L compact, oriented 3-manifold

Is $L = \partial M$, M $\underline{\mathbb{Q}HD}$ ("rational homology disk")?

$S^3 = \partial D^4$

Cohomological considerations
$\Rightarrow$ L is $\underline{\mathbb{Q}HS}$ ("rational homology sphere"), order of $H_1(L)$ is a square

Examples: Many lens spaces S^3/C_n (Casson-Harer, c. 1980)

- M could be $\mathbb{Z}HS$

Other examples as well

Filling a link via <u>smoothing</u>:

$$\begin{array}{c}(\mathcal{X},0)\\ \downarrow f\\ (D,0)\end{array}$$

$f^{-1}(0) \simeq X$

$f^{-1}(t)$ non-singular $(t \neq 0)$

M = Milnor fibre = $f^{-1}(t) \cap B_\epsilon$

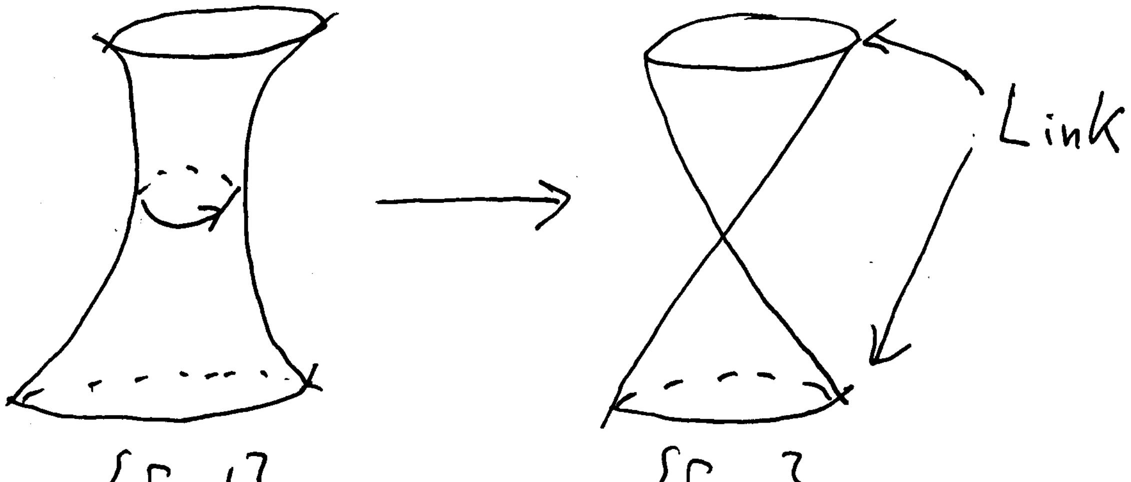
$\partial M = L$

M is <u>Stein manifold</u>

<u>Original example</u>

$\mathcal{X} = \mathbb{C}^3$, X hypersurface sing. $\{f=0\}$

$\Rightarrow M$ simply connected, $\mu = \mathrm{rk}\, H_2(M) \neq 0$.

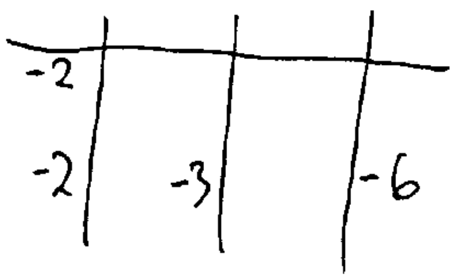


Algebraic geometry version:

$(X,0)\subset(\mathbb{C}^N,0)$ germ of complex NSS ("normal surface singularity")

$L=(X,0)\cap S_\epsilon^{2N-1}$ "link" - 3-manifold

L $\mathbb{Q}HS \Leftrightarrow$ if $\pi:(\tilde{X},E)\to(X,0)$ resolution, E is tree of rational curves



$\longleftrightarrow$ Graph Γ

Γ determines topology of L (+ conversely)

Note: $\partial\tilde{X}=L$, but never $\mathbb{Q}HD$ (for $E\neq\emptyset$)

QUESTION: Which $(X,0)$ have a smoothing with $\mathbb{Q}$HD Milnor fibre?

Examine cyclic quotient singularities
($=$ lens space L)

$0 < r < n$, $(r,n)=1$, $\gamma = \exp(2\pi i/n)$

$$X_{n,r} \equiv \mathbb{C}^2 \Big/ \left\langle \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^r \end{pmatrix} \right\rangle = \text{"}n/r \text{ quotient"}$$

Smoothings?

$\tilde{X}_{n,r}$ always Milnor fibre
(Atiyah, Brieskorn, Artin)

Pinkham (1974) - deformations of $X_{n,1}$ = cone over rat'l normal curve in $\mathbb{P}^n$

Filling a link via <u>smoothing</u>:

$$\begin{array}{l} (\mathcal{X}, o) \\ \downarrow f \\ (D, o) \end{array}$$

$f^{-1}(0) \simeq X$

$f^{-1}(t)$ non-singular $(t \neq 0)$

$M =$ Milnor fibre $= f^{-1}(t) \cap B_\epsilon$

$\partial M = L$

M is <u>Stein manifold</u>

<u>Original example</u>

$\mathcal{X} = \mathbb{C}^3$, X hypersurface sing. $\{f = 0\}$

$\Rightarrow M$ simply connected, $\mu = \mathrm{rk}\, H_2(M) \neq 0$.

