Real Singularities in Complex Geometry

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En honor a Tolya, por sus 60!!!

Anatoly Libgober's work is a beautiful example of multi-facetic mathematics: ideas and results from various sources combine to produce interesting work about singularities, topology of algebraic varieties, mirror symmetry, ... etc.
CONGRATULATIONS TOLYA!!!!

Thus, when deciding what I wanted to talk about in this conference, I decided to go for a lecture with a wide scope.

The aim of this talk is two-folded: I want to explain a point of view that I have been exploring for about 15 years, to study real singularities using complex geometry and conversely, to study complex singularities using real singularities.

This line of research originates in work I did in the early 1990's, by myself and a little later jointly with M. A. Ruas and A. Verjovsky. This is explained in my book "On the topology of isolated singularities on Analytic Spaces" (Birkhauser, 2005).

Previous, related, work by E. Looijenga, Church-Lamotke, N. A'Campo, L. Rudolph, Jacquemard ... and of course Milnor.

More recently there are interesting articles on this and related subjects by various authors:
A. Pichon,
A. Bodin,
D. Massey,
M. Oka,
R. Araujo Dos Santos, also by himself and M. Tibar,
Lopez de Medrano (and with L. Hernández)
J. L. Cisneros....

Also (different viewpoint, but much related) by S. Gusein-Zade, I. Luengo and A. Melle-Hernandez, and also D. Siersma and M. Tibăr

I will speak today about recent work with several colleagues (most of them are now here):

Anne Pichon + Se (Math. Ann. 2008)
Anne Pichon, Arnaud Bodin, Se (just appeared on line in J. LMS)
J. L. Cisneros-Molina, J. Snoussi and Se (to appear in Adv. in Maths. and in I.J.M.)
J. L. Cisneros-Molina, J. Snoussi and Se (to appear in I.J.M.)

Let us start with classical case.

§1. A new look on Milnor's fibration theorem

Consider a holomorphic function

$$f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)$$

with a critical point at $\underline{0}$. Let $V = f^{-1}(0)$ and $K = V \cap \mathbb{S}_\varepsilon$ the link. Milnor's classical theorem (1968) says that we have a locally trivial fibration:

$$\phi := \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \longrightarrow \mathbb{S}^1$$

There is another classical way of associating a local fibration to f, which is equivalent to the previous one and it is implicit in Milnor's work. This is usually called also Milnor's fibration:

Given $\varepsilon > 0$ as above, choose $0 < \delta << \varepsilon$ and set $N(\varepsilon, \delta) = f^{-1}(\partial \mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$. Then:

$$f : N(\varepsilon, \delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1$$

is a locally trivial fibration, equivalent to previous one.

These theorems were generalised first by H. Hamm for complete intersections, and also for holomorphic functions defined on complex varieties with an isolated singularity.

More generally, using that holomorphic functions have Thom's a_f-property (Hironaka 1971), Lê Dũng Tráng gave the following generalisation:

Theorem

(Lê): Let $(X, \underline{0})$ *be the germ of a complex analytic variety and* $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ *holomorphic. Set* $V = f^{-1}(0)$, $K = V \cap \mathbb{S}_\varepsilon$ *the link and* $N(\varepsilon, \delta) = f^{-1}(\partial \mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$*. Then:*

$$f : N(\varepsilon, \delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1$$

is a locally trivial fibration.

We call this the **Milnor-Lê fibration** of f.

It is implicit in Lê's work that if $L_X = X \cap \mathbb{S}_\varepsilon$ denotes the link of $\underline{0}$ in X, then one also has a fibration:

