The Topology of Hyperplane Arrangements

Richard Randell, University of Iowa

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Thanks, Anatoly, for all the great discussions over the years. Happy Birthday!

INTRODUCTION

Throughout this talk $\mathcal{A}$ will be an arrangement of hyperplanes in $\mathbf{C}^\ell$. This means that $\mathcal{A}$ is a finite collection $\{H_1, \ldots, H_n\}$ where $H_i = \alpha_i^{-1}(b_i)$ and each α_i is a linear homogeneous form in the variables $(z_1, \ldots, z_\ell)$. (See [P. Orlik and H. Terao, Arrangements of Hyperplanes] for material on arrangements). We call $\mathcal{A}$ an ℓ-*arrangement*. If all the $b_i = 0$, we say that the arrangement is *central*. Otherwise we call $\mathcal{A}$ an *affine* arrangement. We let M be the complement of the union of the hyperplanes

$$M = \mathbf{C}^\ell \setminus \cup H_i.$$

Our focus here is to describe the topology of M and various related spaces. Surprisingly this turns out to be more difficult than one would expect, given the linear nature of the situation. Since each H_i has real codimension two, we are in a knotting situation, and the fundamental group of M turns out to have considerable interest, in some ways analogous to that of a knot complement. Sometimes M turns out to be aspherical (as in the classical pure braid case–see below), but in general it is not, and so in that way M is not behaving like a knot complement. Asphericity turns out to be rare, and generally a consequence of special geometric or algebraic considerations.

What is true is that in many ways M behaves nicely as a topological space. For example, M has the homotopy type of a CW complex of real dimension ℓ. In fact, this CW complex is *minimal* in that the number of k-cells is equal to the k-th betti number $b_k(M)$. If the arrangement is *real* (the coefficients of the defining polynomial are real), there are techniques for explicitly determining the attaching maps in this CW structure. As a further example of this topological niceness the ordinary cohomology of M is generated by rank one classes (the "logarithmic forms" $d\alpha_i/\alpha_i$).

After we sketch some basic constructions, examples and results below, we intend to highlight a number of areas of current interest, mentioning along the way a somewhat personal list of open problems. We focus on what should be true for all, or at least large classes of, arrangements, rather than on properties of rather special families.

BASIC TOPOLOGY, COMBINATORICS AND ALGEBRA

Coning, De-coning, and fibering

Let's start with a central arrangement in $\mathbf{C}^\ell$, given by a product $Q = \prod(\alpha_i) = 0$ of linear forms. We may regard Q as a smooth function $Q : M \to \mathbf{C} - \{0\}$. In fact, this is just the Milnor fibration, global here because Q is homogeneous. Let $F = Q^{-1}(1)$ be the fiber. Also, one has the Hopf map $h : M \to M^*$, where M^* is the complement of the projective variety defined by Q. Notice that M^* is an affine arrangement in its own right, simply by removing any of the n projective hyperplanes from projective space.

Then one has the following basic diagram (which I first saw in the work of M. Oka):

$$\begin{array}{ccccc} & & \mathbf{C}^* & & \\ & & \downarrow & & \\ F & \longrightarrow & M & \longrightarrow & \mathbf{C}^* \\ & \searrow & \downarrow & & \\ & & M^* & & \end{array}$$

Here the vertical bundle map is the Hopf map and the horizontal bundle map is the Milnor fibration. The diagonal map is the cyclic n-fold cover map $(z_1, \ldots, z_\ell) \to [z_1 : \ldots : z_\ell]$.

We say that the affine arrangement space M^* is the *decone* of M with respect to the hyperplane removed. If we choose coordinates so that the removed hyperplane is given by $z_n = 0$ then M^* is associated to the affine arrangement obtained by taking $z_n = 1$ in the defining equation Q for M. Conversely, given an affine arrangement given by possibly non-homogeneous defining polynomial, we obtain a central arrangement (the *cone*) by homogenization.

Notice that the Hopf map is the projection of a trivial bundle, provided that there is at least one hyperplane, because it is the pullback of a bundle over a contractible base given by removing a single hyperplane from projective space. Thus topologically $M = M^* \times \mathbf{C}^*$.

