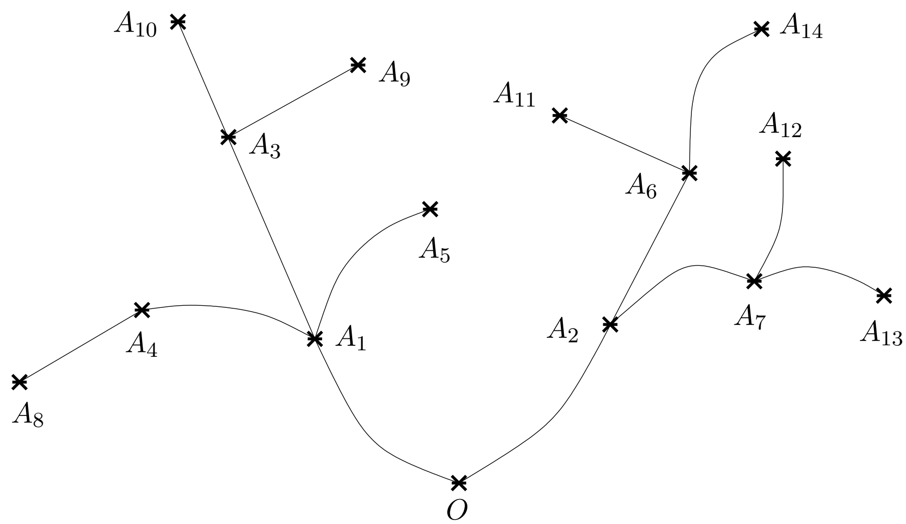
Definition 0.3. — (Campillo, González-Sprinberg, Lejeune-Jalabert, 1993) A **constellation** centered at O is a subset $\mathcal{C} \subset \mathcal{C}_O$ which is stable under p_D.



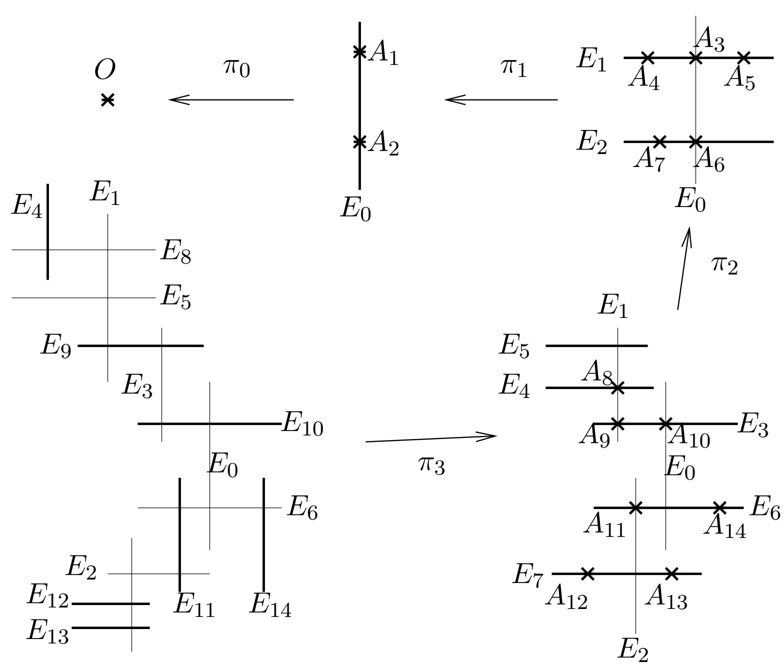
The predecessor map is given by :

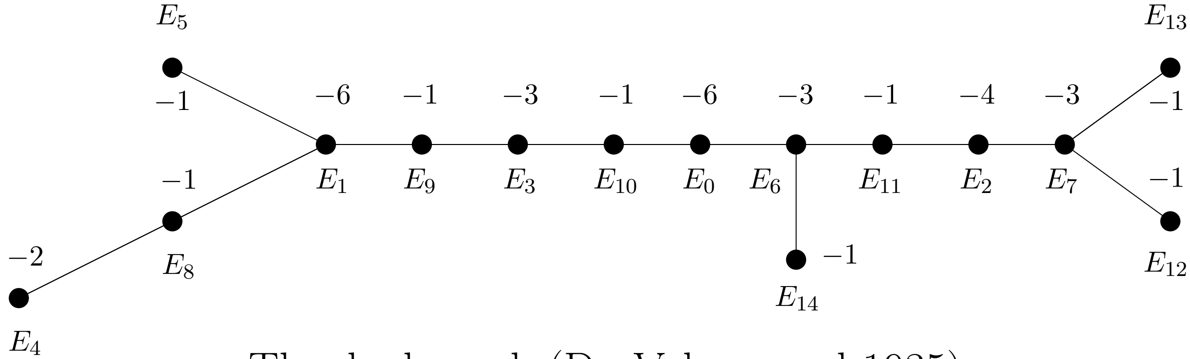
$$\begin{array}{ccccccccccccccc} O & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ O & O & O & A_1 & A_1 & A_1 & A_2 & A_2 & A_4 & A_3 & A_3 & A_6 & A_7 & A_7 & A_6 \end{array}$$

The *free stars* are $A_1, A_2, A_4, A_5, A_7, A_{12}, A_{13}, A_{14}$ and the *satellite* ones are $A_3, A_6, A_8, A_9, A_{10}, A_{11}$.

