Characteristic classes

in singularity theory

Andrzej Weber

joint with

Malgorzata Mikosz and Piotr Pragacz

Characteristic classes

Thom polynomials

in singularity theory

Andrzej Weber

joint with

Malgorzata Mikosz and Piotr Pragacz

Characteristic classes

Thom polynomials

in singularity theory

of maps

Andrzej Weber

joint with

Malgorzata Mikosz and Piotr Pragacz

Characteristic classes

Thom polynomials

in singularity theory

of maps

and bundle sections

Andrzej Weber

joint with

Malgorzata Mikosz and Piotr Pragacz

- **Singularities** $\leftrightsquigarrow$ **Characteristic Classes**
  Local Theory *Global Theory*

- **Singularities** $\leftrightsquigarrow$ **Characteristic Classes**
  Local Theory *Global Theory*
- **Eg. vector field $\vec{v}$ on a manifold M**
  Poincaré–Hopf theorem: $\sum \mathbf{i_x}(\vec{\mathbf{v}}) = \chi(\mathbf{M})$

- **Singularities** $\quad\leftrightsquigarrow\quad$ **Characteristic Classes**
  Local Theory $\qquad\qquad$ *Global Theory*
- **Eg. vector field $\vec{v}$ on a manifold M**
  Poincaré–Hopf theorem: $\sum \mathbf{i}_{\mathbf{x}}(\vec{\mathbf{v}}) = \chi(\mathbf{M})$

$$\Rightarrow$$

- **local contribution to a global invariant**

- **Singularities** *Local Theory* $\leftrightsquigarrow$ **Characteristic Classes** *Global Theory*
- **Eg. vector field $\vec{v}$ on a manifold M**
  Poincaré–Hopf theorem: $\sum \mathbf{i_x}(\vec{\mathbf{v}}) = \chi(\mathbf{M})$

$$\Rightarrow$$

- **local contribution to a global invariant**

$$\Leftarrow$$

- **non vanishing of the global invariant forces the existence of singularities**

- $\xi \to M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
  $[H] = c_1(\xi) \in H^2(M)$

- $\xi \to M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
  $[H] = c_1(\xi) \in H^2(M)$

- **If $c_1(\xi) \neq 0$ then $H \neq \emptyset$**

- $\xi \longrightarrow M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
  $[H] = c_1(\xi) \in H^2(M)$

- **If** $c_1(\xi) \neq 0$ **then** $H \neq \emptyset$

- **Is** H **singular?**

- $\xi \longrightarrow M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
  $[H] = c_1(\xi) \in H^2(M)$

- **If $c_1(\xi) \neq 0$ then $H \neq \emptyset$**

- **Is H singular?**

- **Generically smooth (e.g. for ξ very ample)**

- $\xi \to M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
  $[H] = c_1(\xi) \in H^2(M)$

- **If $c_1(\xi) \neq 0$ then $H \neq \emptyset$**

- **Is H singular?**
- **Generically smooth (e.g. for ξ very ample)**
- **Singularities $\Sigma \subset H$ have to occur in one-dimensional families**

- B – **parameter space**

- B – **parameter space** $\quad M_b \subset M \twoheadrightarrow B$

- B – **parameter space** $M_b \subset M \twoheadrightarrow B$

$$H = \bigcup_{b\in B} H_b = \{s = 0\}$$ **hypersurface**

- B – **parameter space** $M_b \subset M \twoheadrightarrow B$

$$H = \bigcup_{b \in B} H_b = \{s = 0\}$$ **hypersurface**

Singularity locus of the family

$$\Sigma = \bigcup_{b \in B} \Sigma_b = \left\{d\left(s_{|M_b=0}\right)\right\} \subset H$$

- B **– parameter space** $M_b \subset M \twoheadrightarrow B$

$$H = \bigcup_{b \in B} H_b = \{s = 0\}$$ **hypersurface**

Singularity locus of the family

$$\Sigma = \bigcup_{b \in B} \Sigma_b = \left\{d\left(s_{|M_b=0}\right)\right\} \subset H$$

