SMOOTH SUBVARIETIES THROUGH SINGULARITIES OF A NORMAL VARIETY

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Let k be an algebraically closed field.

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Let X be a normal variety over k which admits resolution of singularities. Let $\pi : \tilde{X} \to X$ be a desingularization of X, (this means that π is a proper birational morphism and $\tilde{X}$ is non-singular). Let $X_{\rm reg} = X - {\rm Sing(X)}$ be the nonsingular locus of X.

We assume that π induces an isomorphism between $\pi^{-1}(X_{\rm reg})$ and $X_{\rm reg}$.

Definition 1. *A* **formal subvariety** *of dimension r on X is a morphism*

$$\phi_r : Spec\ k[[x_1, \dots, x_r]] \to X.$$

Let us denote by η the generic point of $Spec\ k[[x_1, \dots, x_r]]$ and by ξ the closed point.

Consider a formal curve ϕ_1 as above. Let $P = \phi_1(\xi)$. Let $\mathcal{M}$ denote the maximal ideal of $\mathcal{O}_{X,P}$.

Definition 2. *We say that ϕ_1 is* **smooth** *if $ord_t \phi_1^* \mathcal{M} = 1$, where ord_t denotes the t-adic valuation in $k[[t]]$.*

Let $\mathcal{L}_r$ be the set of formal subvarieties Y of X containing a formal nonsingular curve ϕ_1 on X such that $\phi_1(\xi) = P \in {\rm Sing}(X)$, $\phi_1(\eta) \in X_{\rm reg}$ and ϕ_1 is transversal to ${\rm Sing}(X)$, that is, ϕ_1 intersects ${\rm Sing}(X)$ transversally in a smooth point of ${\rm Sing}(X)$. This means, by definition, that $ord_t \mathcal{J} = 1$ where $\mathcal{J}$ is the ideal defining ${\rm Sing}(X)$.

Definition 3. *We call the* **special cycle** $Z_{\tilde{X}}$ *the cycle* $Z_{\tilde{X}} = \sum m_i E_i$, *defined by the divisorial part of* $\mathcal{J}\mathcal{O}_{\tilde{X}}$, $\mathcal{J}$ *is as above. The* E_i *are the irreducible components of codimension 1 of the exceptional fiber* $\pi^{-1}(Sing(X))$ *and the* m_i *are nonnegative integers. A component* E_j *such that* $m_j = 1$ *is called a* **reduced component** *of the cycle.*

Proposition 4. *Let* $\pi : \tilde{X} \to X$ *be a desingularization of* X. *Let* $\phi_1 \in \mathcal{L}_1$. *Let* $SuppZ_{\tilde{X}}$ *be the support of the special cycle of* π. *Let us take a point* $R \in SuppZ_{\tilde{X}}$. *The point* R *is the intersection of the strict transform* $\tilde{C}$ *of* C *with* $\pi^{-1}(Sing(X))$ *if and only if there exists* $\phi_1 \in \mathcal{L}_1$ *such that* $R = \pi^{-1}(Sing(X)) \cap Im\tilde{\phi}_1$ *and such that there exists a regular system of parameters* $\{t_1, \ldots, t_s\}$ *of* $\mathcal{O}_{\tilde{X},R}$ *such that* $\mathcal{J} = (t_1)$.

Proof. Let a be the greatest common divisor of the elements in $\mathcal{J}$. Since $\mathcal{O}_{\tilde{X},R}$ is a UFD, $\mathcal{J} = aI$, for some ideal I in $\mathcal{O}_{\tilde{X},R}$ with $\mathrm{ht}I \geq 2$. Let $R = \pi^{-1}(\mathrm{Sing}(X)) \cap \mathrm{Im}\tilde{\phi}_1$, for some $\phi_1 \in \mathcal{L}_1$. Then ϕ_1 factors through a local homomorphism $\tilde{\phi}_1 : \mathcal{O}_{\tilde{X},R} \to k[[t]]$ such that $\mathrm{ord}_t \mathcal{J} = 1$, since ϕ_1 intersects $\mathrm{Sing}(X)$ transversally. Since $R \in SuppZ_{\tilde{X}}$, a is not a unit. We have

