Characteristic classes

in singularity theory

Andrzej Weber

joint with

Malgorzata Mikosz and Piotr Pragacz

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Characteristic classes

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and bundle sections

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$$\Leftarrow$$

- **non vanishing of the global invariant forces the existence of singularities**

- $\xi \to M$ **line bundle,**
  $s \in \Gamma(\xi)$ **section**

  $H = \{s = 0\}$ **hyperplane**
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- **or**

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- **More complicated singularities?**

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Hessian(s)=0

$$(x_1, x_2, \dots, x_n) \mapsto x_1^3 + x_2^2 + \cdots + x_n^2$$

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- **Have to appear in 2-dimensional families.**

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- **Can be expressed by**

$$3c_2(E^* - E) + c_1(E^* - E)t$$

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- **called <u>Thom Polynomias</u>**
- **We study the coefficients of Thom polynomials expanded in some distinguished basis of characteristic classes**

- **Localized Thom polynomial**

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- **1) There exists a polynomial $\mathcal{T}_\eta[c_*, t]$, s.t.**

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- **2) Thom polynomial is expressed by Schur Q-functions:**

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- $\mathrm{I} = (i_1 > i_2 > \ldots > i_n)$ **strict partition**
  $\mathrm{I} \sim$ **the Schubert cell in the Lagrangian Grassmannian**

- **We study the structure of the expansions of Thom polynomials in the basis of Schur Q-functions**

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- **Theorem: All the coefficients $a_{\mathrm{I},j}$ are nonnegative integers.** *Mikosz, Pragacz, AW, 2008*

Schur Q-functions

$$Q_{\mathrm{I}}(E^*) = \widetilde{Q}_{\mathrm{I}}(E^* - E), \qquad \textit{Pragacz}, \textit{Ratajski},$$

where $\widetilde{Q}_I(-)$ is a polynomial on $c_*(-)$ such that:

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- ...

Pfaffian formulas

- **Examples:**

$$\mathcal{T}_{A_1} = 1$$
$$\mathcal{T}_{A_2} = Q_1(E^*) = 2c_1(T^*M/B \otimes \xi^{\frac{1}{2}})$$
$$\mathcal{T}_{A_3} = 3Q_2 + tQ_1$$
$$\mathcal{T}_{A_4} = 3Q_{21} + 12Q_3 + 10tQ_2 + 2t^2Q_1$$

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- **Note: all coefficients are not only nonnegative.**
- **Also, they do not vanish.**

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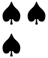
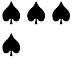
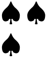
- **Here, we have the partition** $(2,1)$ 
- **The partitions for deeper singularities contain the partition** (2.1)**,**

  e.g. $D_5 = 6Q_{31} + 4t\,Q_{21}$ **= 6**  **+ 4 t** 

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- **. . .**

  1–dimensional family of cubic singularities, – corank of $D^2(s) = 3$

  $$\mathcal{T}_{P_8} = Q_{321} = \begin{array}{lll}\spadesuit & \spadesuit & \spadesuit \\ \spadesuit & \spadesuit & \\ \spadesuit & & \end{array}$$

- **. . . much more complicated singularities**

$$\mathcal{T}_{A_8} = 18840Q_{61} + 20160Q_7 + 3123Q_{421} + 5556Q_{43} + 15564Q_{52}$$
$$+t(71856Q_6 + 3999Q_{321} + 55672Q_{51} + 34780Q_{42})$$
$$+t^2(64524Q_{41} + 24616Q_{32} + 105496Q_5)$$
$$+t^3(36048Q_{31} + 81544Q_4)$$
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- $$\mathcal{T}_{E_8} = 93Q_{421} + 108Q_{43} + 204Q_{52} + 72Q_{61}$$
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- *Kazarian (Rimanyi method + geometric argument)*

- **Yet another positivity result** *Pragacz, AW 2006*

  The nonlocalized Thom polynomial associated to any singularity type η

$$\mathcal{T}_\eta(c_*(T^*M/B), c_1(\xi^*)) \cdot c_n(T^*M/B \otimes \xi)$$

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- **can be expressed as a combination**

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where $\mathrm{I} = (i_1 \geq i_2 \geq i_3 \geq \dots)$ **partition,**
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- **Theorem: The coefficients a_{I} are nonnegative.**
- **Formulas depend on** $n = \dim M$**,**
e.g. $\mathcal{T}_{A_2} = n\, S_{1^{n+1}} + 2\, S_{1^{n-1}2}$

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  joint work with MM,PP,Kazarian
- $\Rightarrow$ *effective cone* **of characteristic classes coming from effective cycles.**

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Fundamenta Mathematicae 195, No. 1 (2007) 85-95

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