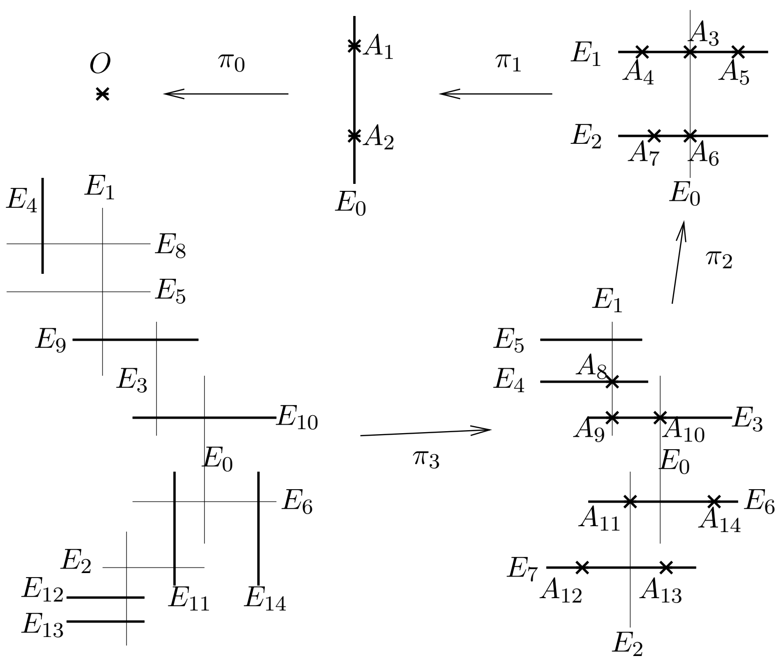


The Enriques diagram (around 1918)





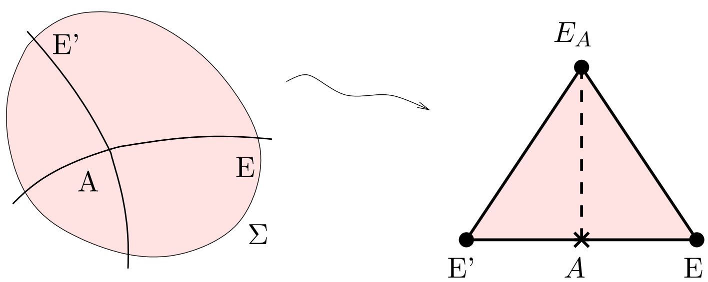
The dual graph (Du Val, around 1935)



The vertices of the two graphs correspond bijectively but the graphs are not even abstractly isomorphic. *How to understand their relation?*

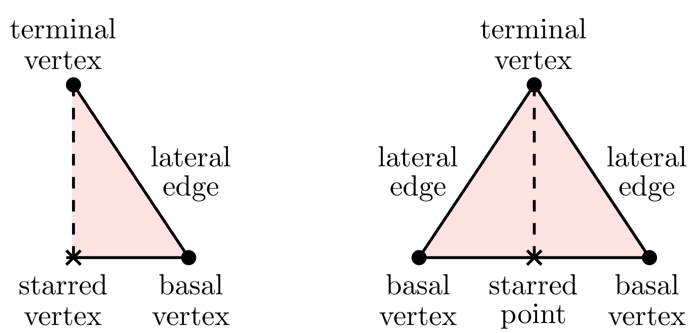
One may construct a 2-dimensional simplicial complex (the **kite**) which contains both the Enriques diagram and the dual graph of a constellation. It is obtained by gluing elementary pieces : **sails** and **ropes**.

Main idea :



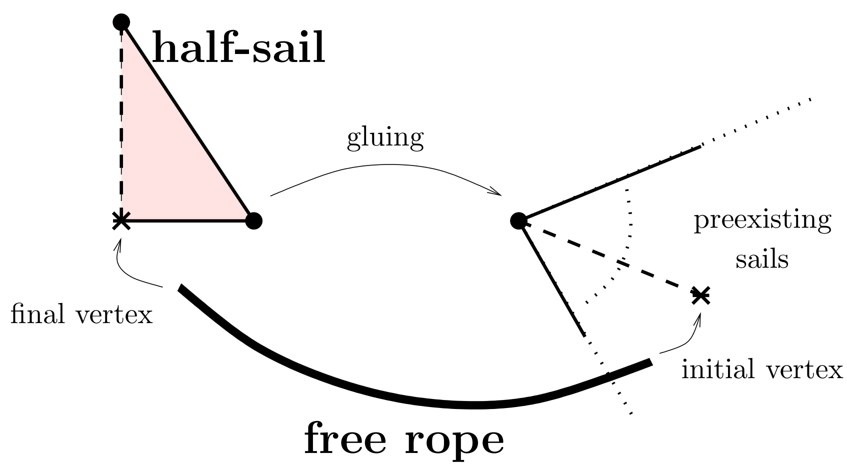
The **simple sail** associated to a normal crossings divisor

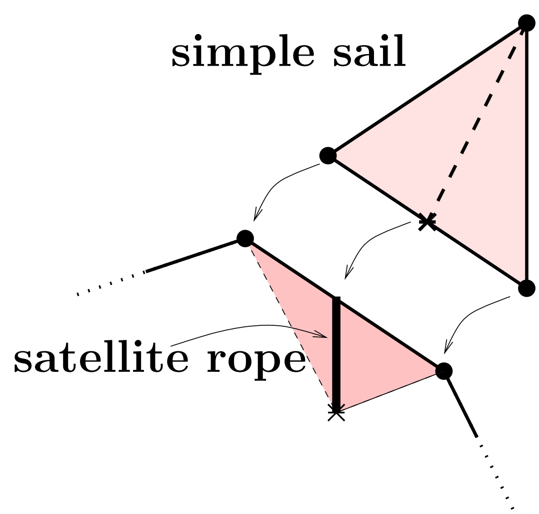
The elementary sails are :