BUT $X_{4,1}$ has special smoothing

(Veronese $\mathbb{P}^2 \subset \mathbb{P}^5$ is cause!)

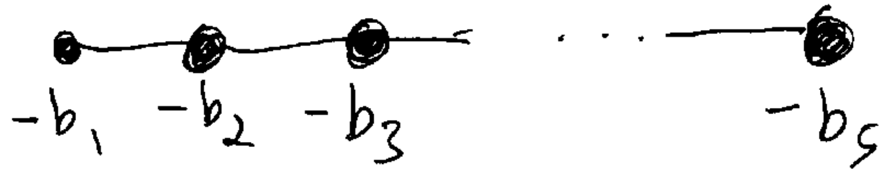
Find

$$M = \mathbb{P}^2 - \text{Quadric} - \text{a QHD}.$$

QUESTION:

Any others?

Resolution graphs



Theorem: (W(1980), Looijenga-W (1986)).

$X_{n,r}$ has a $\mathbb{Q}HD$ smoothing

$$\iff n = p^2, \; r = pq - 1, \; (p,q) = 1.$$

[Later called "$\mathbb{Q}$-Gorenstein smoothings"]

Examples: $p^2/pq-1$

$4/1 \qquad \underset{-4}{\circ} \qquad$ (Pinkham)

$9/2 \qquad \underset{-5}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{-2}{\circ}$

Theorem: (W (1980), Looijenga–W (1986))

$X_{n,r}$ has a $\mathbb{Q}HD$ smoothing

$\Leftrightarrow n = p^2,\ r = pq-1\ ,\ (p,q)=1$.

[Later called "$\mathbb{Q}$-Gorenstein smoothings"]

Examples: $p^2/pq-1$ (Pinkham)

$4/1 \qquad \underset{-4}{\circ}$

$9/2 \qquad \underset{-5}{\circ}\!\!-\!\!\!-\!\!\underset{-2}{\circ}$

Proof: $\Leftarrow$ Let $\zeta = \exp(2\pi i/p)$, $G = \left\langle \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & \zeta^{q} \end{pmatrix} \right\rangle$.

G acts freely on $\mathbb{C}^3 - \{\underline{0}\}$, $f = xy - z^p$ invariant

$$\begin{tikzcd}
\mathbb{C}^3 \arrow[d] & \supset \{f=0\} \arrow[d] & \doteq A_{p-1} \arrow[d] & \chi_{top}(M) = 1 + (p-1) = p \\
\mathbb{C}^3/G \arrow[d, "f = xy - z^p"] & \supset \{f=0\}/G \arrow[d] & \doteq X_{p^2,\, pq-1} & \chi_{top}(M/G) = \frac{p}{|G|} = 1 \\
\mathbb{C} & \supset \{0\} & &
\end{tikzcd}$$

$\pi_1(M/G) = G$

$\Rightarrow M/G$ is $\mathbb{Q}HD$

$\Rightarrow \exists$ invariant $K \cdot K \in \mathbb{Q}$, must be in $\mathbb{Z}$.

Generalize quotient construction:

$(\mathcal{X}, 0)$ isolated, Cohen-Macaulay, dim 3

G finite gp. of autom. of $\mathcal{X}$, free off $\{0\}$.

$f : \mathcal{X} \to \mathbb{C}$ G-invariant

Same diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d] & \supset \{f^{-1}(0)\} \arrow[d] \\ \mathcal{X}/G \arrow[d] & \supset \{f^{-1}(0)\}/G \arrow[d] \\ \mathbb{C} & \ni \{0\} \end{tikzcd}$$

, G acts freely on $M_f = M$

$\chi_{top}(M/G) = \chi_{top}(M)/|G|$

Want: $|G| = \chi_{top}(M)$.

Bonus:

If M simply connected,

new Milnor fibre has known π_1,

perhaps quotient of known space.