We draw arrangements as in the picture of $Q = (x+z)(x-z)(y+z)(y-z)(x-y)z$.

COMBINATORICS

Define a poset $\mathcal{L}(\mathcal{A})$ whose elements are the intersections of the hyperplanes, with $X \leq Y$ if and only if $Y \subset X$. The rank of X, denoted $rk(X)$ is defined to be the complex codimension of X in the ambient space.

Consider again the simple example, deconed to an affine arrangement.

$$Q = (x+1)(x-1)(y+1)(y-1)(x-y).$$

There are five rank one elements and four rank two elements.

A basic question is: to what extent does the combinatorics determine the topology? A first answer was given by the following theorem of Orlik and Solomon:

Theorem 1 *(P. Orlik and L. Solomon, 1980) The lattice of $\mathcal{A}$ determines the cohomology ring of M.*

In particular, the lattice determines the Poincaré polynomial $P_{\mathcal{A}}(t)$. To begin to understand this result we need some algebra, and we need to examine the beginnings of the subject.

HISTORY AND ALGEBRA

The first hyperplane arrangement studied was the braid arrangement, $Q = \prod(z_i - z_j)$, which was studied by Fadell and Neuwirth in order to better understand the braid groups. (The fundamental group of the associated M is the pure braid group.) In 1969 V. I. Arnol'd studied this braid space as the complement of the discriminant for polynomial equations, and showed that the cohomology algebra was generated by logarithmic one forms $d\alpha_i/\alpha_i$ and was in fact isomorphic to the graded subalgebra of the DeRham complex generated by these forms. He conjectured that such a result should be true always (i.e. for any complex hyperplane arrangement), and this was soon proved by E. Brieskorn, in 1971.

In particular, there is the result

Theorem 2 *(Arnol'd for braid case, Brieskorn in general) $H^*(M)$ is the the subalgebra of the DeRham complex generated by logarithmic one forms associated to the hyperplanes.*

In the course of his work Brieskorn observed a fundamental property of arrangements, that cohomology is given by the direct sum of local contributions. That is, consider any element X of the lattice. Now X has rank p and is the intersection of a collection $\mathcal{A}_X = \{H_i\}, i \in S$. If one sets M_X to be the complement of the arrangement $\mathcal{A}_X$, then

Theorem 3 *(Brieskorn) There is an isomorphism*

$$H^p(M, \mathbf{Z}) \cong \oplus H^p(M_X, \mathbf{Z})$$

the sum taken over all X of rank p.

The key element in the proof is the Lefschetz hyperplane theorem. As a consequence of this result one is able to compute easily the cohomology groups of the complement from a Moebius function attached to the lattice of the arrangement. For example, we have the Poincaré polynomials shown in the illustration.

Now Orlik and Solomon considered the exterior algebra over the ring $R = \mathbf{Z}[a_i]$ where a_i is a formal generator in degree one (corresponding to the one form $d\alpha/\alpha$ considered by Arnol'd). For any lattice element X which is the intersection of H_i, i as above, consider $a_X = \wedge_i(a_i)$ and let ∂a_X be given by the usual boundary formula. Let I be the ideal in R generated by these ∂a_X. Then the quotient algebra R/I is determined by the intersection lattice of the arrangement. It is generally called the *Orlik-Solomon algebra*–Orlik and Solomon showed that it is isomorphic to the cohomology algebra of the complement, giving the combinatorial result above.

MORE TOPOLOGY

Now given that the cohomology (even over $\mathbf{Z}$) is determined by the intersection lattice, it is natural to ask what else is so determined. In particular, is the homotopy type of the complement determined by the intersection lattice. The answer to this turns out to be no, as shown by an example of Rybnikoff (see also Artal, Carmona, Cogolludo and Marco). This example, consisting of two thirteen hyperplane arrangements of rank four, obtained by "amalgamating" two copies of the MacLane matroid in complex conjugate ways, gives two complex hyperplane arrangements with isomorphic intersection lattices but non-isomorphic fundamental groups.

So the combinatorics does not determine homotopy type–

Theorem 4 *(Rybnikoff) There are complex hyperplane arrangements with isomorphic intersection lattices and non-homotopy equivalent complements.*

By the way, the determination that the fundamental groups in these examples are non-isomorphic is rather specific to these examples, and quite subtle, involving the lower central series and more. It would be of interest to have invariants of hyperplane arrangements which separate these two examples.