- Σ **defines a cohomology class**
- $$[\Sigma] = \underline{c_n(T^*M/B \otimes \xi)}_{\ |H} \in H^{2n}(H)$$

- B – **parameter space** $\quad M_b \subset M \twoheadrightarrow B$

$$H = \bigcup_{b\in B} H_b = \{s=0\} \qquad \textbf{hypersurface}$$

Singularity locus of the family

$$\Sigma = \bigcup_{b\in B} \Sigma_b = \left\{d\left(s_{|M_b=0}\right)\right\} \subset H$$

- Σ **defines a cohomology class**
- $[\Sigma] = \underline{c_n(T^*M/B \otimes \xi)}_{\,|H} \in H^{2n}(H)$
- **or**

$$\underline{c_n(T^*M/B \otimes \xi)} \cdot c_1(\xi) \in H^{2n+2}(M)$$

- **More complicated singularities?**

  A_2 **singularity**

  Hessian(s)=0

  $(x_1, x_2, \dots, x_n) \mapsto x_1^3 + x_2^2 + \cdots + x_n^2$

- **More complicated singularities?**

  A_2 singularity

  Hessian(s)=0

  $(x_1, x_2, \ldots, x_n) \mapsto x_1^3 + x_2^2 + \cdots + x_n^2$
- **Codimension $n+2$ in the jet space**

- **More complicated singularities?**

  A_2 singularity

  Hessian(s)=0

  $(x_1, x_2, \ldots, x_n) \mapsto x_1^3 + x_2^2 + \cdots + x_n^2$
- **Codimension $n+2$ in the jet space**
- **Have to appear in 2-dimensional families.**

- **The condition: Hessian(s)=0**
  $\Rightarrow$ **vanishing** $D^2(s) : TM/B \to T^*M/B \otimes \xi$

- **The condition: Hessian(s)=0**
  $\Rightarrow$ **vanishing** $D^2(s) : TM/B \to T^*M/B \otimes \xi$
- **Localized cohomology class**

$$[\Sigma_{A_2}]_{loc} = \underline{c_1(T^*M/B \otimes \xi - TM)}_{|\Sigma} \in H^2(\Sigma)$$

- **The condition: Hessian(s)=0**
  $\Rightarrow$ **vanishing** $D^2(s) : TM/B \to T^*M/B \otimes \xi$
- **Localized cohomology class**

$$[\Sigma_{A_2}]_{loc} = \underline{c_1(T^*M/B \otimes \xi - TM)}_{|\Sigma} \in H^2(\Sigma)$$

- **Nonlocalized class** $[\Sigma_{A_2}] \in H^{2n+2}(H)$
  given by

$$\underline{c_1(T^*M/B \otimes \xi - TM)} \cdot c_n(T^*M/B \otimes \xi)$$

- **The condition: Hessian(s)=0**
  $\Rightarrow$ **vanishing** $D^2(s) : TM/B \to T^*M/B \otimes \xi$
- **Localized cohomology class**

$$[\Sigma_{A_2}]_{loc} = \underline{c_1(T^*M/B \otimes \xi - TM)}_{|\Sigma} \in H^2(\Sigma)$$

- **Nonlocalized class** $[\Sigma_{A_2}] \in H^{2n+2}(H)$
  given by

$$\underline{c_1(T^*M/B \otimes \xi - TM)} \cdot c_n(T^*M/B \otimes \xi)$$

- **or in** $H^{2n+4}(M)$

$$-,,- \qquad \cdot c_1(\xi)$$

- A_3 **singularity:** $x_1^4 + x_2^2 + \cdots + x_n^2$
codimension $n+3$ **in the jet space**