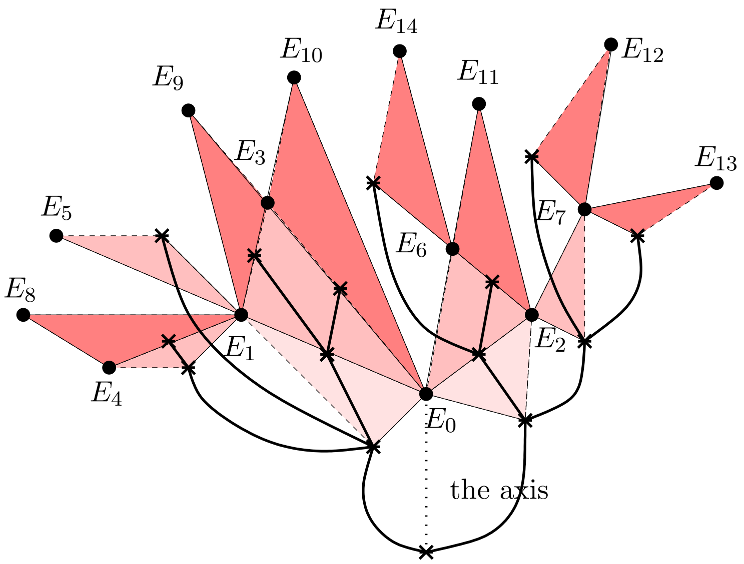
half-sail **simple sail**

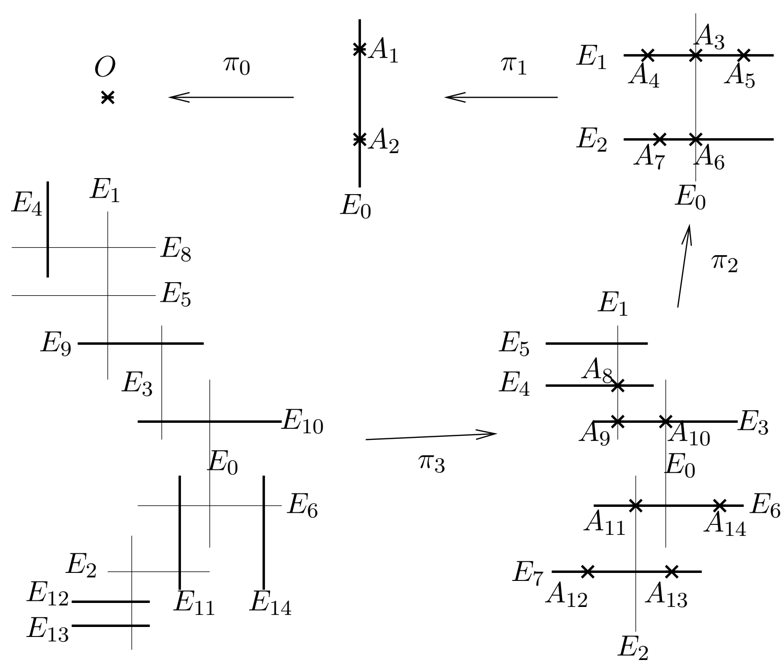
The elementary sails and the ropes may be attached in the following way :