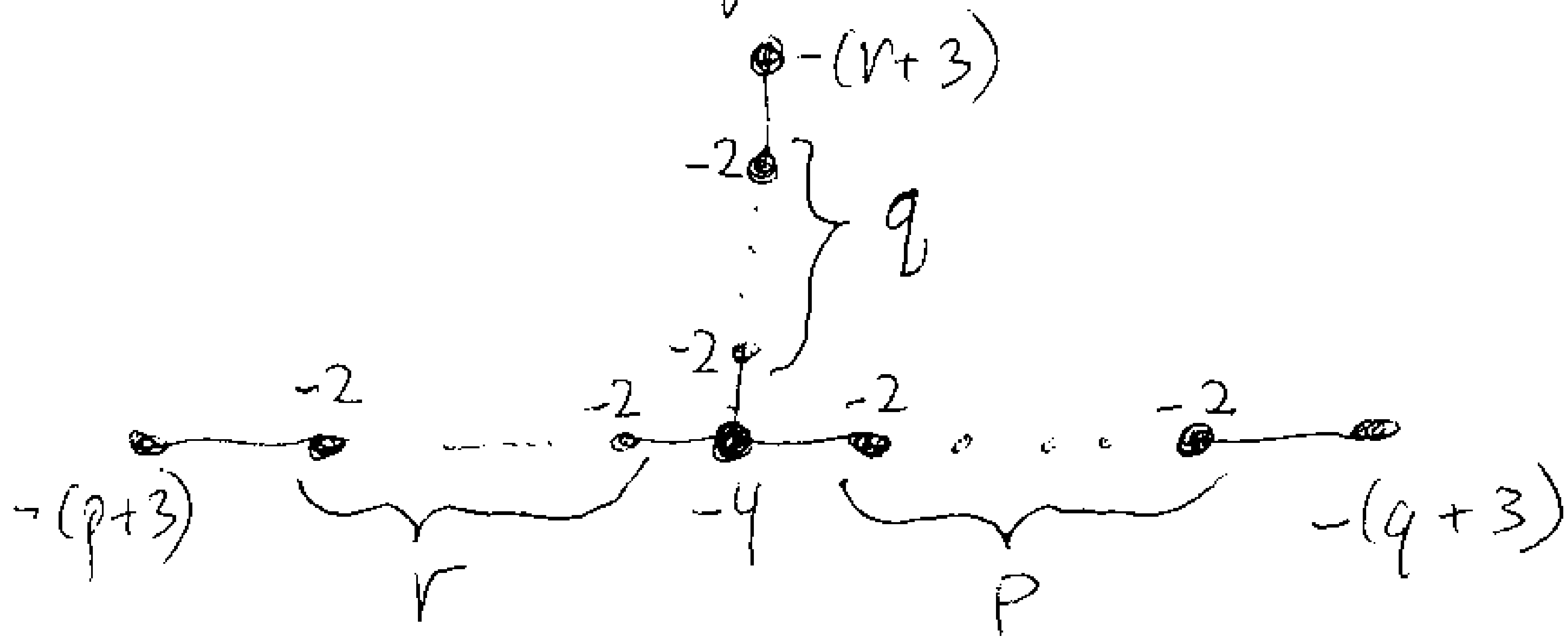
Example (W, 1981).

$\mathcal{X} = \mathbb{C}^3$, $f = xy^{p+2} + yz^{q+2} + zx^{r+2}$, $p, q, r \geq 0$

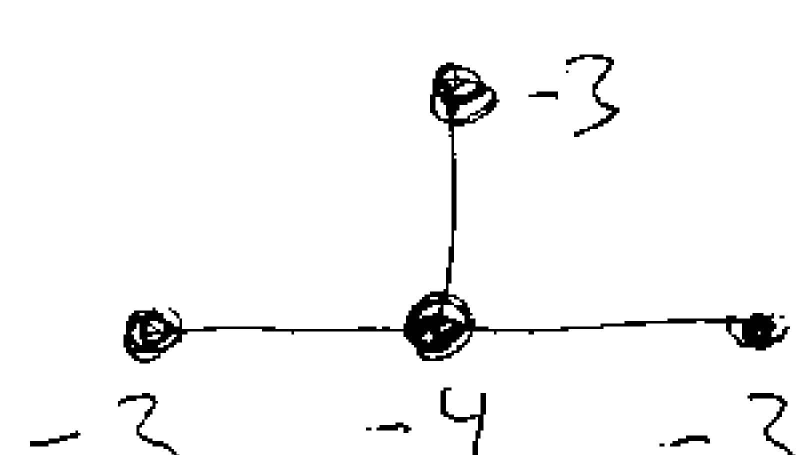
G appropriate diagonal cyclic group
(of order $\mu(f) + 1 = (p+2)(q+2)(r+2) + 1$)

$\Rightarrow \{f=0\}/G$ as $\mathbb{Q}HD$ smoothing

Resolution graph



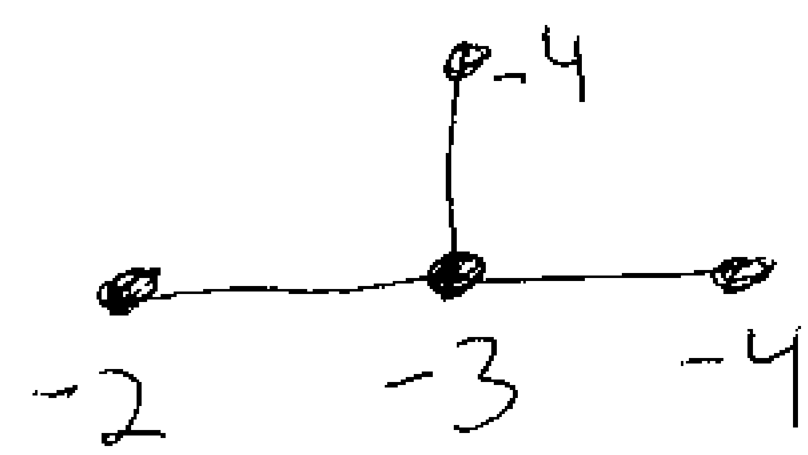
Triply infinite family, based on $p = q = r = 0$