- $\underline{A_3}$ **singularity:** $x_1^4 + x_2^2 + \cdots + x_n^2$
  codimension $n+3$ **in the jet space**
- **Localized cohomology class**

$$[\Sigma_{A_3}]_{loc} = 6c_1^2 + (6n-1)c_1 u + \frac{3n^2-n}{2}u^2$$

where $c_* = c_*(T^*M/B)$ **and** $u = c_1(\xi)$

- **A_3 singularity:** $x_1^4 + x_2^2 + \cdots + x_n^2$
  codimension $n+3$ in the jet space
- **Localized cohomology class**

$$[\Sigma_{A_3}]_{loc} = 6c_1^2 + (6n-1)c_1 u + \frac{3n^2 - n}{2} u^2$$

  where $c_* = c_*(T^*M/B)$ **and** $u = c_1(\xi)$
- **Can be expressed by**

$$3c_2(E^* - E) + c_1(E^* - E)t$$

  where $E = TM/B \otimes \xi^{-\frac{1}{2}}$
  $t = c_1(\xi^{-\frac{1}{2}}) = -\frac{1}{2}u$

- **Our goal is to study the characteristic classes associated to singularity types,**

- **Our goal is to study the characteristic classes associated to singularity types,**
- **called Thom Polynomias**
- **We study the coefficients of Thom polynomials expanded in some distinguished basis of characteristic classes**

Localized Thom polynomial

Localized Thom polynomial

- **1) There exists a polynomial $\mathcal{T}_\eta[c_*, t]$, s.t.**

$$[\Sigma_\eta]_{loc} = \mathcal{T}_\eta[c_*(T^*M/B), c_1(\xi^*)] \qquad \textit{Kazarian}$$

Localized Thom polynomial

- **1) There exists a polynomial $\mathcal{T}_\eta[c_*, t]$, s.t.**

$$[\Sigma_\eta]_{loc} = \mathcal{T}_\eta[c_*(T^*M/B), c_1(\xi^*)] \qquad \textit{Kazarian}$$

- **2) Thom polynomial is expressed by Schur Q-functions:**

$$\mathcal{T}_\eta = \sum_{\mathrm{I},j} a_{\mathrm{I},j}\; Q_{\mathrm{I}}(E^*)\cdot\left(\frac{t}{2}\right)^j ,$$

where $E = TM/B \otimes \xi^{-\frac{1}{2}}$

$$t = c_1(\xi^{-\frac{1}{2}}) = -\frac{1}{2}u$$

Localized Thom polynomial

- **1) There exists a polynomial $\mathcal{T}_\eta[c_*, t]$, s.t.**

$$[\Sigma_\eta]_{loc} = \mathcal{T}_\eta[c_*(T^*M/B), c_1(\xi^*)] \qquad \textit{Kazarian}$$

- **2) Thom polynomial is expressed by Schur Q-functions:**

$$\mathcal{T}_\eta = \sum_{\mathrm{I},j} a_{\mathrm{I},j}\; Q_{\mathrm{I}}(E^*)\cdot\left(\frac{t}{2}\right)^j \;,$$

where $E = TM/B \otimes \xi^{-\frac{1}{2}}$

$$t = c_1(\xi^{-\frac{1}{2}}) = -\frac{1}{2}u$$

- $\mathrm{I} = (i_1 > i_2 > \ldots > i_n)$ **strict partition**
  $\mathrm{I} \sim$ **the Schubert cell in the Lagrangian Grassmannian**

- **We study the structure of the expansions of Thom polynomials in the basis of Schur Q-functions**

$$\mathcal{T}_\eta = \sum_{\mathrm{I},j} a_{\mathrm{I},j}\, Q_{\mathrm{I}}(E^*) \cdot \left(\frac{t}{2}\right)^j$$

- **We study the structure of the expansions of Thom polynomials in the basis of Schur Q-functions**

$$\mathcal{T}_\eta = \sum_{\mathrm{I},j} a_{\mathrm{I},j}\; Q_{\mathrm{I}}(E^*) \cdot \left(\frac{t}{2}\right)^j$$

- **Theorem: All the coefficients $a_{\mathrm{I},j}$ are nonnegative integers.** *Mikosz, Pragacz, AW, 2008*

- **Schur Q-functions**

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz}, \textit{Ratajski},$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

$\widetilde{Q}_{\mathrm{I}}(L^*)$ **= the class of Schubert cell**
in the Lagrangian Grassmannian,
L **– tautological Lagrangian bundle**