$$\phi := \frac{f}{|f|} : L_X \setminus K \longrightarrow \mathbb{S}^1$$

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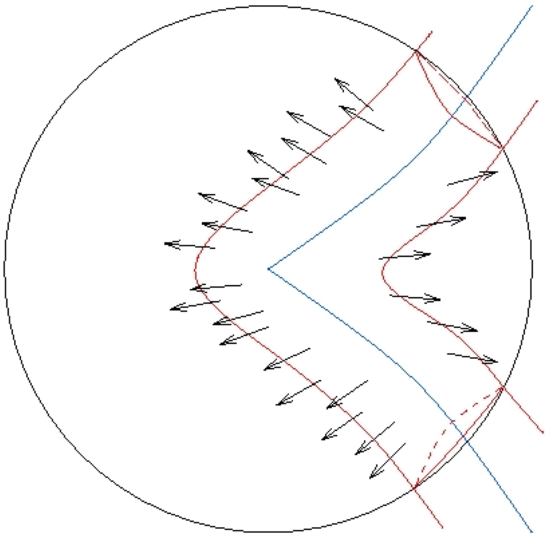

- Now speak about my work with Cisneros-Molina and Snoussi (article in Adv. Math.): we use real singularities to get new insights of the geometry of complex hypersurface singularities.

  As before, consider $(X, \underline{0})$, complex analytic germ, $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ hol. Set $V = f^{-1}(0)$; K the link of f in X.

- For each $\theta \in [0, \pi)$, consider the real line $\mathcal{L}_\theta \subset \mathbb{C}$ passing through the origin with an angle θ with respect to the real axis. Set

$$X_\theta = \{z \in X | f(z) \in \mathcal{L}_\theta\}$$

  Each X_θ is real analytic hypersurface with singular set

$$\operatorname{Sing} V \cup (X_\theta \cap \operatorname{Sing} X).$$

- One has $X \cap \mathbb{B} = \cup X_\theta$ and $V = \cap X_\theta = X_{\theta_1} \cap X_{\theta_2}$

- Call family $\{X_\theta\}$ the **canonical pencil** of f.

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One has:

Theorem (uniform conical structure)

1. *The X_θ are all homeomorphic.*
2. *If $\{S_\alpha\}$ is a Whitney stratification of X adapted to V, then the intersection of the strata with each X_θ determines a Whitney-strong stratification of X_θ.*
3. *There is a uniform conical structure for all X_θ, i.e., a homeomorphism*

$$h\colon (X \cap \mathbb{B}_\varepsilon, V \cap \mathbb{B}_\varepsilon) \to \big(Cone(L_X), Cone(K)\big),$$

which restricted to each X_θ defines a homeomorphism

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- **Idea of proof: It has three main ingredients:**
  a) Every holomorphic *f* has the strict Thom property w.r.t. every Whitney stratification (Hironaka + Teissier and others)
  b) Verdier's theory of stratified rugose vector fields
  c) M. H. Schwartz construction of stratified, radial vector fields.
- Putting these together, can construct (with some extra work!) a stratified, rugose vector field on small ball $\mathbb{B}_\varepsilon \cap X$ s.t.
  i) It is tangent to each X_θ
  ii) It is transversal to all spheres in $\mathbb{B}_\varepsilon$ centered at $\underline{0}$.
  iii) All its solutions converge to $\underline{0}$. □
- Notice each X_θ is a union $X_\theta = E_\theta \cup V \cup E_{\theta+\pi}$ corresponding to the two half lines of $\mathcal{L}_\theta \setminus \{0\}$. We get:

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Theorem (Fibration Theorem)

1. *One has a commutative diagram of fibre bundles*

$$\begin{array}{ccc} (X \cap \mathbb{B}_\varepsilon) \setminus V & \xrightarrow{\Phi} & \mathbb{S}^1 \\ & \searrow^{\Psi} & \downarrow \pi \\ & & \mathbb{RP}^1 \end{array}$$

   where $\Psi(x) = (Re(f(x)) : Im(f(x)))$ *with fibre* $(X_\theta \cap \mathbb{B}_\varepsilon) \setminus V$, $\Phi(x) = \frac{f(x)}{|f(x)|}$ *and* π *is the natural two-fold covering.*

2. *Restriction of* Φ *to the link* $L_X \setminus K$ *is usual Milnor fibration.*

3. *Restriction of* Φ *to the Milnor tube* $f^{-1}(\partial \mathbb{D}_\eta) \cap \mathbb{B}_\varepsilon$ *is the equivalent Milnor-Lê fibration.*

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Idea of pf.

