Rigidity and Chern numbers of singular varieties

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Rigid Operators

Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator commuting with an S^1 action. D is **rigid** if

$$\mathrm{Ind}_{S^1} D = \mathrm{Tr}\ e^{i\theta}|_{\ker D} - \mathrm{Tr}\ e^{i\theta}|_{\mathrm{coker} D} = const$$

Example: $D = d + d^* : \Omega^{even}(M) \to \Omega^{odd}(M)$. Then

$$\ker D = H^{even}(M)$$
$$\mathrm{coker} D = H^{odd}(M)$$

A rotation on M acts trivially on the cohomology. So $\mathrm{Ind}_{S^1} D \equiv \chi(M)$.

In more interesting cases, the S^1 action may be non-trivial on $\ker D$ and $\mathrm{coker} D$, even when $\mathrm{Ind}_{S^1} D = const$.

Example: (Atiyah-Hirzebruch)

If M is spin, and $\Delta_\pm$ the spin bundles, then the Dirac operator $D : \Gamma(\Delta_+) \to \Gamma(\Delta_-)$ is rigid.

Interpretation from Physics

Index theory arises naturally in super-symmetric quantum mechanics.

Hilbert space of states $\to L^2$ sections of a vb
Supercharges $\to$ Dirac operators
Ground state $\to$ Harmonic sections the vb

In passing to super-string theory, the left side still makes sense. The right side should be interpreted as Dirac operators and vectorbundles on the *loop space* $\mathcal{L}M$.

The loop space $\mathcal{L}M$ comes equipped with a natural circle action whose fixed points are in $1-1$ correspondence with M.

If $D_{\mathcal{L}}$ is a Dirac operator on $\mathcal{L}M$, then by formal application of the Atiyah-Bott-Lefschetz fixed point formula:

$$\text{Ind} D_{\mathcal{L}} = \int_M \widehat{A}(M)\text{ch}(E) = \text{Ind} \not{D} \otimes E$$

for some complicated vectorbundle E associated to the normal bundle of M in $\mathcal{L}M$.

From this physics perspective, Witten predicted the rigidity of $\not{D} \otimes E$ for a number of exotic vectorbundles E.

For $\not{D} \otimes E$ to make sense as a Dirac operator on $\mathcal{L}M$, M must satisfy various topological constraints.

Example: To define the signature operator on the loop space, $\mathcal{L}M$ must be orientable, so M must be spin.

Complex elliptic genus: $\mathrm{Ind}\bar{\partial} \otimes E_{q,y}$.

$$E_{q,y} = y^{-n/2} \bigotimes_{n=1}^{\infty} \Lambda_{-yq^{n-1}} T''M \otimes \Lambda_{-yq^n} T'M \otimes \bigotimes_{m=1}^{\infty} S_{q^m} T''M \otimes S_{q^m} T'M$$

An operator on the loop space when $c_1(M) = 0$.

Riemann Roch formula: $\mathrm{Ind}\bar{\partial} \otimes E_{q,y} = \mathrm{Ellip}(M) \cap [M]$ where

$$\mathrm{Ellip}(M) = \prod_j \frac{x_j \vartheta(\frac{x_j}{2\pi i} - z, \tau)}{\vartheta(\frac{x_j}{2\pi i}, \tau)}$$

(Liu, Witten): $\overline{\partial} \otimes E_{q,y}$ is rigid when $c_1(M) = 0$.

Generalization for toric varieties: Make replacements

$$\Lambda_{-yq^n} T'M \to \bigotimes_{i=1}^{\ell} \Lambda_{-y^{a_i}q^n} \mathcal{O}(D_i)$$
$$S_{q^n} T'M \to \bigotimes_{i=1}^{\ell} S_{q^n} \mathcal{O}(D_i), ...$$

etc. D_i are the toric divisors.

Call this $E_{q,y}(a_1, ..., a_\ell)$.

(Hattori, W): $\overline{\partial} \otimes E_{q,y}(a_1, ..., a_\ell)$ is rigid when $a_1 D_1 + ... + a_\ell D_\ell = 0$. Moreover, the equivariant index vanishes identically.

Chern numbers of singular varieties

X almost complex. The Chern numbers of X are:

$$c_{i_1,\ldots,i_n} = \int_X c_1(X)^{i_1} \cdots c_n(X)^{i_n}$$

where $i_1 + 2i_2 ... + ni_n = \dim X$.

Question: what combinations of Chern numbers make sense for singular varieties X?

Partial answer: At minimum, all Chern numbers encoded in the Todd-genus (by Hartog).