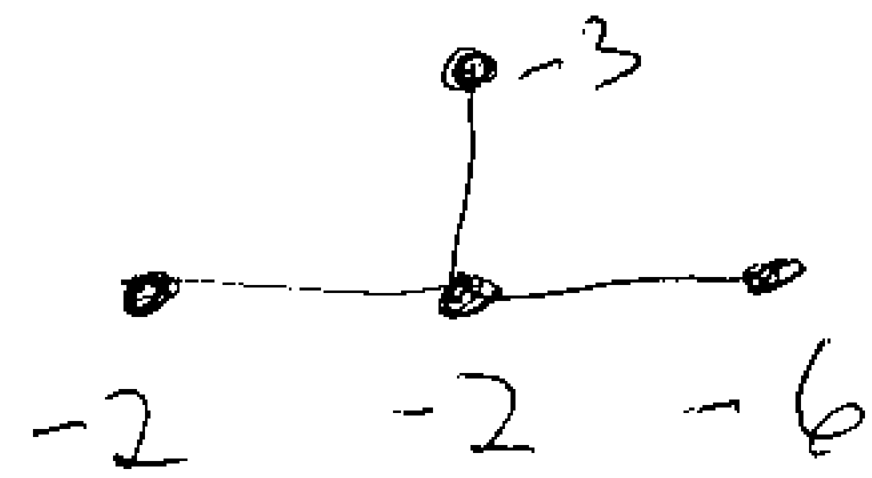
Unpublished examples (early 1980's)

Two other triply-infinite families:

Type $\mathcal{N}$, based on



Type $\mathcal{M}$, based on

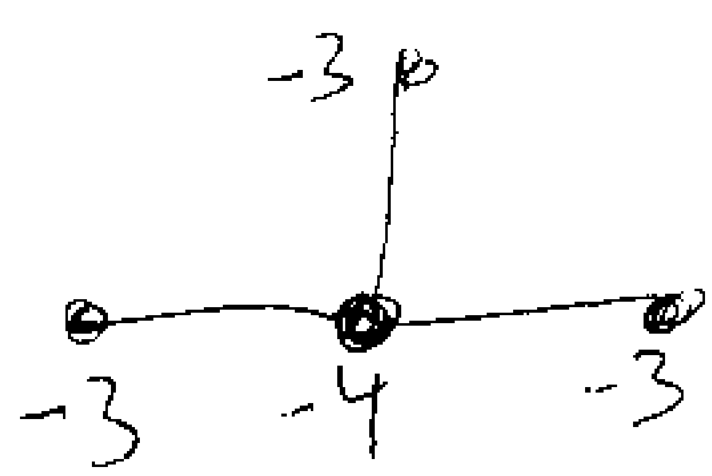


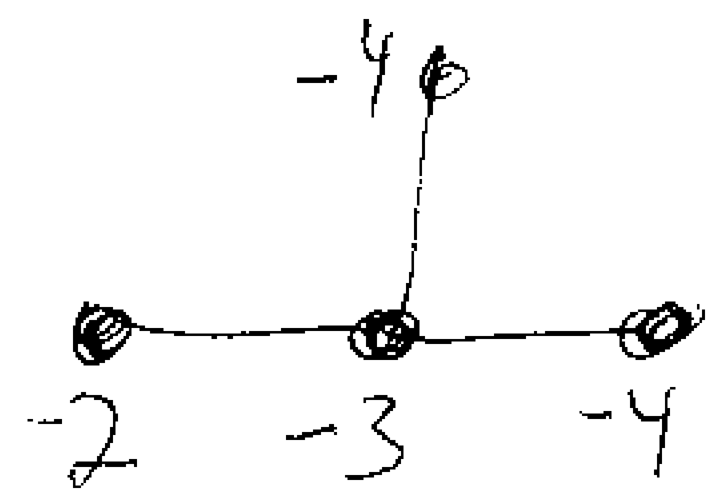
1) All graphs $\perp$-shaped
2) $\mathcal{N}$ from quotient construction
($\mathfrak{X} \subset \mathbb{C}^4$ hypersurface)
3) $\mathcal{M}$ via "smoothings of negative weight"

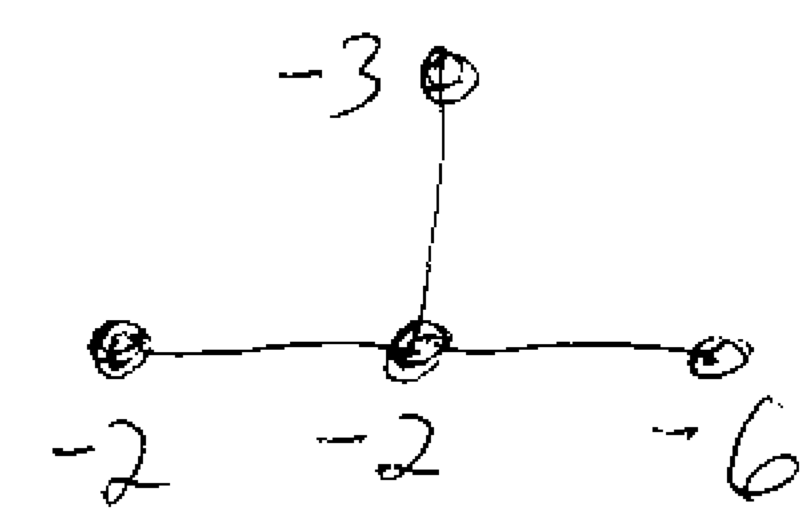
Summary:

$p^2/pq-1$ cyclic quotients

Three triply-infinite families, based on







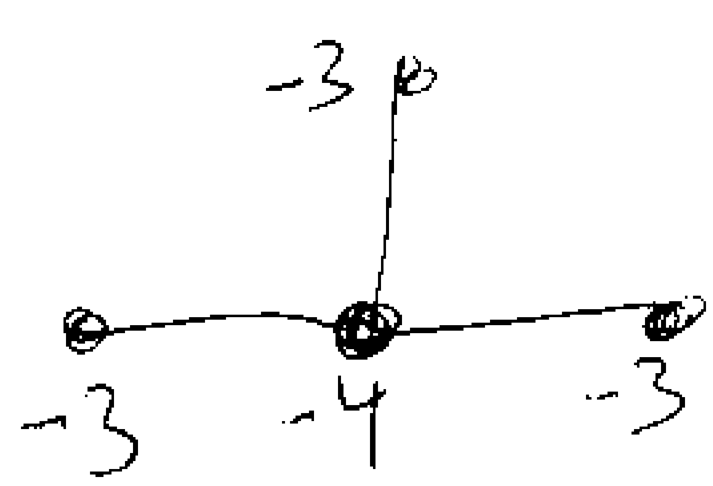
$W_{p,q,r}$ $\quad$ $N_{p,q,r}$ $\quad$ $M_{p,q,r}$

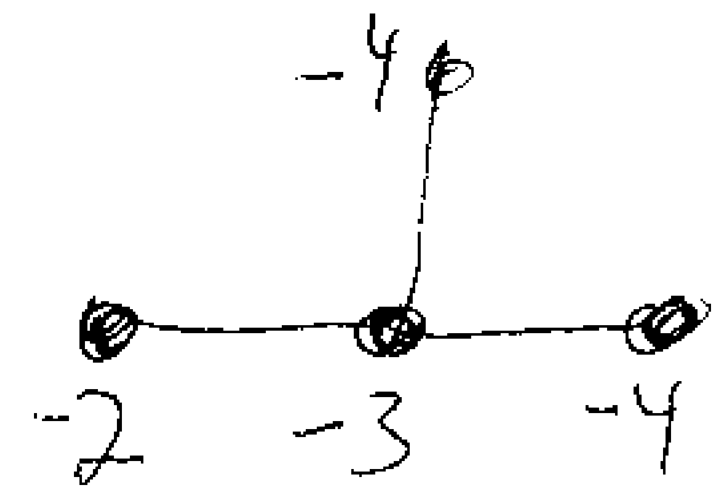
"Basic" examples are quotients of simple elliptic singularities

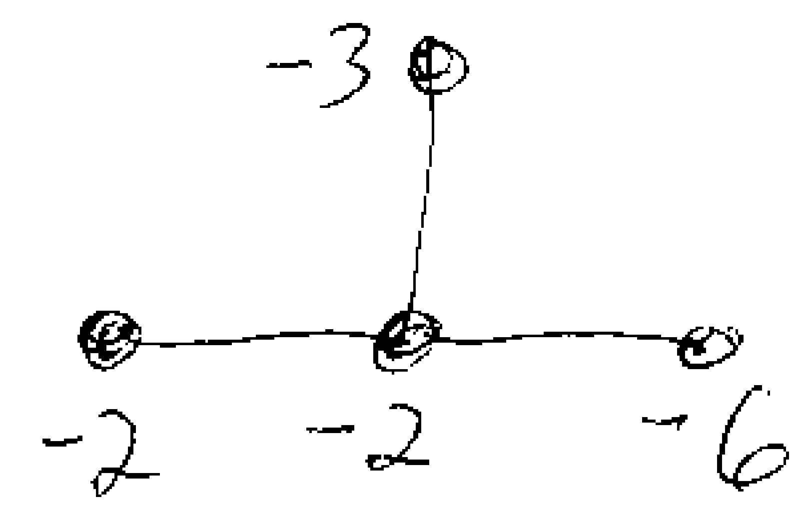
Summary:

$p^2/pq-1$ cyclic quotients

Three triply-infinite families, based on







$\mathcal{W}_{p,q,r}$ $\quad$ $\mathcal{N}_{p,q,r}$ $\quad$ $\mathcal{M}_{p,q,r}$

"Basic" examples are quotients of simple elliptic singularities

Wait – there are more!

Examples with valency 4 node:

Let $S = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$, ζ prim. 7^{th} root