1. a) Use uniform conical structure to construct a complete vector field on $(X \cap \mathbb{B}_\varepsilon) \setminus V$ which is tangent to each sphere centered at $\underline{0}$ and transversal to all $X_\theta \Rightarrow$ fibration restricted to each sphere.
2. b) Use the previous vector field (that gives conical structure) to get fibration on radial direction.
3. c) Put the two previous vector fields together in $(X \cap \mathbb{B}_\varepsilon) \setminus V$ to construct a stratified, rugose vector field which is tangent to each X_θ, transversal to all spheres and transversal to all Milnor tubes $f^{-1}(\partial \mathbb{D}_\delta)$. This gives equivalence of the two fibrations.

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Our proofs actually show:

Corollary

Assume germ $(X, \underline{0})$ is irreducible and consider its Milnor fibration $\phi = \frac{f}{|f|} : L_X \setminus K \longrightarrow \mathbb{S}^1$. Then:

1. *Every pair of fibres of ϕ over antipodal points of $\mathbb{S}^1$ are glued together along K producing the link of X_θ, which is homeo. to link of $\{ Re\, f = 0 \}$.*
2. *If X and f have isolated singularity at $\underline{0}$, then homeo is in fact diffeo and the link of each X_θ is diffeomorphic to the double of the Milnor fibre of f (regarded as a smooth manifold with boundary K).*

Example: for map $(z_1, z_2) \overset{f}{\mapsto} z_1^2 + z_2^q$ one gets that the link of $(Re\, f)$ is a closed, oriented surface in the 3-sphere of genus $q - 1$.

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Notice fibres of ψ are the $X_\theta \setminus V$; each "half" is a fiber of Φ. This can be improved: Consider real analytic map

$$\begin{array}{rccc} \Psi: & X \setminus V & \to & \mathbb{RP}^1 \\ & z & \mapsto & (Re(f(z)) : Im(f(z)))\,. \end{array}$$

Let $\tilde{X}$ be the analytic set in $X \times \mathbb{RP}^1$ defined by $Re(f)t_2 - Im(f)t_1 = 0$, where $(t_1 : t_2)$ denotes points in $\mathbb{RP}^1$

The first projection induces a real analytic map:

$$e_V : \tilde{X} \to X\,;$$

this is the real blow-up of V in X (Mather). It induces a real analytic isomorphism $\tilde{X} \setminus e_V^{-1}(V) \cong X \setminus V$.

Inverse image of V by e_V is $V \times \mathbb{RP}^1$.

We prove:

Theorem

Let $\tilde{X}$ be the space obtained by the real blow-up of V. Then:

The projection $\tilde{\Psi} : \tilde{X} \to \mathbb{RP}^1$ is a topological fibre bundle with fibre X_θ.

Restriction of projection to $\tilde{X} \setminus e_V^{-1}(V) \cong X \setminus V$ coincides with Ψ.

So the blow up $\tilde{X}$, and the map $\tilde{\Psi}$ provide **a compactification** of the Milnor fibration of f

§2. Milnor fibrations for real singularities

Subject goes back to Milnor's book . Consider real analytic map-germ

$$f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)\ ,\ n \geq p\,.$$

Milnor proved that if f is a submersion on a punctured neighbourhood of the origin $\underline{0} \in \mathbb{R}^n$, then one has a fibre bundle:

$$\varphi\colon \mathbb{S}_\varepsilon^{n-1} \setminus K_\varepsilon \to \mathbb{S}^{p-1}\,,$$

where $K_\varepsilon = f^{-1}(0) \cap \mathbb{S}_\varepsilon^{n-1}$ is the link.

Idea: iso. crit. pt. $\Rightarrow$ fibration on a tube. Then Curve Sel. Lemma yields to vector field that inflates tube into sphere.