So what does determine the topology? In this vein there are are number of results. First, one can in obvious ways consider the moduli space of all arrangements of n hyperplanes in $\mathbf{C}^\ell$. Within this space there are the subsets of those arrangements with constant intersection lattice. Such subsets need not be path-connected (there are examples), but there is the following result.

Theorem 5 *(R.) The diffeomorphism type of the complement (and even of the Milnor fibration) is constant within smooth families with constant intersection lattice.*

Such smooth families are called lattice isotopies.

So how does one understand the structure of the complement as a CW or simplicial complex? First of all, work of Milnor shows that the complement has the homotopy type of an ℓ-dimensional CW complex. In the case of *real* arrangements (coefficients are real), there is a canonical regular cell complex, called the Salvetti complex, associated to the complement. In the case of real arrangements in $\mathbf{C}^2$, for example, this complex has one vertex for each chamber, two edges for each edge of the picture, and $2k$ two-cells for each point of multiplicity k, where $k > 1$.

Theorem 6 *The Salvetti complex has the homotopy type of the complement.*

The Salvetti complex allows one to write down the fundamental group of the complement, and has been crucial in work of L. Paoluzzi and L. Paris, for example, on understanding (from an arrangements point of view) the faithful Lawrence-Krammer-Bigelow representations of the braid groups.

The Salvetti complex elegantly uses the geometry of the arrangement as a template for constructing the complex. On the other hand, it has lots of cells. It is now known, however, that the complement has a minimal cell structure:

Theorem 7 *(Dimca-Papadima, R.) There is a CW structure for M for which the number of cells in each dimension is equal to the betti number in that degree.*

As one might expect, the proofs use Morse theory, together with the fact that the ranks of the homology groups can be read from the intersection lattice. In these minimal CW complexes the attaching maps are thus homologically trivial. (One can see this exemplified in standard presentations for the fundamental group, for which the relators lie in the commutator subgroup.) It is a natural question to ask exactly how the cells are attached in general. Recent work of Yoshinaga, Salvetti-Settepanella, Delucchi-Settepanella has clarified this issue, and there are now classes of arrangements (though by no means all) for which the attaching maps can be written down. This works (with some effort) for the "discriminantal" arrangements which arise in the theory of hypergeometric functions and in the LKB representations.

It is worth noting that the existence of these minimal structures on the complement provides a way in which arrangements complements differ from knot complements, which hardly ever have minimal CW structures–the complement of a knot in S^3 has the homology of a circle, so a minimal CW structure would be a 0-cell and a 1-cell, implying that the homotopy type of the complement was that of the circle, which implies that the knot in question is the unknot (by Dehn's lemma).

FUNDAMENTAL GROUPS

Fadell and Neuwirth studied the pure braid arrangement via iterated fiberings. Using these one can show for the pure braid groups P_ℓ:

- the complement of the pure braid arrangement is an Eilenberg-MacLane $K(P_\ell, 1)$ space.

- P_ℓ has finite cohomological dimension.

- P_ℓ has type FL (it has the homotopy type of a finite CW complex).

- P_ℓ is torsion-free.

- P_ℓ is residually nilpotent, and the lower central series is well-behaved (T. Kohno, Falk-R.).

These observations motivated considerable further study. It is known now that not all arrangement complements are Eilenberg-MacLane spaces (Hattori), and that not all have fundamental groups of type FL (Arvola). The other three questions above are still interesting for general arrangements.

FUNDAMENTAL GROUPS, COVERS AND LOCAL SYSTEMS

"Topological properties of the curve $f(x,y) = 0$ can be derived by studying the surface F, $z^k = f(x,y)$ ..." , Oscar Zariski

"If $\Gamma = \pi_1(P^2 - C)$, $\Gamma' = [\Gamma, \Gamma]$,$\Gamma'' = [\Gamma', \Gamma']$,It would be interesting to investigate the structure of Γ'/Γ'' as a $\mathbf{Z}[\Gamma/\Gamma']$ module." David Mumford, in comments in Zariski's book "Algebraic Surfaces" .