- **Schur Q-functions**

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz}, \textit{Ratajski},$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

$\widetilde{Q}_{\mathrm{I}}(L^*)$ **= the class of Schubert cell in the Lagrangian Grassmannian,**

L **– tautological Lagrangian bundle**

- $\widetilde{Q}_i = c_i$

- **Schur Q-functions**

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz}, \textit{Ratajski},$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

$\widetilde{Q}_{\mathrm{I}}(L^*)$ **= the class of Schubert cell in the Lagrangian Grassmannian,**
L – tautological Lagrangian bundle

- $\widetilde{Q}_i = c_i$
- $\widetilde{Q}_{ij} = c_i c_j - 2(c_{i+1}c_{j-1} + c_{i+2}c_{j-2} - \dots)$

- **Schur Q-functions**

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz}, \textit{Ratajski},$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

$\widetilde{Q}_{\mathrm{I}}(L^*)$ **= the class of Schubert cell in the Lagrangian Grassmannian,**
L – tautological Lagrangian bundle

- $\widetilde{Q}_i = c_i$
- $\widetilde{Q}_{ij} = c_i c_j - 2(c_{i+1}c_{j-1} + c_{i+2}c_{j-2} - \dots)$
- $\widetilde{Q}_{ijk} = c_i \widetilde{Q}_{jk} - c_j \widetilde{Q}_{ik} + c_k \widetilde{Q}_{ij}$

- **Schur Q-functions**

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz, Ratajski,}$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

$\widetilde{Q}_{\mathrm{I}}(L^*)$ **= the class of Schubert cell in the Lagrangian Grassmannian,**
L **– tautological Lagrangian bundle**

- $\widetilde{Q}_i = c_i$
- $\widetilde{Q}_{ij} = c_i c_j - 2(c_{i+1}c_{j-1} + c_{i+2}c_{j-2} - \dots)$
- $\widetilde{Q}_{ijk} = c_i \widetilde{Q}_{jk} - c_j \widetilde{Q}_{ik} + c_k \widetilde{Q}_{ij}$
- . . .

  Pfaffian formulas

- **Examples:**

$$\mathcal{T}_{A_1} = 1$$
$$\mathcal{T}_{A_2} = Q_1(E^*) = 2c_1(T^*M/B \otimes \xi^{\frac{1}{2}})$$
$$\mathcal{T}_{A_3} = 3Q_2 + tQ_1$$
$$\mathcal{T}_{A_4} = 3Q_{21} + 12Q_3 + 10tQ_2 + 2t^2Q_1$$

- **Examples:**

  $\mathcal{T}_{A_1} = 1$

  $\mathcal{T}_{A_2} = Q_1(E^*) = 2c_1(T^*M/B \otimes \xi^{\frac{1}{2}})$

  $\mathcal{T}_{A_3} = 3Q_2 + tQ_1$

  $\mathcal{T}_{A_4} = 3Q_{21} + 12Q_3 + 10tQ_2 + 2t^2Q_1$
- **Note: all coefficients are not only nonnegative.**

- **Examples:**

  $\mathcal{T}_{A_1} = 1$

  $\mathcal{T}_{A_2} = Q_1(E^*) = 2c_1(T^*M/B \otimes \xi^{\frac{1}{2}})$

  $\mathcal{T}_{A_3} = 3Q_2 + tQ_1$

  $\mathcal{T}_{A_4} = 3Q_{21} + 12Q_3 + 10tQ_2 + 2t^2Q_1$
- **Note: all coefficients are not only nonnegative.**
- **Also, they do not vanish.**

- $\mathcal{T}_{D_4} = Q_{21}$ **– corank of** $D^2(s) = 2$

- $\mathcal{T}_{D_4} = Q_{21}$ – **corank of** $D^2(s) = 2$
- **For the rank condition on** $D^2(s)$ **we obtain a single** Q**-function.**

- $\mathcal{T}_{D_4} = Q_{21}$ – **corank of** $D^2(s) = 2$
- **For the rank condition on** $D^2(s)$ **we obtain a single** Q**-function.**
- **Here, we have the partition** $(2, 1)$ ♠♠
♠

