Separating sets in isolated complex singularities

Joint work with Alexandre Fernandes and Lev Birbrair

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The Lipschitz category

Topic:

The metric theory of complex analytic (or algebraic) germs.

The Lipschitz category is the appropriate category for this.

Definition (The Lipschitz category)

A map $f : Y \to Z$ of metric spaces is Lipschitz if $\exists K$:

$$\frac{1}{K} d_Y(p,q) \le d_Z(f(p), f(q)) \le K d_Y(p,q).$$

Bi-Lipschitz means bijective and Lipschitz.

Two metrics on X are Lipschitz equivalent if the identity map $(X, d_1) \to (X, d_2)$ is bi-Lipschitz.

In the Lipschitz category we consider them to be "the same."

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Metrics on germs

Let (X, p) be a complex algebraic germ, $x_1, \ldots, x_N$ generators of local ring $\mathcal{O}_{X,p}$.

$$\text{Then } (x_1, \ldots, x_N)\colon (X, p) \to \mathbb{C}^N \quad \text{is an embedding.}$$

Definition

- Outer metric on X is given by distance in $\mathbb{C}^N$.
- Inner metric on X is arc length in X (Riemannian metric).

Proposition

Inner metric is determined by outer metric. In the Lipschitz category these metrics on X are independent of choices.

If you change generating set of $\mathcal{O}_{X,p}$, the identity map $(X, \text{old metric}) \to (X, \text{new metric})$ is bi-Lipschitz.

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A germ (Y, p) is *metrically trivial* if it is equivalent to a metric cone:

$$(Y, p) \cong (\{ry : y \in \Sigma, r \in [0, 1]\}, 0) \quad \text{where} \quad \Sigma \subset S^{n-1} \subset \mathbb{R}^n$$

The first example of non-triviality of complex germs was found by Birbrair and Fernandes: *for $k > 1$ and odd, the A_k surface singularity $A_k = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^{k+1} = 0\}$, has a separating set, and is hence non-trivial.*

Later we showed, using mostly other techniques, that for weighted homogeneous surface singularities non-triviality is very common.

It appears now that separating sets are very common.

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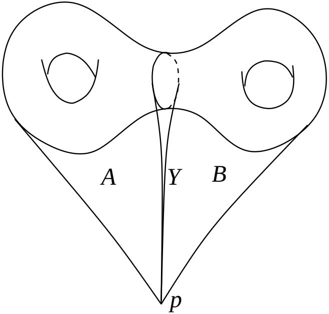
Separating set

Let (X, p) be a real k-dimensional semialgebraic germ. A separating set $(Y, p) \subset (X, p)$ is a subgerm of zero $(k-1)$–density which (locally) separates (X, p) into pieces of positive k–density.



k–Density

If $(X,0) \subset (\mathbb{R}^n,0)$ is a rectifiable subset, the k–density of (X,p) is

$$\Theta^k(X,p) := \lim_{\epsilon \to 0} \frac{\mathcal{H}^k\big(X \cap B^n(\epsilon)\big)}{\mathrm{vol}\big(B^k(\epsilon)\big)}\,.$$

Here $\mathcal{H}^k$ is k-dimensional Hausdorff measure.

In the situations that interest us the limit exists.

But, more generally, use $\liminf$ and $\limsup$ to define lower and upper k-density and define a separating set to be a set of zero upper $(k-1)$–density that locally divides (X,p) into sets of positive lower k–density.

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Fact

In the semi-algebraic category, separating sets are preserved by bi-Lipschitz maps (inner metric)

The reason is that separating sets can be defined equally well in the inner metric, and so long as things are semi-algebraic, one gets the same definition. This follows from:

Pancake Decomposition Theorem (Kurdyka)

A semialgebraic set has a finite semi-algebraic decomposition into pieces whose inner and outer metrics are Lipschitz equivalent.

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Of course, implicit in our discussion so far is that separating sets detect metric non-triviality:

Theorem

If Σ is a compact manifold, the metric cone $C\Sigma$ on Σ has no separating set.

In particular, an isolated singularity germ which has a separating set is metrically non-trivial (not bi-Lipschitz homeomorphic to a metric cone).

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Theorem 1

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Let $(X,0) \subset (\mathbb{C}^3,0)$ be an isolated weighted homogeneous singularity with weights $w_1 \geq w_2 > w_3$. Suppose $X \cap \{z=0\}$ is reducible. Then $(X,0)$ has a separating set.

Example (A_k again)

$A_k := \{(x,y,z) \in \mathbb{C}^3 : x^2 + y^2 + z^{k+1}\}$
has weights $(k+1, k+1, 2)$ or $(\frac{k+1}{2}, \frac{k+1}{2}, 1)$.
$\{z=0\}$ is the union of two lines: $\{x = \pm iy\}$. So A_k has a separating set if $k > 1$.