$$1 \leq \mathrm{ord}_t a \leq \mathrm{ord}_t a + \mathrm{ord}_t I = 1.$$

Thus, $\mathrm{ord}_t a = \mathrm{ord}_R a = 1$, $\mathrm{ord}_t I = \mathrm{ord}_R I = 0$, the function a is a part of a regular system of parameters of $\mathcal{O}_{\tilde{X},R}$ and we have that $\mathcal{J} = (a)$. Conversely, if there exists a regular system of parameters $\{t_1, \ldots, t_s\}$ of $\mathcal{O}_{\tilde{X},R}$ such that $\mathcal{J} = (t_1)$ the projection on X of any formal curve $\tilde{\phi}_1$ on $\tilde{X}$ through R whose parametrization sends t to t_1 is a smooth curve on X through P. By taking $\tilde{\phi}_1$ such that its generic point lies in $\pi^{-1}(X_{\mathrm{reg}})$, we get a curve in $\mathcal{L}_1$.

Note II. Consider a formal curve ϕ : *Spec* $k[[t]] \to X$. Let us denote by η the generic point of *Spec* $k[[t]]$ and by ξ the closed point. Let $P = \phi(\xi)$. Let $\mathcal{M}$ denote the maximal ideal of $\mathcal{O}_{X,P}$.

Note that any formal curve ϕ factors through *Spec* $\hat{\mathcal{O}}_{X,P}$. If ϕ is a smooth formal curve, the natural image of *Spec* $k[[t]]$ in *Spec* $\hat{\mathcal{O}}_{X,P}$ is a regular subscheme of *Spec* $\hat{\mathcal{O}}_{X,P}$, but the converse is not true since the map ϕ might be composed with a map of *Spec* $k[[t]]$ to itself, such as $t \to t^2$.

Let $\mathcal{L}$ be the set of formal nonsingular curves ϕ on X such that $\phi(\xi) = P$ and $\phi(\eta) \in X_{\mathrm{reg}}$.

Assume that P is an isolated singularity. We have a map of sets $f_{\tilde{X}} : \mathcal{L} \to \pi^{-1}(P)$ which sends $\phi \in \mathcal{L}$ to the intersection point of the strict transform $\tilde{C}$ with $\pi^{-1}(P)$, where $C = \mathrm{Im}(\phi_1)$.

Definition II.1. *We call the* **maximal cycle** $Z_{\tilde{X}}$ *the cycle* $Z_{\tilde{X}} = \sum m_i E_i$, *defined by the divisorial part of* $\mathcal{M}\mathcal{O}_{\tilde{X}}$, *where* $\mathcal{M}$ *is the maximal ideal of* $\mathcal{O}_{X,P}$; *the* E_i *are the irreducible components of codimension 1 of the exceptional fiber* $\pi^{-1}(P)$ *and the* m_i *are nonnegative integers. A component* E_j *such that* $m_j = 1$ *is called a* **reduced component** *of the cycle.*

In this case, Proposition 4 becomes

Corollary II.2. *Let $\pi : \tilde{X} \to X$ be a desingularization of X. Let $SuppZ_{\tilde{X}}$ be the support of the maximal cycle of π. If $R \in SuppZ_{\tilde{X}}$ then $R \in f_{\tilde{X}}(\mathcal{L})$ if and only if there exists a regular system of parameters $\{t_1, \ldots, t_r\}$ of $\mathcal{O}_{\tilde{X},P}$ such that $\mathcal{M}\mathcal{O}_{\tilde{X},R} = (t_1)$.*

Proof. This is a special case of Proposition 4 when $P = \mathrm{Sing}(X)$.