Two main weaknesses (both pointed out by Milnor):

- 1) Too stringent: Map-germs satisfying Milnor's condition are non-generic. (Work by Looijenga, Church-Lamotke determines possible pairs (n, p). Also A'Campo, Perron, L. Rudolph)
- 2) No control on projection map: unlike complex case where φ is $f/|f|$, now the proof only gives the existence of the projection map φ but says nothing about the mapping itself.
- Work by many authors around producing examples and/or conditions to insure that a real analytic map germ has a Milnor fibration on a sphere with projection map $f/|f|$.
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- First explicit families of singularities having a Milnor fibration on spheres with projection map $f/|f|$ were by Seade (1997):

$$(z_1, ..., z_n) \mapsto z_1^{a_1}\bar{z}_{\sigma(1)} + ... + z_n^{a_n}\bar{z}_{\sigma(n)}\ ,\ a_i \geq 2,$$

where σ a permutation of the set $\{1, ..., n\}$

- If σ is identity, these are topologically equivalent to usual Pham-Brieskorn sing. (Ruas-S-Verjovsky)

$$z_1^{a_1-1} + ... + z_n^{a_n-1}\ .$$

Hence called **Twisted Pham-Brieskorn singularities**

- Singularities inspired by "moment angle manifolds" studied by Lopez de Medrano et al: those correspond to linear vector fields (more general actions of $\mathbb{C}^k$); these correspond to vector flds. $(z_{\sigma(1)}^{a_1}, ..., z_{\sigma(n)}^{a_n})$. Rich geometry behind the screen!

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$$(z_1, ..., z_n) \mapsto z_1^{a_1}\bar{z}_{\sigma(1)} + ... + z_n^{a_n}\bar{z}_{\sigma(n)}\ ,\ a_i \geq 2,$$

  where σ a permutation of the set $\{1, ..., n\}$
- If σ is identity, these are topologically equivalent to usual Pham-Brieskorn sing. (Ruas-S-Verjovsky)

$$z_1^{a_1-1} + ... + z_n^{a_n-1}\ .$$

  Hence called **Twisted Pham-Brieskorn singularities**
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- Main theorem gives an equivalence: $f\bar{g}$ has isolated critical value $\iff$ *multilink* $L_f \cup -L_g$ *is fibered* (in the sense of Eisenbud-Neumann) $\iff$ given any resolution of the holomorphic germ fg, then for each rupture vertex (j) of the decorated dual graph one has $m_j^f \neq m_j^g$.
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Milnor fibrations for meromorphic germs

- A specially interesting class of real analytic functions are $h = f\bar{g}$ with f, g holomorphic (Lee Rudolph, Pichon-Seade)

  Notice that away from $\{g = 0\}$ one has (noticed by L. Rudolph):

$$\frac{f\bar{g}}{|f\bar{g}|} = \frac{f\bar{g}(g/g)}{|f\bar{g}(g/g)|} = \frac{|g|^2 f/g}{|g|^2|f/g|} = \frac{f/g}{|f/g|} \,,$$

  Hence germs $f\bar{g}$ are much related to meromorphic germs f/g.

- From different but analogous point of view, interesting works (1998-...) by Gusein-Zade, Luengo and Melle for meromorphic germs (also Siersma and Tibar), and more recently by Arnoud Bodin, Anne Pichon and myself, for meromorphic germs.

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Let U open neigbh. of $0 \in \mathbb{C}^n$ and let $f, g : U \longrightarrow \mathbb{C}$ be holomorphic without common factors, such that $f(0) = g(0) = 0$. Indetermination locus is

$$I = \{z \in U \mid f(x) = 0 \text{ and } g(x) = 0\} .$$

Consider meromorphic map:

$$F = f/g : U \setminus I \to \mathbb{C}P^1$$

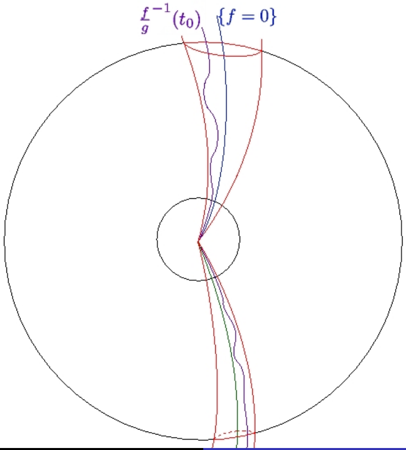
For each $c \in \mathbb{C}$ the fiber $F^{-1}(c)$ is

$$F^{-1}(c) = \{x \in U \mid f(x) - cg(x) = 0\} \setminus I .$$

Articles by S. M. Gusein-Zade et al studied fibrations on "local Milnor tubes" $\mathbb{B}_\epsilon \cap F^{-1}(\partial \mathbb{D}_\delta)$ around critical values.