Anatoly (on a bench near UI campus in Chicago, about 1979–it was Lib**30**ber) “I’m going to do that.” .

Recall that we have the diagram

$$\begin{array}{ccccc} & & \mathbf{C}^* & & \\ & & \downarrow & & \\ F & \rightarrow & M & \rightarrow & \mathbf{C}^* \\ & \searrow & \downarrow & & \\ & & M^* & & \end{array}$$

The space F in this diagram is closely related to the F of Zariski, in the arrangements (completely reducible) case, where the degree k of the cover is just the number n of hyperplanes in the associated central arrangement. Of course, covers of the affine arrangement M^* correspond to subgroups of the fundamental group, and homology of the covers corresponds to homology of M^* with local coefficients. Our understanding of these owes much to the work of Anatoly Libgober.

ABELIAN COVERS OF HYPERPLANE COMPLEMENTS

For many reasons alluded to above, one is especially interested in (co)-homology of an arrangement complement, with coefficients in a local system. There are two particular cases:

- (Covers) $H_i(M; \mathbf{Z}[\pi/\pi'])$ where π' is a subgroup of $\pi = \pi_1(M)$, corresponding to covers. Generally one takes $H = \pi/\pi'$ to be abelian. Thus one has Alexander invariants.

- (Complex local systems) Here there are rich connections with various topics, including hypergeometric functions.

COVERS (FROM Z to A)

I want to mention a few results here which give a feel for the subject and as well a feel for the contributions of Libgober to these and related topics. It should be mentioned too that Anatoly's work applied much more generally than to just arrangement varieties–usually to complements of general hypersurfaces.

A first result generalizes work of Zariski, who showed that for a finite abelian cover of P^2 branched over a plane curve with only nodes and cusps as singularities, the first betti number of the desingularization of $cl(F)$ vanishes unless both the degree of the curve and the degree of the covering are divisible by six. Libgober's result (Duke Math. J., 1982) is

Theorem 8 *(Libgober) For an irreducible curve the global Alexander polynomial divides the product of the local Alexander polynomials for all branches of singularities of the curve.*

The connection between the two results is due to the influence of roots of the Alexander polynomial on the betti numbers. An interesting example here is the complement of the 4-strand braid arrangement. This is an arrangement of six hyperplanes. One associates the reducible curve in P^2 with six linear components, which has only double and triple points. The Milnor fiber F is the six-fold cover of the complement. It has an excess of two in its first betti number (which is seven, as determined by Artal), giving an example in which the conclusion of Zariski's theorem fails to hold. Libgober's theorem shows why this behavior is possible.

Before we move on to local systems, I want to mention a subtle related question. From the above we know that the Milnor fiber F associated to a central arrangement as in the main diagram above has interesting homology, which has a subtle reliance on properties of the arrangement. We'll see more on this below. It turns out that in all known cases, $H_i(F; \mathbf{Z})$ is torsion-free. (There are examples of G. Denham-D. Cohen-Suciu of homology torsion when one allows multiplicity on the hyperplanes). For local reasons, however, it is reasonable to conjecture that the actual Milnor fiber associated to an arrangement has torsion-free homology groups.

COMPLEX LOCAL SYSTEMS

For the remainder of the talk we will be concerned with complex local systems–we will focus on the rank one case. We discuss briefly the relationship between higher homotopy and local homology, and then we will describe the characteristic and resonance varieties of an arrangement, objects of great current interest in the subject.

Recall that the cohomology ring of M is generated in degree one, so that it is somewhat like that of a torus. As a consequence one cannot use homology to detect non-trivial higher homotopy classes:

Theorem 9 *(R.) On arrangement complements the image of the Hurewicz homomorphism*

$h{:}\pi_i(M) \to H_i(M)$

is trivial for all $i > 1$.

It follows that $H_2(M) \cong H_2(\pi)$.