Corollary II.3. *Let $\pi : \tilde{X} \to X$ be a desingularization of X. For any irreducible component E of $\pi^{-1}(P)$, let ord_E be the divisorial valuation of the function field of X given by the filtration of $\mathcal{O}_{\tilde{X},E}$ by the powers of its maximal ideal. Let*

$$\mathcal{L}_E = \{\phi \in \mathcal{L} : f_{\tilde{X}}(\mathcal{L}) \in E\}.$$

Then,

1. *The components E such that $\mathcal{L}_E \neq \emptyset$ are those for which $ord_E\mathcal{M}\mathcal{O}_{\tilde{X},E} = 1$.*
2. *The set $\mathcal{L}$ is the disjoint union of the $\mathcal{L}_E$.*

Proof. Immediate consequence of Corollary II.2, [1, 1.2].

We generalize [1, 1.3] as follows:

Definition II.4.

1. *A* **family** *of formal nonsingular curves on X through P is any of the nonempty subsets $\mathcal{L}_E$ defined in Corollary II.3, (a).*
2. *A* **general hyperplane section** *H of X through P is a Cartier divisor of X with local equation $f = 0$, $f \in \frac{\mathcal{M}_P}{\mathcal{M}_P{}^2}$, such that H intersects transversally $Supp\, Z_{\tilde{X}}$ and such that $H \cap Sing\, X = \emptyset$.*
3. *A* **first order family** *is a family which contains an irreducible component of a general hyperplane section of X through P.*

Proposition II.5. *Let $\mathcal{L}_E$ be a family of formal nonsingular curves on X through P. Let $\pi : \tilde{X} \to X$ be the desingularization of X at P. If $\mathcal{L}_E$ is a first order family, there exists a reduced component F such that $f_{\tilde{X}}(\mathcal{L}_E) = F \cap (\tilde{X})_{\mathrm{Reg}} \cap Supp\, Z_{\tilde{X}}$. Otherwise, there exists a singular point $Q \in \tilde{X}$ such that $f_{\tilde{X}}(\mathcal{L}_E) = Q$.*

Proof. By Corollary II.2, $f_{\tilde{X}}(\mathcal{L}_E)$ is contained in a single reduced component E of $\mathrm{Supp}\, Z_{\tilde{X}}$. If $\dim E = 1$, $\mathcal{M}\mathcal{O}_X$ is invertible at any $Q \in f_{\tilde{X}}(\mathcal{L}_E)$. $\mathcal{L}_E$ is a first order family if and only if $\pi(E)$ is a curve. By II.2, $f_{\tilde{X}}(\mathcal{L}_E) = F \cap (\tilde{X})_{\mathrm{Reg}} \cap \mathrm{Supp}\, Z_{\tilde{X}}$.

Note III. Let $\mathcal{L}_s$ be the set of formal nonsingular subvarieties

$\phi_s : \mathrm{Spec}\, k[[x_1, \ldots, x_s]] \to X$ of dimension s (see Definition 1) on X containing a formal nonsingular curve ϕ_1 on X such that $\phi_1(\xi) = P \in \mathrm{Sing}(X)$, $\phi_1(\eta) \in X_{\mathrm{reg}}$ and ϕ_1 is transversal to $\mathrm{Sing}(X)$. There exists $Y \in \mathrm{Spec}\, k[[x_1, \ldots, x_s]]$ such that $\phi_s(Y) \in X_{\mathrm{reg}}$. Assume that $\pi : \tilde{X} \to X$ is a birational map. If $Y \in \mathcal{L}_s$, $C = \mathrm{Im}(\phi_1)$ and $R \in SuppZ_{\tilde{X}} \cap \tilde{C}$ then a formal parametrization of Y factors through a local homomorphism $\mathcal{O}_{\tilde{X},R} \to k[[t_1, \ldots, t_s]]$. Note that $t_1 = x_1$ $t_i = f_i(x_1, ..., x_s)$, where f_i, $2 \leq i \leq s$, is a rational function of the form $t_i = \frac{P_i(x_1, \ldots, x_s)}{t_1^{d_i}}$, $2 \leq i \leq s$.

References

[1] G. González-Sprinberg and M. Lejeune-Jalabert, *Families of smooth curves on singularities and wedges*, Ann. Pol. Math. **LXVII.2 (1997)**, 179-190.