These are not actual tubes since they may contain $0 \in U$ in their closure. These are "pinched tubes".

Natural to ask whether one has for meromorphic maps, fibrations of Milnor type ”on the sphere”.



This question was addressed independently by Bodin-Pichon (Math. Res. Lett. 2007) and Pichon-Seade (Math. Ann. 2008), from different viewpoints. One has:

Theorem (Bodin-Pichon)

If the meromorphic germ $F = f/g$ is semitame, then

$$\frac{F}{|F|} = \frac{f/g}{|f/g|} : \mathbb{S}_\epsilon \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1 \tag{1}$$

is a fiber bundle, where $L_f = \{f = 0\} \cap \mathbb{S}_\epsilon$ and $L_g = \{g = 0\} \cap \mathbb{S}_\epsilon$ are the oriented links of f and g.

For semi-tameness: set

$$M(F) = \{x \in U \setminus I \mid \exists \lambda \in \mathbb{C}, \mathrm{grad}(f/g)(x) = \lambda x\}$$

This consists of the points of "non-transversality" between the fibres of f/g and the spheres $\mathbb{S}_r$ centered at the origin of $\mathbb{C}^n$.

Now let B be the set of all $c \in \mathbb{C}P^1$ such that there exists a sequence of points $(x_k)_{k \in \mathbb{N}}$ in $M(F)$ such that

$$\lim_{k \to \infty} x_k = 0 \quad \text{and} \quad \lim_{k \to \infty} F(x_k) = c\,.$$

Then **semitame** means that B is contained in the set $\{0, \infty\}$.

Pichon-Seade proved that if map $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$ and the Thom a_f property, then one has the Milnor-Lê fibration of $f\bar{g}$,

$$N(\epsilon,\delta) := [\mathbb{B}_\epsilon \cap (f\bar{g})^{-1}(\partial \mathbb{D}_\delta)] \xrightarrow{f\bar{g}} \partial \mathbb{D}_\delta \cong \mathbb{S}^1 \,, \tag{2}$$

and when f/g is semitame, this fibration is equivalent to the Milnor fibration on the sphere, given by $\frac{f\bar{g}}{|f\bar{g}|}$.

Then next natural question is: how these fibrations on the sphere relate to the fibrations on local Milnor (pinched) tubes studied by Gusein-Zade et al. We have the following theorem (Bodin-Pichon-Seade, to appear in Journal L.M.S.):

Theorem

If the germ f/g is semitame and (IND)-tame, then the Milnor fibration for f/g on the sphere is obtained from the local Milnor fibrations of f/g at 0 and ∞ by a gluing process, which is a fiberwise reminiscent of the classical connected sum of manifolds

(IND)-tame stands for "tameness" near the indetermination locus.

- In other words, if f/g is semitame and $(IND)-tame$, to get the Milnor fibre F_ϕ for $\phi = \frac{f\bar{g}}{|f\bar{g}|}$ on $\mathbb{S}_\varepsilon$ we do as follows:
- a) We take fibres $F_0 = (f/g)^{-1}(t_0) \cap \mathbb{B}_\varepsilon$, $F_\infty = (f/g)^{-1}(t_\infty) \cap \mathbb{B}_\varepsilon$ for t_0 and t_∞ "sufficiently close" to $0, \infty$ respectively.
- b) We take $0 < \varepsilon' << \varepsilon$, and consider $\widehat{F}_0 := F_0 \setminus Int(F_0 \cap \mathbb{S}_{\varepsilon'})$, and $\widehat{F}_\infty := F_\infty \setminus Int(F_\infty \cap \mathbb{S}_{\varepsilon'})$.
- c) One has a common boundary component $\widehat{F}_0 \cap \mathbb{S}_{\varepsilon'} \cong \widehat{F}_\infty \cap \mathbb{S}_{\varepsilon'}$ and F_ϕ is obtained by gluing $\widehat{F}_0$ and $\widehat{F}_\infty$ along their intersection with $\mathbb{S}_{\varepsilon'}$.