On the other hand, M. Yoshinaga recently proved (with $M'' = M \bigcap H_{generic}$)

Theorem 10 *(Yoshinaga) Suppose* (M, M'') *is an arrangement pair and* N *is a non-resonant local system of rank* r *on* M. *Then* $h_{M''}$: $\pi_{n-1}(M'') \otimes_\pi N \to H_{n-1}(M''; N)$ *is onto.*

Thus it is natural to ask the following question:

Question: Let $\mathcal{A}$ be any complex hyperplane arrangement in $\mathbf{C}^\ell$, $\rho \in \pi_k(M)$ with $k \leq \ell$. When is there a rank one local system $\mathcal{L}$ on M so that for the appropriate Hurewicz map

$$h : \pi_k(M) \otimes_Z \mathcal{L}_m \to H_k(M; \mathcal{L})$$

one has $h(\rho) \neq 0$?

For the purposes of Yoshinaga's theorem, the term non-resonant may as well mean "cohomology concentrated in top degree". The precise formulation is due to A. Varcenko in the study of hypergeometric functions. One wants to consider local systems derived from weights on each hyperplanes. Resonant then means that the weights satisfy certain numerical conditions on certain "dense edges" (a very checkable condition). Esnault-Schechtman-Viehweg and Schechtman-Terao-Varcenko showed that in the nonresonant case the twisted Brieskorn complex cohomology is isomorphic to the twisted Aomoto complex cohomology (see later). The latter is combinatorial in the lattice and the weights, and Orlik and Terao showed that the homology of this complex in concentrated in the top degree in the nonresonant case.

CHARACTERISTIC AND RESONANCE VARIETIES

The *characteristic varieties* on any space X are the jumping loci for the cohomology of X with coefficients in rank one local systems. That is,

$$V_d^i(X) = \{t \in Hom(\pi_1(X), \mathbf{C}^*) : dim_{\mathbf{C}} H^i(X, \mathbf{C}_t) \geq d\}$$

Here the π_1-module structure of $\mathbf{C}_t$ is given by the representation $t : \pi_1(X) \to \mathbf{C}^*$. In the arrangements case, and since $\mathbf{C}^* = Aut(\mathbf{C})$ is abelian the representation factors through $H_1(M)$ which has basis small loops transverse to the hyperplanes. Thus we can think of a rank one local system as associated with an assignment of a weight $w_i \in \mathbf{C}^*$ to each hyperplane H_i, that is, as an element of the character torus $Hom(\pi_1(X), \mathbf{C}^*) \cong ((\mathbf{C}^*)^{\mathbf{n}}$.

Example 11 *Consider the pure braid arrangement on four strands. This is an arrangement (central but non-essential) of six hyperplanes in* $\mathbf{C}^4$ *. It has four triple points in rank two, each with Alexander polynomial* $(t-1)(t^3-1)$*. These cube roots of unity, and the fact that there are six hyperplanes and therefore the Milnor fiber is a six-fold cover, allows the "extra" rank one homology.*

A second variety associated to an arrangement is the *resonance variety* defined by M. Falk. One looks at the Orlik-Solomon algebra A over a field and for any $a \in A_1$ converts it to a cochain complex by setting the i-th cochain group to be the degree i part of A, and letting the differential be multiplication by a. The resonance varieties are then

$$R_d^i(A) = \{a \in A : dim H^i(A, a) \geq d\}$$

These two varieties are related as follows. Arapura showed very generally that the irreducible components of the characteristic varieties of X are algebraic subtori of the character torus, possibly translated by unitary characters. Falk conjectured that the resonance varieties were always linear; this follows from the result below, proved by Libgober-Yuzvinsky and D. Cohen-Suciu.

Theorem 12 *(Libgober-Yuzvinsky, D. Cohen-Suciu) The tangent cone at the identity to* $V_d^i(X)$ *is equal to* $R_d^i(A)$*.*

Now the resonance varieties come from the Orlik-Solomon algebra and are therefore combinatorially determined. Thus the algebraic subtori in the characteristic varities are determined by the intersection lattice. Examples of Suciu show that there are arrangements for which the characteristic varieties have components (positive-dimensional) which do not pass through the identity. It's unknown if such are combinatorially determined.

The properties of the resonance varieties are closely connected with combinatorics of sub-arrangements of $\mathcal{A}$. For example, Libgober and Yuzvinsky used the Vinberg classification of generalized Cartan matrices to place strong restrictions and elegant structure on the underlying arrangement.

Thank you.