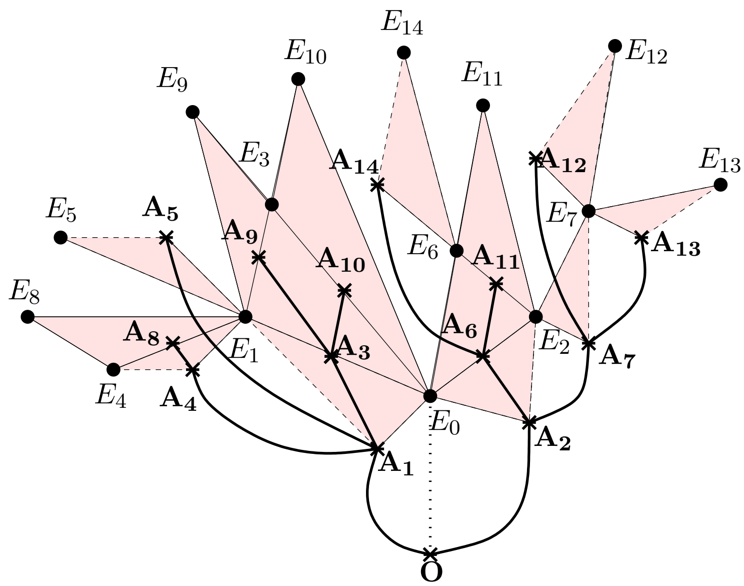


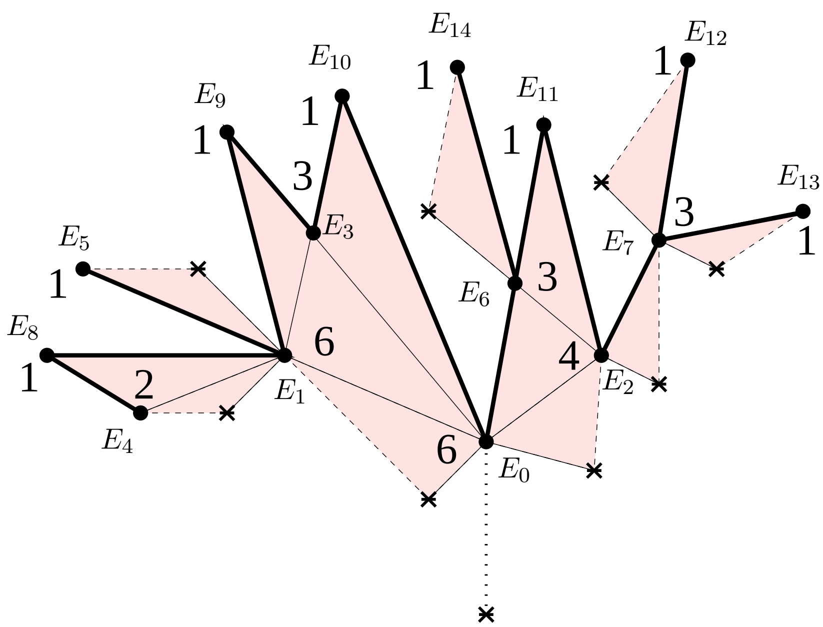
To each finite constellation is associated a kite :





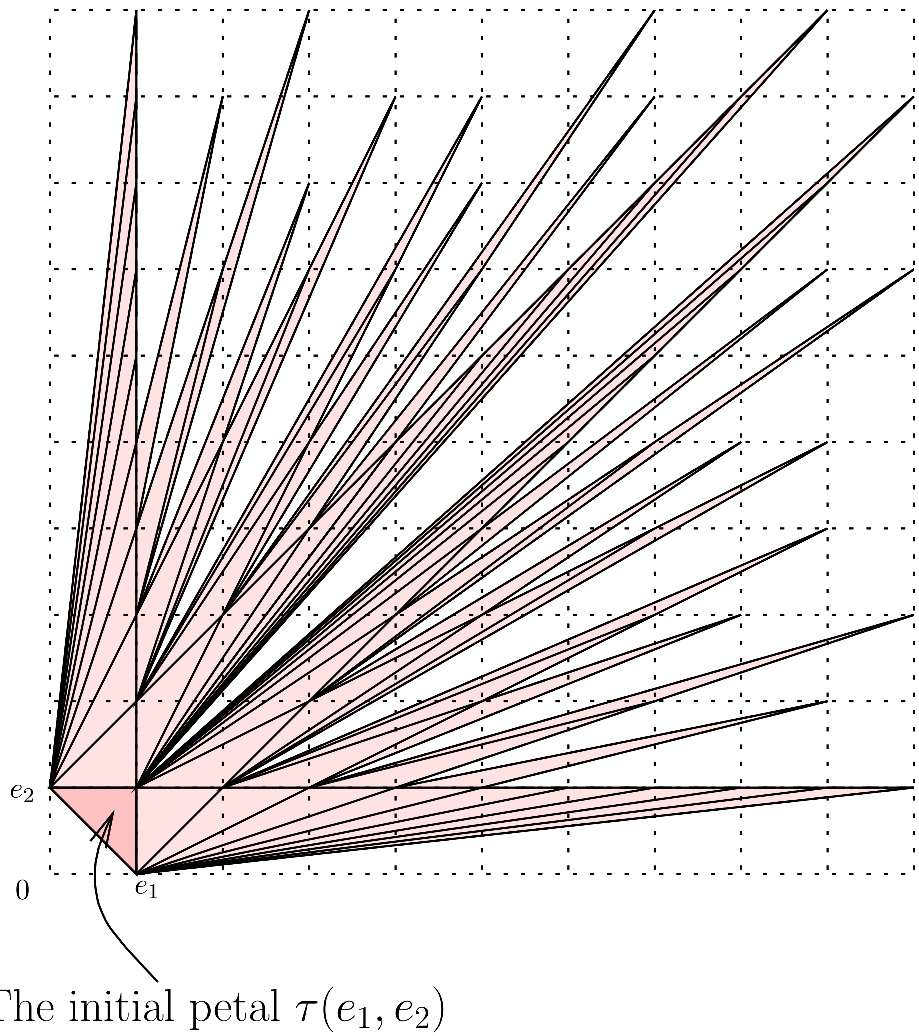
The **Enriques diagram** *and the* **dual graph** *embed canonically into the kite of a constellation :*





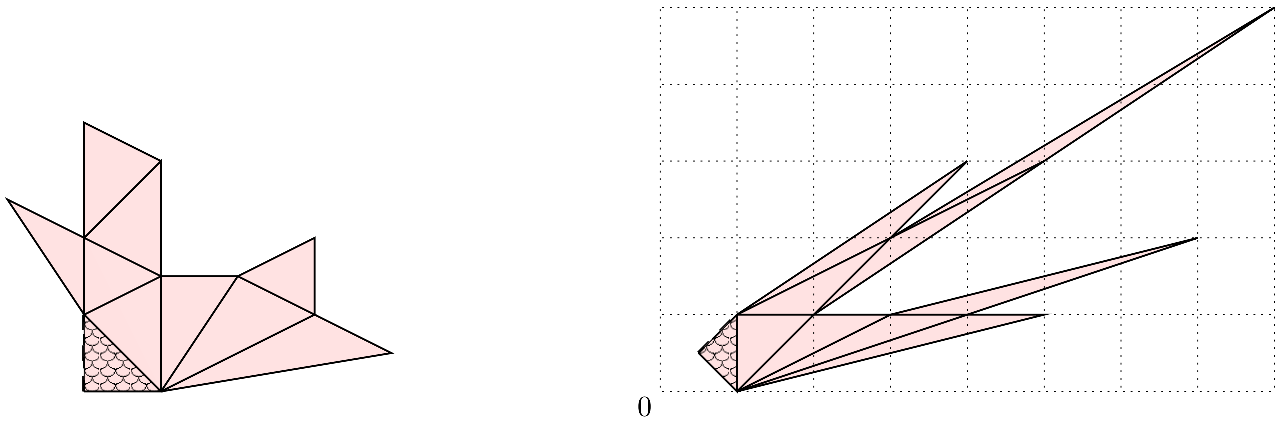
How to see the segments which *go straight* in the Enriques diagram ?