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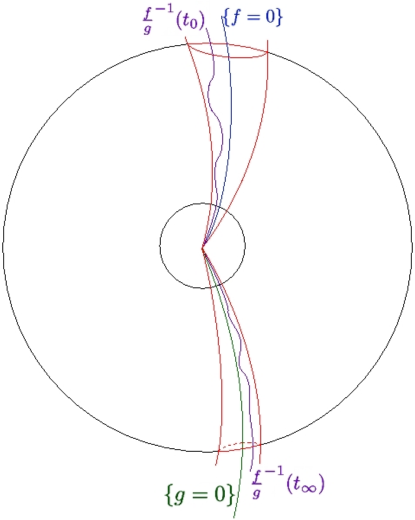
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$\{f = 0\}$
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General case of real analytic map-germs

We return to the general case of real analytic map-germs $(\mathbb{R}^n, \underline{0}) \xrightarrow{f} (\mathbb{R}^p, 0)$, $n \geq p$. Three natural questions:

Question (Q1)

When does such map-germ has a Milnor-Lê fibration:
$f : N(\varepsilon, \delta) \to \partial\mathbb{D}_\delta$ *?; $N(\varepsilon, \delta)$ is the Milnor tube* $f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$.

Question (Q2)

When does such map-germ has a Milnor fibration:
$\frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^{p-1}$ *?*

Question (Q3)

If both fibrations exist, when are they equivalent?

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Q1 arises in Milnor's work for *f* with isolated critical point, and from Pichon-Seade [PS] for *f* with isolated critical value.

In isolated singularity case the Milnor-Lê fibration exists always. For isolated critical value, it was noticed in [PS] that if $dim V > 0$, then Thom a_f property $\Rightarrow$ fibration on tube.

Question (Q4)

If $dim V > 0$, does transversality of the intersection $f^{-1}(\partial \mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$, for all $\delta > 0$ sufficiently small, imply Thom's a_f property ? We do not know the answer.

- Q2 was first studied by Jacquemard (1989). Seade (1997) studied the Twisted Pham-Brieskorn singularities mentioned above $z_1^{a_1}\bar{z}_{\sigma(1)} + ... + z_n^{a_n}\bar{z}_{\sigma(n)}$. These opened a gate.

  Work by Ruas-Seade-Verjovsky, then Ruas- Dos Santos, and more recently work by Dos Santos (also with Tibar), Cisneros-Molina (Polar weighted), M. Oka, D. Massey. (& my work with Cisneros-M. and Snoussi, that I'll speak about)
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With Cisneros-Molina and Snoussi (to appear in Int. J. Maths.) we proved:

Theorem

If the two fibrations exist, then they are equivalent.

This result is a consequence of the theory we develop, which is analogous to what I explained above for holomorphic germs.

- **Basic question is**:

  What is the underlying structure, existing in the holomorphic, meromorphic and other settings, that makes it possible to have a Milnor fibration, and the equivalence of the two fibrations, when the two of them exist?

- **Answer is: *d*-regularity.**
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- **Answer is: *d*-regularity.**
- Let me explain.

- Consider real analytic map-germ $f\colon (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^p, 0)$ with critical point at $\underline{0}$; a submersion at each $x \notin V = f^{-1}(0)$.

  Equip V with a Whitney stratification and let $\mathbb{B}_\varepsilon$ be small open ball in $\mathbb{R}^n$, centred at $\underline{0}$: every sphere in this ball, centred at $\underline{0}$, meets transversely every stratum of V. (Bertini-Sard theorem)

- Define a family of real analytic spaces: For each $\ell \in \mathbb{RP}^{p-1}$, consider the line $\mathcal{L}_\ell \subset \mathbb{R}^p$ through the origin defined by ℓ and set
$$X_\ell = \{x \in U \,|\, f(x) \in \mathcal{L}_\ell\}\,.$$
- $\{X_\ell\}$ is a family of real analytic varieties parametrised by $\mathbb{RP}^{p-1}$. One has that: i) each $\{X_\ell\} \setminus V$ is non-singular; ii) they all meet at V; and iii) their union is $\mathbb{R}^n$.