- $\mathcal{T}_{D_4} = Q_{21}$ – **corank of** $D^2(s) = 2$
- **For the rank condition on** $D^2(s)$ **we obtain a single** Q**-function.**
- **Here, we have the partition** $(2, 1)$ ♠♠ / ♠
- **The partitions for deeper singularities contain the partition** (2.1)**,**

  e.g. $D_5 = 6Q_{31} + 4t\, Q_{21}$ **= 6** ♠♠♠ / ♠ **+ 4 t** ♠♠ / ♠

- $\mathcal{T}_{D_4} = Q_{21}$ – **corank of** $D^2(s) = 2$
- **For the rank condition on** $D^2(s)$ **we obtain a single** Q**-function.**
- **Here, we have the partition** $(2,1)$ ♠♠
♠
- **The partitions for deeper singularities contain the partition** (2.1)**,**

  e.g. $D_5 = 6Q_{31} + 4t\,Q_{21}$ **= 6** ♠♠♠
♠ **+ 4 t** ♠♠
♠
- **...**

  1–dimensional family of cubic singularities, – corank of $D^2(s) = 3$

  $\mathcal{T}_{P_8} = Q_{321}$ **=**

  ♠♠♠
♠♠
♠

- **...much more complicated singularities**

$$\mathcal{T}_{A_8} = 18840Q_{61} + 20160Q_7 + 3123Q_{421} + 5556Q_{43} + 15564Q_{52}$$
$$+t(71856Q_6 + 3999Q_{321} + 55672Q_{51} + 34780Q_{42})$$
$$+t^2(64524Q_{41} + 24616Q_{32} + 105496Q_5)$$
$$+t^3(36048Q_{31} + 81544Q_4)$$
$$+t^4(8876Q_{21} + 34936Q_3)$$
$$+t^5 7848Q_2$$
$$+t^6 720Q_1$$

- **... much more complicated singularities**
  $\mathcal{T}_{A_8} = 18840Q_{61} + 20160Q_7 + 3123Q_{421} + 5556Q_{43} + 15564Q_{52}$
  $+t(71856Q_6 + 3999Q_{321} + 55672Q_{51} + 34780Q_{42})$
  $+t^2(64524Q_{41} + 24616Q_{32} + 105496Q_5)$
  $+t^3(36048Q_{31} + 81544Q_4)$
  $+t^4(8876Q_{21} + 34936Q_3)$
  $+t^5 7848Q_2$
  $+t^6 720Q_1$
- $\mathcal{T}_{E_8} = 93Q_{421} + 108Q_{43} + 204Q_{52} + 72Q_{61}$
  $+t(99Q_{321} + 216Q_{51} + 414Q_{42})$
  $+t^2(246Q_{41} + 246Q_{32})$
  $+t^3 126Q_{31}$
  $+t^4 24Q_{21}.$

- **...much more complicated singularities**
  $$\mathcal{T}_{A_8} = 18840Q_{61} + 20160Q_7 + 3123Q_{421} + 5556Q_{43} + 15564Q_{52}$$
  $$+t(71856Q_6 + 3999Q_{321} + 55672Q_{51} + 34780Q_{42})$$
  $$+t^2(64524Q_{41} + 24616Q_{32} + 105496Q_5)$$
  $$+t^3(36048Q_{31} + 81544Q_4)$$
  $$+t^4(8876Q_{21} + 34936Q_3)$$
  $$+t^5 7848Q_2$$
  $$+t^6 720Q_1$$
- $$\mathcal{T}_{E_8} = 93Q_{421} + 108Q_{43} + 204Q_{52} + 72Q_{61}$$
  $$+t(99Q_{321} + 216Q_{51} + 414Q_{42})$$
  $$+t^2(246Q_{41} + 246Q_{32})$$
  $$+t^3 126Q_{31}$$
  $$+t^4 24Q_{21}.$$
- *Kazarian (Rimanyi method + geometric argument)*