Example (More generally:)

$V(p,q,r) := \{(x,y,z) \in \mathbb{C}^3 : x^p + y^q + z^r\}$ has a separating set if $p \leq q < r$ and $gcd(p,q) > 1$.

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Briançon Speder example

Example (Briançon Speder family)

$$BS_t := \{(x, y, z) \in \mathbb{C}^3 : x^5 + z^{15} + y^7 z + txy^6 = 0\}, \quad t \in \mathbb{C}$$

Weighted homogeneous with weights $(3, 2, 1)$.

$BS_t \cap \{z = 0\}$ is the curve $\{x(x^4 + ty^6) = 0\}$.
This has 3 components if $t \neq 0$, so

BS_t has separating sets if $t \neq 0$.

Theorem (Lipschitz non-triviality in a topological trivial family)

BS_0 has no separating set.

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$X \subset \mathbb{C}^3$ is a weighted homogeneous germ with weights $w_1 \geq w_2 > w_3$. $X \cap \{z = 0\}$ is reducible. Then $(X, 0)$ has a separating set.

- $\Sigma := X \cap S^5$, the link of the singularity, is a 3-manifold.
- $\Sigma \cap \{z = 0\} = V \cup W$, disjoint closed sets.
- In Σ, let Y_0 be the conflict set $Y_0 = \{x \in \Sigma : d(x, V) = d(x, W)\}$.
- $Y := \mathbb{R}^* Y_0 \cup \{0\}$ using $\mathbb{R}^*$ in the $\mathbb{C}^*$–action. Y divides X into pieces A and B.
- Tangent cone $T_0 Y \subset z$–axis. So it has real dimension ≤ 2. It follows that the 3–density $\Theta^3(Y, 0)$ is zero.
- $T_0 A$ and $T_0 B$ each contains a complex plane. It follows that $\Theta^4(A) > 0$, $\Theta^4(B) > 0$.

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Theorem 2

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Let (X, p) be a complex isolated singularity of complex dimension n. Suppose that the tangent cone T_pX is separated by an analytic subset S of dimension $< n$. Then (X, p) has a separating set with tangent cone in S.

Example (Dimension n)

The Brieskorn singularity
$V(p_0, \dots, p_n) := \{(x_0, \dots, x_n) : x_0^{p_0} + \cdots + x_n^{p_n}\}$
with $2 \leq p_0 = p_1 < p_2 \leq p_3 \cdots \leq p_n$ has tangent cone consisting of p_0 intersecting planes. So it has separating sets.

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Example: Quotient singularities

If $G \subset \mathrm{GL}_2 \mathbb{C}$ is a finite subgroup which acts freelly on $\mathbb{C}^2$, then the tangent cone of $X = \mathbb{C}^2/G$ is irreducible only for:
• the homogeneous cyclic quotients $\mathbb{C}^2/\mu_r$ with $\mu_r \subset \mathbb{C}^*$ acting diagonally, and
• the simple singularities of type D and E.
Thus all other quotient singularities have separating sets.

This is a rich class of examples: Cyclic quotients are classified by pairs (r, s), with $0 < s < r$ and $gcd(r, s) = 1$.
There are examples with arbitrarily many separating sets.

The other quotients are classified by tuples $(n; p_1, q_1; p_2, q_2; p_3, q_3)$ with $(p_1, p_2, p_3) = (2, 2, p)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$ and $0 < q_i < p_i, \quad gcd(p_i, q_i) = 1, \quad n + \sum \frac{q_i}{p_i} > 0.$

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The above examples show that Theorem 2 is quite powerful. Could it be that every separating set arises through this theorem?

Answer: No: The Briançon-Speder singularity BS_t has tangent cone C^2, but has separating sets if $t \neq 0$.

We will describe a resolution.

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Theorem 3

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A semialgebraic germ (X, p) has a semialgebraic separating set if and only if its metric tangent cone has a semialgebraic separating subset of codimension > 1.

Metric Tangent Cone

The metric tangent cone $\mathcal{T}_pX$ of a semialgebraic germ (X,p) was studied in depth by Bernig and Lytchak (the definition goes back to Gromov, and versions are used in many fields).

Definition

$$\mathcal{T}_pX := \lim_{t\to\infty}{}^{Gromov-Hausdorff}(X, p, \frac{1}{t}d)$$

Note that $\mathcal{T}_pX$ is metrically a strict cone. But even if (X,p) is a complex germ, $\mathcal{T}_pX$ may not be a complex cone; in fact it is not clear that it always admits a complex structure (probably not).

Example

The D_4 singularity $V(2,3,3)$ is metrically conical [BFN], from which follows: $\mathcal{T}_0D_4 \cong D_4$. But D_4 is not a complex cone, since then its link would be the total space of an S^1–bundle (it is not).

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- [Birbrair-Mostowski] Normal embedding theorem
- For an normally embedded semialgebraic set $\mathcal{T}_p X = T_p X$
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Thank You