The affine structures of the elementary sails may be glued canonically by embedding the *complete sails* into the **lotus** :

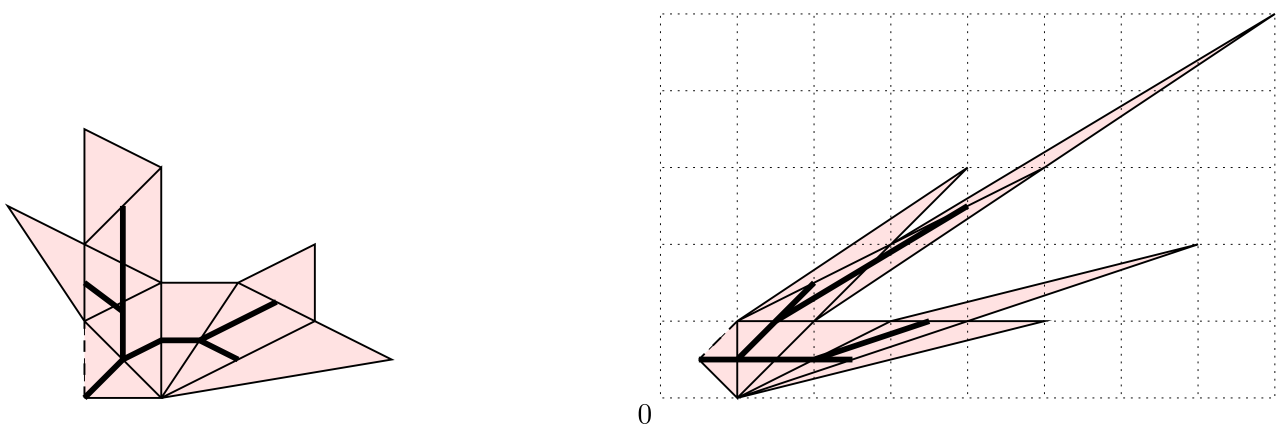


The lotus associated to the base (e_1, e_2)

A complete sail and its canonical affine embedding in the lotus :



The satellite ropes glued onto a complete sail, as well as their embeddings in the lotus :



One may give a **valuative interpretation** of the kite.

Denote by $\mathcal{O}$ the local ring of S at O, by $\mathcal{M}$ its maximal ideal and by F its field of fractions.

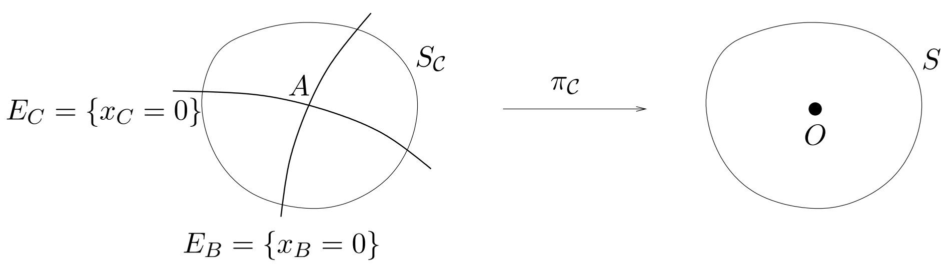
Definition 0.4. — A **valuation of** F **dominating** O is a function $\nu : F \to \mathbb{R}_+ \cup \{\infty\}$ such that :

1. $\nu(xy) = \nu(x) + \nu(y)$ for any $x, y \in F$;
2. $\nu(x + y) \geq \min(\nu(x), \nu(y))$ for any $x, y \in F$;
3. $\nu(\mathcal{M}) \subset \mathbb{R}_+^* \cup \{\infty\}$.

$\mathcal{V}_{S,O}$:= the set of valuations of F dominating O.
$\mathcal{A}_{S,O} := \{ \nu \in \mathcal{V}_{S,O} \mid \min \nu(\mathcal{M}) = 1 \}$.

Favre and Jonsson (2004) : $\mathcal{V}_{S,O}$ admits a natural functional space topology which is locally compact. Then $\mathcal{A}_{S,O}$ is a compact $\mathbb{R}$-*tree*.

Main observation : each germ of normal crossings divisor on a model above O determines an affine cone of *monomial valuations*.



Definition 0.5. — A valuation $\nu \in \mathcal{V}_{S,O}$ is called **monomial** with respect to ν_B and ν_C if there exist $(b, c) \in \mathbb{R}^2_+ \setminus 0$ such that for any $f \in F^*$:

$$\nu(f) = \min\{b \cdot m + c \cdot n \mid c_{m,n} \neq 0,\ f \circ \pi_\mathcal{C} = \sum_{(m,n)} c_{m,n} x_B^m x_C^n\}$$

