Symbolic Homotopy Construction

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Abstract

The classical Theorem of Bézout yields an upper bound for the number of finite solutions to a given polynomial system, but is very often too large to be useful for the construction of a start system, for the solution of a polynomial system by means of homotopy continuation. The BKK bound gives a much lower upper bound for the number of solutions, but unfortunately, constructing a start system based on this bound seems as hard as solving the original given polynomial system. This paper presents a way for computing an upper bound together with the construction of a start system. The first computation is performed symbolically. Due to this symbolic computation, the constructed start system can be solved numerically more efficiently. The paper generalizes current approaches for homotopy construction towards the BKK bound.

Key words. Bézout number, BKK bound, homotopy continuation

1 Introduction

Continuation methods can be applied to compute all solutions to a given polynomial system $F = (f_1, f_2, \ldots, f_n)^T$, with $f_k \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ for $k = 1, 2, \ldots, n$. Therefore, together with a start system G, whose solutions are known, the system F is embedded in a homotopy $\mathcal{H}$:

$$(1) \qquad \mathcal{H}(\vec{x}, t) = \gamma(1-t)^k G(\vec{x}) + t^k F(\vec{x}) = \vec{0}, \ \ \gamma, t \in \mathbb{C}, \ \ k \in \mathbb{N}_0, \text{ see [2].}$$

As the continuation parameter t varies from 0 to 1, one can apply standard numerical continuation methods [1, 19] to trace the solution paths.

The total degree d is defined as the product of all degrees $d_k = \deg(f_k)$, for $k = 1, 2, \ldots, n$. The classical Theorem of Bézout [16] in projective space states that, if the system F has a finite number of solutions, this number equals the total degree d. The term 'in projective space' means that d includes finite solutions and solutions at infinity as well, which are for most applications of no importance. It is our aim to compute all finite solutions, without the calculation of the solutions at infinity.

In order to avoid the computation of solutions at infinity, Morgan and Sommese [13] proposed to apply the multi-projective version of Bézout's theorem [16]. In [18], Wampler, Morgan and Sommese explained how to construct an m-homogeneous start system. For a special class of polynomial systems, Li, Sauer and Yorke [10] developed the Random Product Homotopy, well suited to solve polynomial systems belonging to this class. In [17], Verschelde, Beckers and Haegemans extended the use of the Newton Homotopy [1] to more than one solution path. The problem is to construct a trivial to solve start system in order to compute efficiently all finite solutions.

In [5], Canny and Rojas proved the Vertex Coefficient Theorem. They show that the BKK bound, named after Bernshteĭn [3], Kushnirenko [9] and Khovanskiĭ [8], is an exact bound for the number of solutions in $\mathbb{C}_0^n$, $\mathbb{C}_0 = \mathbb{C} \setminus \{0\}$, when only *certain* coefficients of the system are generally chosen. This BKK bound is often much better than the Bézout number for the same system. However there are two difficulties for applying the BKK bound. First, computing the BKK bound for general dimensions is very complicated. The second major problem is that no algorithm seems to be available at the moment for the construction of a trivial to solve system that has exactly a number of nonsingular solutions equal to the BKK bound and that can be useful for homotopy continuation.

In this paper, a new upper bound for the number of solutions in $\mathbb{C}^n$ will be introduced, which is not difficult to compute. The construction of a start system follows then immediately. This paper generalizes the current approaches for constructing start systems to be used for polynomial continuation towards the BKK bound. This means that in general our upper bound lies between the bounds obtained by current practical approaches [10, 13] and the BKK bound [3, 5, 8, 9].

The paper consists of a symbolic and a numerical part. The upper bound will be computed symbolically in the next section, while in the third section a construction algorithm will be presented, based on the symbolically computed upper bound. Then the start system G will be constructed and solved numerically. The latter is performed efficiently by the application of the results of the symbolic computations. Practical applications follow. Our conclusions are stated in the last section.

2 On the number of finite solutions

This section is organized as follows. First the definition of the BKK bound will be given. Then, based on the supporting set structure, a new upper bound for the number of finite solutions can be computed, which leads immediately to the construction of a start system.

2.1 The BKK bound

Bernshteĭn [3], Kushnirenko [9] and Khovanskiĭ [8] introduced an upper bound for the number of solutions in $\mathbb{C}_0^n$ of a system of Laurent polynomials. Denote a Laurent polynomial $f \in \mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ by $f = \sum_{q \,\in\, \mathbb{Z}^n} c_q \, x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$, $c_q \in \mathbb{C}$, using a multi-index notation.

Definition 2.1 The *support of* f, denoted by *supp*(f), is the set of all $q \in \mathbb{Z}^n$, for which $c_q \neq 0$.

Definition 2.2 The *Newton polytope of* f is the convex hull of supp(f) in $\mathbb{R}^n$.

To the system $F = (f_1, f_2, \ldots, f_n)^T$, an n-tuple of Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)^T$ is associated, where each P_k is the respective Newton polytope of f_k, for $k = 1, 2, \ldots, n$. Let $P_1 + P_2 = \{x_1 + x_2 \mid x_1 \in P_1, x_2 \in P_2\}$ be the sum of two polytopes P_1 and P_2.

Definition 2.3 (See [3].) The *BKK bound* is defined as the mixed volume $V(\mathcal{P})$:

$$(2) V(\mathcal{P}) = (-1)^{n-1} \sum_i V_n(P_i) + (-1)^{n-2} \sum_{i<j} V_n(P_i + P_j) + \cdots + V_n(P_1 + P_2 + \cdots + P_n)$$

where $V_n(P)$ stands for the standard n-dimensional Lebesgue measure.