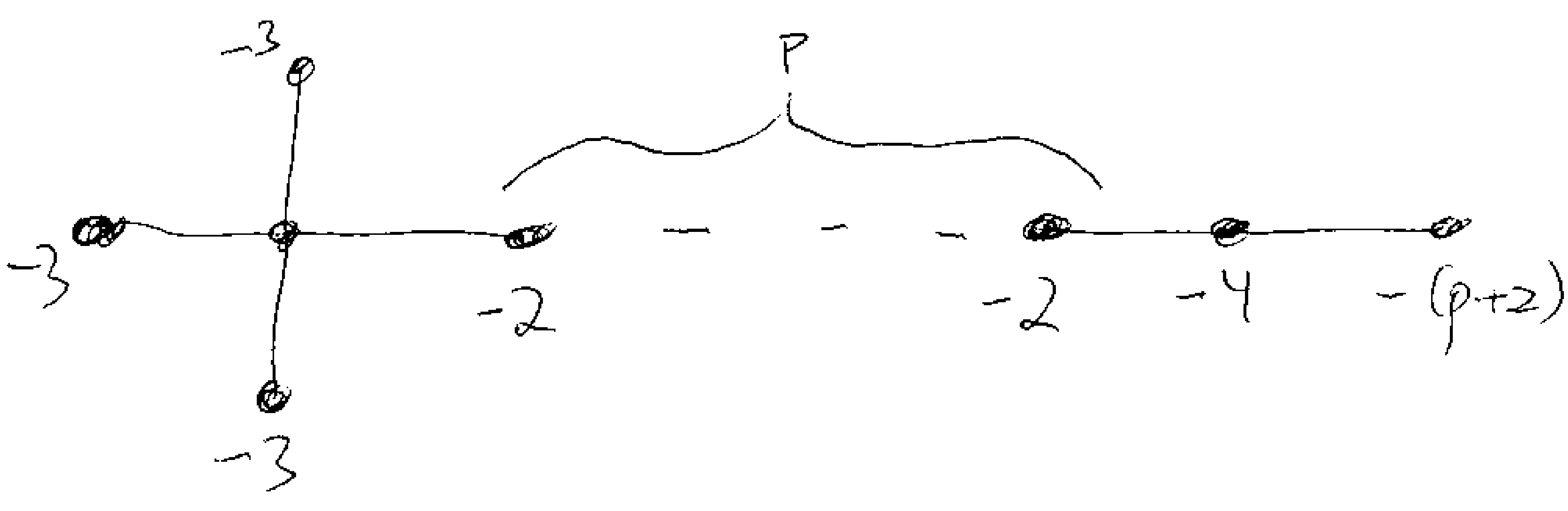
$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}$, ω prim. 6^{th} root.

Then $G = \langle S, T \rangle \subset GL(3, \mathbb{C})$ non-abelian, order $7 \cdot 18$, acts FREELY on $\mathbb{C}^3 - \{0\}$.

$f = x^5 y + y^5 z + \omega z^5 x$ is G-invariant

Then $\{f=0\}/G$ has a $\mathbb{Q}HD$ smoothing, as above.

Have 1-parameter family: $p=0$ case of



Found similar examples, valency 4 nodes, outer weights 2-4-4 and 2-3-6.

Also, a couple of other 1-parameter families known.

"Secret unpublished list" - by 1984

De Jong and van Straten
(1998 Duke article)
- theoretical way to check
(if sandwich singularity) if $\exists$
$\mu = 0$ smoothing

Remarks:

1. Possible (X, o) rational surface singularities

2. Only weighted homogeneous examples

3. $\perp$-shaped taut, valency 4 examples for specific cross-ratios

Remarks:

1. Possible $(X,0)$ rational surface singularities

2. Only weighted homogeneous examples

3. $\perp$-shaped taut, valency 4 examples for specific cross-ratios

WHO CARES? (in 1985?)

1988 – Kollár, Shepherd-Barron

2006 – Lee and Park – new surfaces of general type, $p_g = 0$.

Symplectic interest

Fintushel-Stern (1997)

- "rational blow-down"

(Z, ω) symplectic 4-manifold

- replace tubular neighborhoods of chains of 2-spheres $\leftrightarrow p^2/pq-1$, by $\mathbb{Q}$HD's

- new (Z^*, ω^*) has Seiberg-Witten invariants computable from (Z, ω)

Payoff: Exotic smooth structures on 4-manifolds

→ Symplectic fillings of L
(e.g., lens spaces)

→ Possible configurations of
2-spheres for rational blow-down

→ Symplectic fillings of L
(e.g., lens spaces)

→ Possible configurations of
2-spheres for rational blow-down

With A. Stipsicz, Z. Szabó

Theorem ([SSW], 2008):

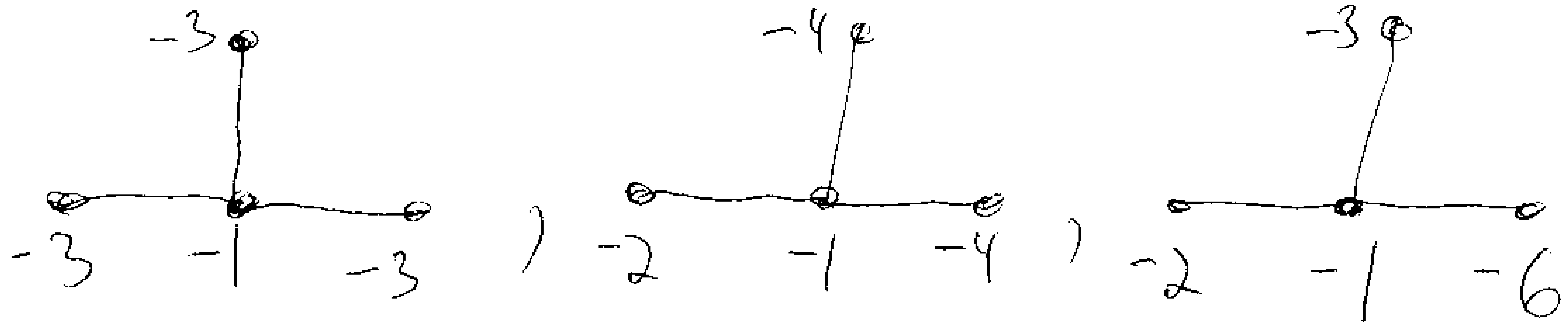
Only possible graphs for smoothing
(or symplectic filling) are among:

$p^2/pq-1$

$W_{p,q,r}$, $N_{p,q,r}$, $M_{p,q,r}$

Type $\mathcal{A}$, $\mathcal{B}$, or $\mathcal{C}$

Last 3 types built from algorithm
on



$\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$

Remarks: $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and (QHD) smoothing

1. $\exists$ known examples in each class
2. $\exists$ examples which do not occur in each class
3. $\exists$ non-star-shaped graphs in every class.