Denote by :

$$b\,\nu_B \oplus c\,\nu_C$$

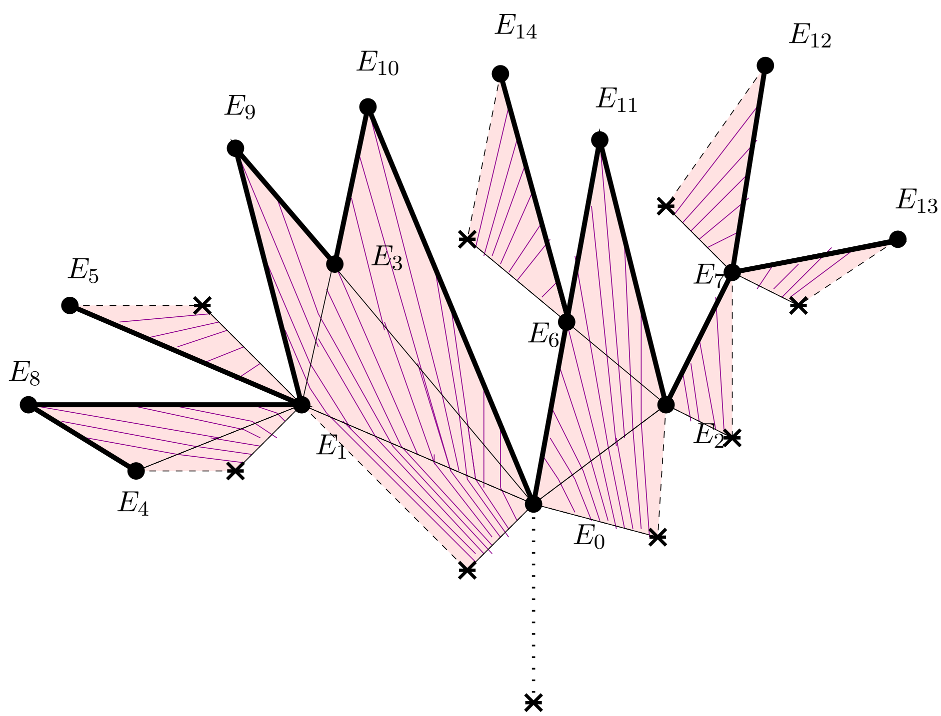
the previous monomial valuation.

The simple sail corresponding to the satellite star A embeds in the valuation space as the convex hull of the valuations ν_A, ν_B, ν_C. Indeed :

Lemme 0.6. — *One has the following equality :* $\nu_A = \nu_B \oplus \nu_C$.

Proposition 0.7. — *The kite of a constellation embedds canonically into $\mathcal{V}_{S,O}$ preserving the affine structures of its complete sails.*

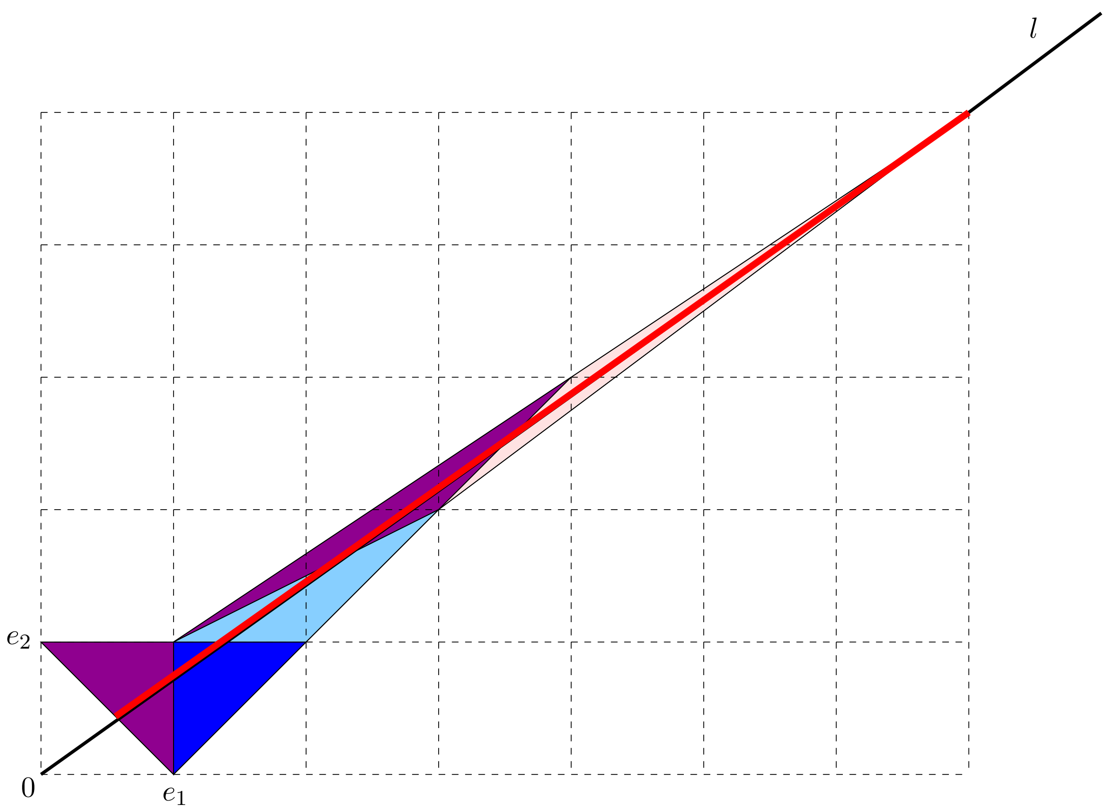
The valuative tree $\mathcal{A}_{S,O}$ appears as *the projectivisation of the kite of the firmament* :



Relation with **continued fractions**.

$$[x_1, x_2, ...] := x_1 + \cfrac{1}{x_2 + \cfrac{1}{\cdots}}.$$

To each *half-line* contained in the cone generated by a basis of a lattice one associates its **sheath** (the union of petals crossed by the half-line) :