Naive approach: Find a nice birational model for X, use Chern numbers from that model.

Different minimal models should be related by flops; this led Totaro to ask:

What combinations of Chern numbers are invariant under classical flops?

Ans: The Chern numbers encoded in the complex elliptic genus.

Sketch of proof: If X_1 and X_2 differ by a classical flop, $X_1 - X_2$ is complex cobordant to a $\mathbb{P}^3$-fibration of the form $\mathbb{P}(E)$, where $E \to B$ is a vb. and the almost complex structure on the fibers is defined so that $c_1(\mathbb{P}^3) = 0$.

Fibration tangent bundle splits as $TB \oplus N$, where $N = T\mathbb{P}^3$ fiberwise. The integrand of the elliptic genus splits as:

$$\text{Ellip}(B) \cdot \text{Ellip}(N)$$

Integrate this first along the fiber. Ellip(N) evaluates to the equivariant elliptic genus of $\mathbb{P}^3$ with resp. to the standard torus action, where we evaluate the equivariant weights at the Chern roots of E.

By rigidity, this equals 0. Consequently

$$\text{Ellip}(X_1)[X_1] = \text{Ellip}(X_2)[X_2]$$

If the singularities of X look locally like the product of a smooth variety B and the 3-fold node, the two minimal resolutions of X are related by a classical flop. Consequently, the elliptic genus of X (and corresponding combinations of Chern numbers) is well-defined.

Question: What about more general singular varieties?

Borisov and Libgober introduce elliptic class of pairs (X, D) :

For $D = \sum a_i D_i$, the elliptic class of (X, D) is $\mathrm{Ellip}(X, D) =$

$$\prod_j \frac{x_j \vartheta(\frac{x_j}{2\pi i} - z, \tau)}{\vartheta(\frac{x_j}{2\pi i}, \tau)} \times$$
$$\prod_i \frac{\vartheta(\frac{D_i}{2\pi i} - (a_i + 1)z, \tau)\vartheta(z, \tau)}{\vartheta(\frac{D_i}{2\pi i} - z, \tau)\vartheta((a_i + 1)z, \tau)}$$

This satisfies the push-forward formula:

$$f_* \mathrm{Ellip}(X_1, D_1) = \mathrm{Ellip}(X_2, D_2)$$

where f a blow-up and

$$K_{X_1} - D_1 = f^*(K_{X_2} - D_2).$$

Proof is closely connected to rigidity of toric elliptic genus.

By result of Wlodarczyk, birational spaces can be connected by a sequence of blow-ups and blow-downs. Can therefore define **singular elliptic genus** as

$$\mathrm{Ellip}(X, D) \cap [X],$$

where X is a resolution with exceptional divisor D.

Works for log-terminal singularities, since the coefficients $a_i + 1 \neq 0$.

Question: What about when some discrepancies $a_i = -1$? How do we define $\mathrm{Ellip}(X, D) \cap [X]$?

Solution for normal surface singularities that are not strictly log-canonical:

Exceptional components with -1 discrep. have locally toric structure.

Introduce perturbation $a_i + \varepsilon b_i$ to coefficients of excep. divisor D, where b_i satisfy:

$$\sum b_i D_i \cdot D_j = 0$$

whenever $a_j = -1$. Take

$$\lim_{\varepsilon \to 0} \mathrm{Ellip}(X, D(\varepsilon)) \cap [X]$$

Limit exists and is independent of choice of b_i.

Sketch of proof: Assume one component E with -1 discrep. $U_E \hookrightarrow X$ toric embedding of an analytic neighborhood of E.

Let $D_1(\varepsilon)$ and $D_2(\varepsilon)$ be two possible perturbation divisors.

$$\lim_{\varepsilon\to 0} \text{Ellip}(X, D_1(\varepsilon)) \cap [X] - \text{Ellip}(X, D_2(\varepsilon)) \cap [X]$$

$$= \lim_{\varepsilon\to 0} \text{Ind } \overline{\partial} \otimes E_{q,y}(\vec{a}_1(\varepsilon)) - \text{Ind } \overline{\partial} \otimes E_{q,y}(\vec{a}_2(\varepsilon))$$

The above vb's are defined on X, and $\vec{a}_i(\varepsilon)$ are defined so that the corresponding divisors have vanishing 1st Chern class.

By rigidity, both indices vanish for all ε.

Overall Theme

Defining singular Chern data using a smooth birational model introduces a redundancy in descriptions.

Difference between two possible descriptions is encoded by data coming from a toric CY divisor pair.

By rigidity, contribution from this toric data vanishes.