Proof: Donaldson diagonalizability

$\Rightarrow$ graph embeds in (-1)-diagonal lattice of same rank.

$+$

Calculations!

Proof: Donaldson diagonalizability
$\Rightarrow$ graph embeds in (-1)-diagonal lattice of same rank.

+

Calculations!

Question: How to go beyond C^∞-structure:

If L $\mathbb{Z}/2$-HS, $\exists\, \mu(L) \in \mathbb{Z}/(16)$, obstruction to $L = \partial M$, M spin $\mathbb{Q}$HD.

Neumann (1980) – $\exists\, \bar{\mu}(L) \in \mathbb{Z}$ lift
Stipsicz (2007) – $\bar{\mu}(L)$ obstruction for $\mathbb{Q}$HD smoothing (odd discrim.).

Smoothings of Negative Weight

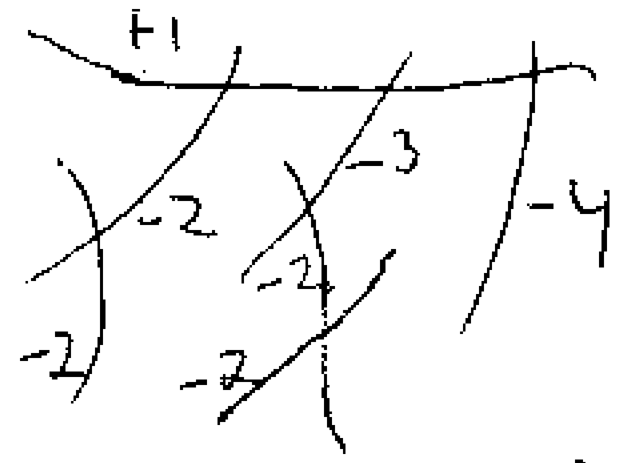
(Orlik-Wagreich, Pinkham)

$0 \in X$ weighted homogeneous ($= \mathbb{C}^*$-action)

$\exists X \subset \bar{X}$ - add curve at ∞

Resolve cyclic quot. sings at ∞:

$0 \in X \subset \bar{X}^*$ - add "dual curve"



Smooth $\bar{X}^*$, fixing dual curve

$\Rightarrow$ smooth proj surface Y, with dual curve C

Sometimes:

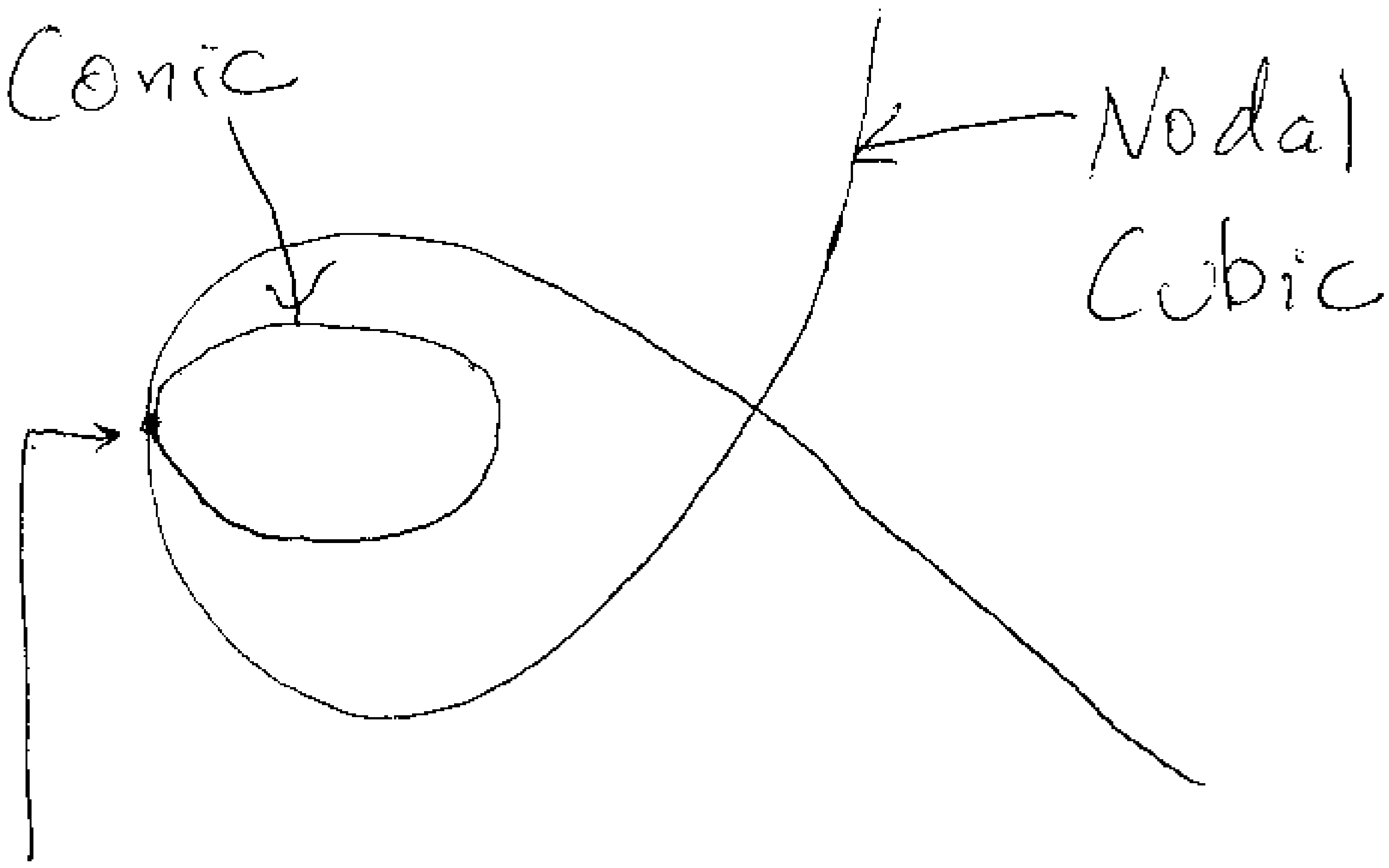
Existence of smoothing of X $\overset{?}{\Longleftrightarrow}$ Existence of (Y, C)