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Definition

The family $\{X_\ell \,|\, \ell \in \mathbb{P}^{p-1}\}$ is **the canonical pencil** of f.

Definition

f is d**-regular** if $\exists\, \varepsilon > 0$ small, such that each sphere in $\mathbb{R}^n$ centred at $\underline{0}$ of radius $\leq \varepsilon$ meets transversally every manifold $X_\ell \setminus V$.

Examples: a) Every holomorphic map-germ into $\mathbb{C}$ is d-regular.

b) If meromorphic germ f/g is semitame and $f\bar{g}$ has isolated critical value, then $f\bar{g}$ is d-regular.

c) Polar weighted and real quasi-homogeneous germs are d-regular.

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Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be real analytic and a submersion away from $V = f^{-1}(0)$. The following are equivalent:

1. *The map f is d-regular at 0.*
2. *One has a commutative diagram of smooth fibre bundles*

$$\begin{array}{ccc} \mathbb{B}_\varepsilon \setminus V & \xrightarrow{\Phi = \frac{f}{|f|}} & \mathbb{S}^{p-1} \\ & \searrow^{\Psi} & \downarrow \pi \\ & & \mathbb{RP}^{p-1} \end{array}$$

   where $\Psi(x) = (f_1(x) : \cdots : f_p(x))$ has fibre $(X_\ell \cap \mathbb{B}_\varepsilon) \setminus V$; Φ and Ψ being fibrations over their images.
3. *Let $K_\eta = V \cap \mathbb{S}^{n-1}_\eta$. Restricting to $\mathbb{S}^{n-1}_\eta \setminus K_\eta$ one has a corresponding diagram of smooth fibre bundles on*

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   where $\Psi(x) = (f_1(x) : \cdots : f_p(x))$ has fibre $(X_\ell \cap \mathbb{B}_\varepsilon) \setminus V$; Φ and Ψ being fibrations over their images.
3. *Let $K_\eta = V \cap \mathbb{S}_\eta^{n-1}$. Restricting to $\mathbb{S}_\eta^{n-1} \setminus K_\eta$ one has a corresponding diagram of smooth fibre bundles on*

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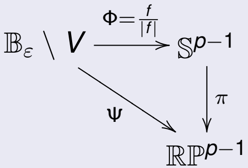

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- Work by our joint Ph. D. student **Haydeé Aguilar**: consider $\mathbb{C}^3 \xrightarrow{f} \mathbb{C}$ defined by:

$$(z_1, z_2, z_3) \to \bar{z}_1 \bar{z}_2 (z_1^p + z_2^q) + z_3^r \,, \quad \text{with } p, q, r \geq 2$$

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Also, A. Pichon recently found an example of real anal. map $\mathbb{C}^3 \to \mathbb{C}$ such that:

a) Its link K is the link of a complex hypersurface iso. sing. in $\mathbb{C}^3$, but:

b) Its open book can not be defined by a holomorphic map. (More precisely, its Milnor fibres can not be the fibres of a smoothing of any normal, Gorenstein, smoothable surface singularity: this uses the Laufer-Steenbrink formula)

Several other problems being worked out, for instance in relation with contact structures, etc.

Summarizing:

- Much of the development of singulatities theory in 1960s and 1970s came from problems in manifolds theory (producing "exotic" differentiable manifolds, open-books, foliations theory (work of Lawson-Durfee), etc.) For this, complex (hypersurface) singularities, and Milnor's fibration, played key-roles.

- Now we have a certain amount of knowledge about open-books and Milnor fibrations associated to real analytic singularities.

- Can this be used to say something new in manifolds theory?

MUCHAS GRACIAS

y

FELICIDADES TOLYA!!!!