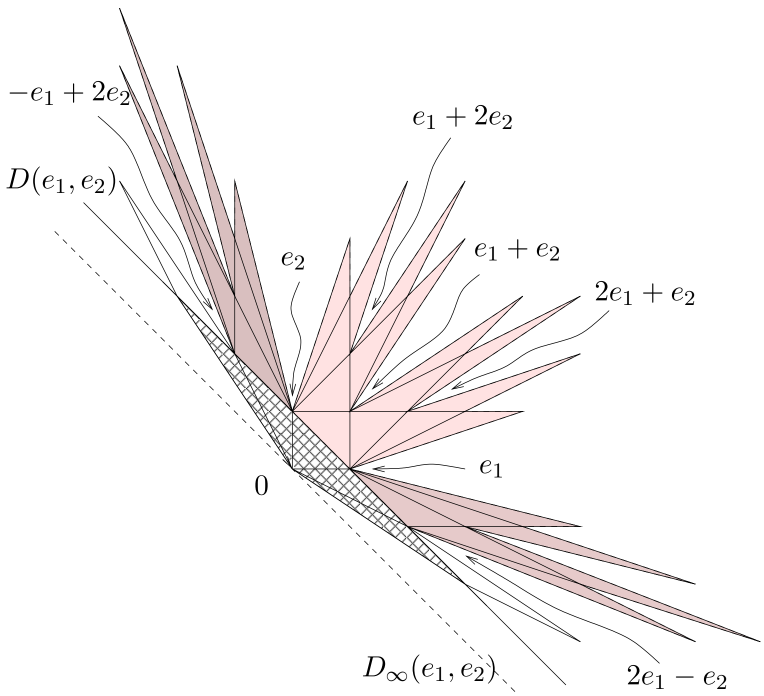


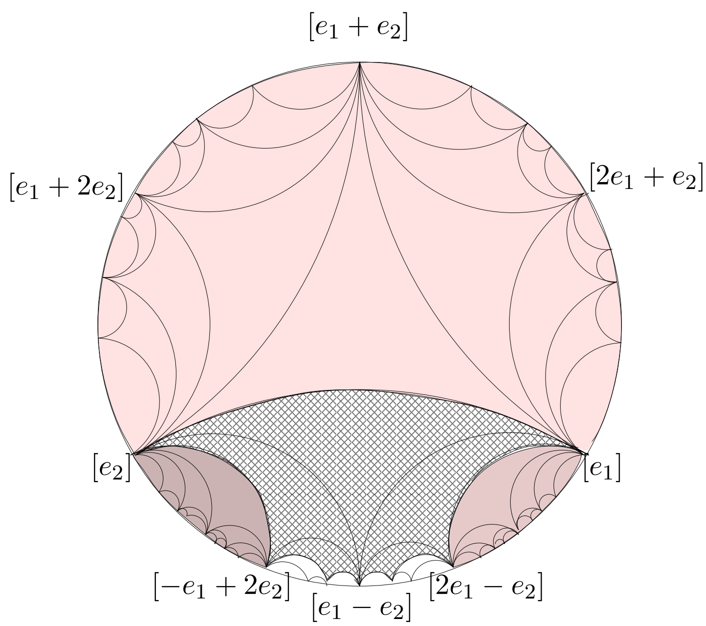
Associated sequence of symbols : R, L, L, R, R

Associated sequence of numbers : 1,2,2

$7/5 = [1, 2, 2]$

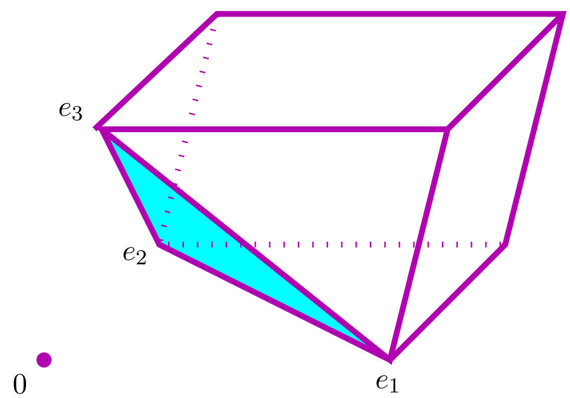
Relation with hyperbolic geometry :



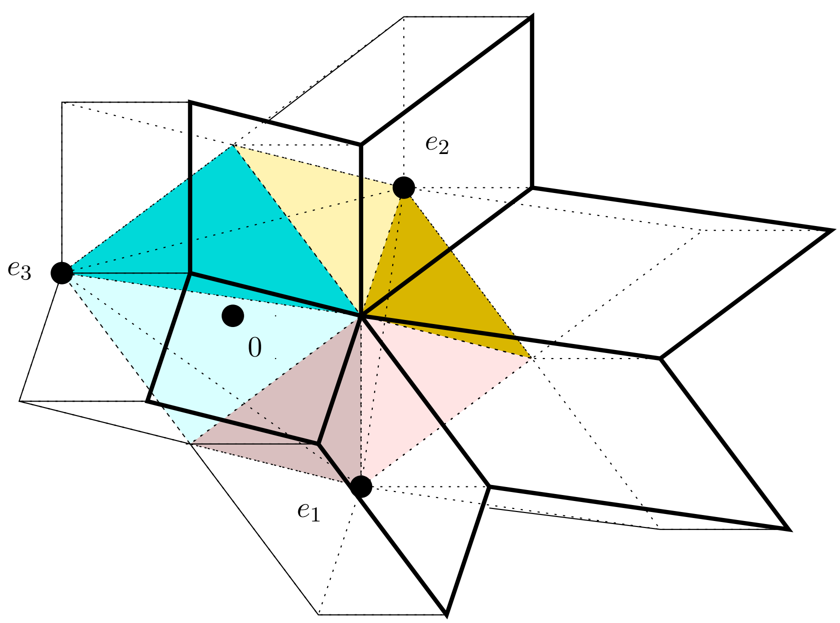


The notion of **lotus** may be generalized to *higher dimensions*.