- **Yet another positivity result** *Pragacz, AW 2006*

  The nonlocalized Thom polynomial associated to any singularity type η

$$\mathcal{T}_\eta(c_*(T^*M/B), c_1(\xi^*)) \cdot c_n(T^*M/B \otimes \xi)$$

- **<u>Yet another positivity result</u>** *Pragacz, AW 2006*

  The nonlocalized Thom polynomial associated to <u>any</u> singularity type η

$$\mathcal{T}_\eta(c_*(T^*M/B), c_1(\xi^*)) \cdot c_n(T^*M/B \otimes \xi)$$

- **can be expressed as a combination**

$$\sum_{\mathrm{I}} a_{\mathrm{I}} S_{\mathrm{I}}(c_*(T^*M/B - \xi^*)),$$

  where $\mathrm{I} = (i_1 \geq i_2 \geq i_3 \geq \dots)$ **partition,**
  $\mathrm{I} \sim$ **Schubert cell in Grassmannian**
  $S_{\mathrm{I}}(c_*)$ – **Schur function**

- **Yet another positivity result** *Pragacz, AW 2006*

The nonlocalized Thom polynomial associated to any singularity type η

$$\mathcal{T}_\eta(c_*(T^*M/B), c_1(\xi^*)) \cdot c_n(T^*M/B \otimes \xi)$$

- **can be expressed as a combination**

$$\sum_{\mathrm{I}} a_{\mathrm{I}} S_{\mathrm{I}}(c_*(T^*M/B - \xi^*)),$$

where $\mathrm{I} = (i_1 \geq i_2 \geq i_3 \geq \dots)$ **partition,**
$\mathrm{I} \sim$ **Schubert cell in Grassmannian**
$S_{\mathrm{I}}(c_*)$ **– Schur function**

- **Theorem: The coefficients** a_{I} **are nonnegative.**

- **Yet another positivity result** *Pragacz, AW 2006*

  The nonlocalized Thom polynomial associated to any singularity type η

$$\mathcal{T}_\eta(c_*(T^*M/B), c_1(\xi^*)) \cdot c_n(T^*M/B \otimes \xi)$$

- **can be expressed as a combination**

$$\sum_{\mathrm{I}} a_{\mathrm{I}} S_{\mathrm{I}}(c_*(T^*M/B - \xi^*)),$$

  where $\mathrm{I} = (i_1 \geq i_2 \geq i_3 \geq \dots)$ **partition,**
  $\mathrm{I} \sim$ **Schubert cell in Grassmannian**
  $S_{\mathrm{I}}(c_*)$ **– Schur function**

- **Theorem: The coefficients** a_{I} **are nonnegative.**

- **Formulas depend on** $n = \dim M$**,**
  e.g. $\mathcal{T}_{A_2} = n\, S_{1^{n+1}} + 2\, S_{1^{n-1}2}$

- **Positivity of Schur expansion does not follow from positivity of Q-expansion.**

- **Positivity of Schur expansion <u>does not</u> follow from positivity of Q-expansion.**
- **Another family of inequalities, obtained from various models of** *classifying space of Legendrian singularities*.

joint work with MM,PP,Kazarian

- **Positivity of Schur expansion <u>does not</u> follow from positivity of Q-expansion.**
- **Another family of inequalities, obtained from various models of** *classifying space of Legendrian singularities***.**

  joint work with MM,PP,Kazarian
- $\Rightarrow$ *effective cone* **of characteristic classes coming from effective cycles.**

P. Pragacz, A. Weber: *Positivity of Schur function expansions of Thom polynomials,*
Fundamenta Mathematicae 195, No. 1 (2007) 85-95

P. Pragacz, A. Weber: *Thom polynomials of invariant cones, Schur functions, and positivity,* **in: Algebraic Cycles, Sheaves, Shtukas, and Moduli, Trends in Mathematics, Birkhäuser, 2007, 117-129.**

M. Mikosz, P. Pragacz, A. Weber: Positivity of Thom polynomials II; the Lagrange singularities, Fund. Math. 202 (2009).