$\exists ?\ (Y, C)$

- in cases of interest, Y rational

Blow-down

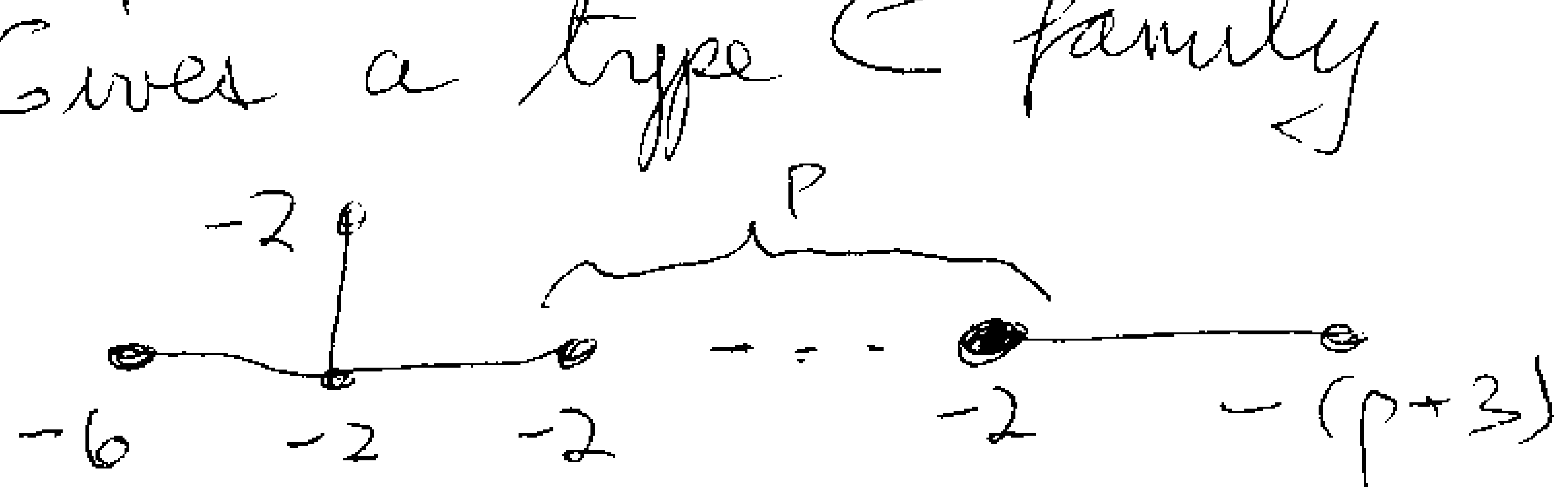
$\exists ?\ C' \subset \mathbb{P}^2$ with certain singularities



Intersection multiplicity 6

$$\left(y^2 - x(1-x)^2\right)\left(y^2 - x(1-2x)\right) = 0$$

Gives a type $\mathbb{C}$ family



Why?

For

Theorem (Bhupal-Stipsicz: 2009):
In the weighted homogeneous case (= star-shaped graph), Wahl's list of graphs is complete.

Proof: Examine star-shaped examples of type $\mathcal{A}, \mathcal{B}, \mathcal{C}$, eliminate all graphs except known ones.

Combination of symplectic methods + smoothing of neg. wt.

+

Calculations

What next?

1. Why these 3-dimensional singularities $\mathfrak{X}$, functions f?

2. Pin down analytic types in valency 4 cases

3. Smoothing components for $(X,0)$

4. Non-star-shaped in $\mathcal{A}, \mathcal{B}, \mathcal{C}$.