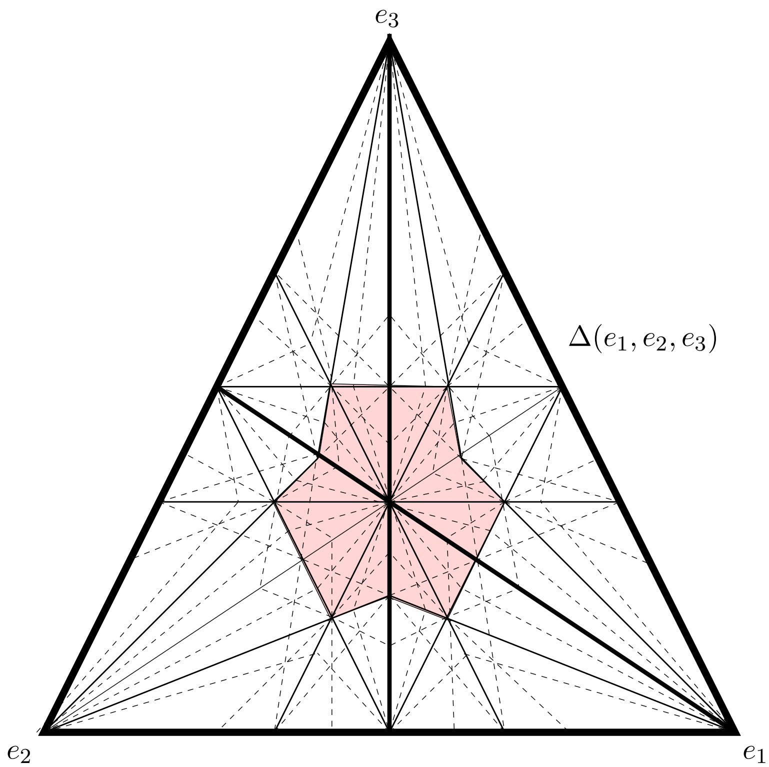
We will make drawings in dimension 3. Denote by (e_1, e_2, e_2) the initial base. The initial **petal** :



One repeats this construction starting from the bases of the form $(e_i, e_i + e_j, e_i + e_j + e_k)$, getting at the second step $3! = 6$ petals :



Look now at the first four levels of the lotus from the origin :

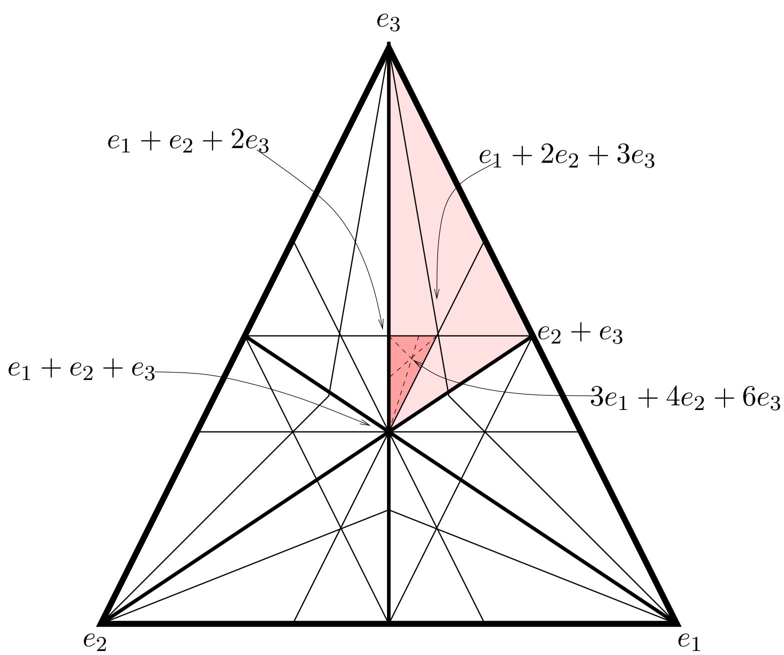


Combinatorially, one simply does *barycentric subdivisions*.

The notion of **sheath of a half-line** may also be extended.

The vertices of each triangle are canonically ordered and canonically labelled by the vertices of the initial triangle. One gets an *arrangement* of the elements of the basis (e_1, e_2, e_3).

Therefore, the sheath of a half-line is representable by *a sequence of arrangements* of sets of decreasing cardinals. This *continued arrangement expansion* generalises the continued fraction expansion.



The sequence of arrangements associated to the half-line proportional to $3e_1 + 4e_2 + 6e_3$ is :

$$(3, 1, 2), (2, 3, 1), (2, 3, 1).$$

The notion of *sheath* allows to *measure geometrically* the **contact** of *monomial curves* in $\mathbb{C}^n$.

If

$$t \to (t^{m_1}, ..., t^{m_n})$$

is a monomial curve, then :

$$(m_1, ..., m_n) \in \mathbb{N}^n$$

is its vector of *exponents*.

To two monomial curves one associates *the intersection of their sheaths*.

Happy Birthday Anatoly ! ! !