For a more detailed discussion about these definitions, we refer to the appendix of [15]. **B**ernshteĭn [3], **K**ushnirenko [9] and **K**hovanskiĭ [8] proved the following

Theorem 2.1 *Let F be a system of Laurent polynomials, with Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)^T$. Then the number of isolated solutions in $\mathbb{C}_0^n$ is bounded by the mixed volume $V(\mathcal{P})$.*

This theorem justifies the name *BKK bound* for the mixed volume $V(\mathcal{P})$.

Example 2.1 Consider the following polynomial system:

$$(3) \qquad F(\vec{x}) = \begin{cases} f_1 & : \; x_1^2 + x_1 x_2 + 3x_1 - 1 = 0 \\ f_2 & : \; x_1^2 + 2x_1 x_2 + x_2 + 1 = 0 \end{cases}$$

The total degree equals 4, while there are only 3 finite solutions.
Figure 1 pictures the Newton polytopes needed for the calculation of the mixed volume. The powers of x_1 and x_2 are denoted by q_1 and q_2 respectively.
Let $\mathcal{P} = (P_1, P_2)$, then the mixed volume $V(\mathcal{P})$ is computed as follows:

$$(4) \qquad V(\mathcal{P}) = -(V_2(P_1) + V_2(P_2)) + V_2(P_1 + P_2) = -(1 + \frac{3}{2}) + \frac{11}{2} = 3$$

where V_2 stands for the standard area. Thus the BKK bound equals 3.

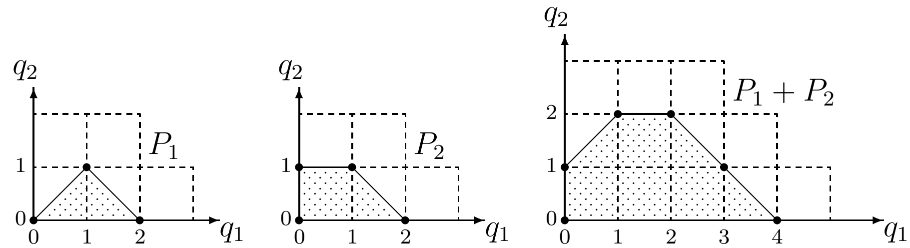

Figure 1: Newton polytopes P_1, P_2 supporting f_1, f_2 respectively.

The Vertex Coefficient Theorem, proved by Canny and Rojas [5], states that the BKK bound depends strongly on coefficients corresponding to vertices and boundaries of the Newton polytope and is only weakly dependent on its remaining coefficients. This means that the BKK bound is exact when only certain coefficients are generally chosen. The BKK bound indicates the lowest number of paths that must be traced in a homotopy continuation environment, for the computation of all solutions in $\mathbb{C}_0^n$. However, it is not clear at the moment how such an ideal homotopy can be constructed. Therefore, we propose a different upper bound, which leads immediately to the construction of a trivial to solve polynomial system.

2.2 The set structure

Instead of associating an n-tuple $\mathcal{P}$ of polytopes to the system F, a set structure $\mathcal{S}$ will be used to compute an upper bound.

Let X denote $\{x_1, x_2, \ldots, x_n\}$, the set of unknowns of a polynomial system of n equations.

Definition 2.4 A *set structure* S is defined as $\mathcal{S} = (S_1, S_2, \ldots, S_n)^T$, where each S_k is a set of subsets of X, for $k = 1, 2, \ldots, n$.

Definition 2.5 Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ and S be a set of subsets of X. Then S is said to be *supporting* for the polynomial f if it satisfies the following:

1. For each term $c_q x_k^q$ of the polynomial f, there are q sets of S that contain x_k.
2. For each term $c_q x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$ of the polynomial f, there exist q_1 sets of S that contain x_1 such that, if they are removed from S, the resulting set of subsets $\tilde{S}$ is supporting for the term $c_q x_2^{q_2} \cdots x_n^{q_n}$.

Definition 2.6 Given a polynomial system $F = (f_1, f_2, \ldots, f_n)$, with f_k a polynomial in n unknowns, for $k = 1, 2, \ldots, n$.
The set structure $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ is said to be *supporting* for the polynomial system F if each set S_k is supporting for the respective polynomial f_k, for all $k = 1, 2, \ldots, n$.
Then $\mathcal{S}$ is the *supporting set structure* for the polynomial system F.

1	$\{x_1\}$	$\{x_1, x_2\}$
2	$\{x_1\}$	$\{x_1, x_2\}$

Table 1: The supporting set structure $\mathcal{S}$ for F

Example 2.2 For the system presented in Example 2.1, the supporting set structure $\mathcal{S}$ is displayed in Table 1.

As with m-homogenization [13], there are many ways to choose the set structure $\mathcal{S}$, but in practice, this choice follows from the structure of the polynomial system. Figure 2 shows the pseudo code for a heuristic construction of the supporting set of sets for one polynomial. By using the algorithm proposed in Figure 2, a supporting set structure for a polynomial system can be constructed. The application of the algorithm is illustrated in Figure 3. It satisfies the conditions of Proposition 3.1. However, this algorithm is only a proposal. It can happen that *better* supporting set structures exist, which are not generated by this algorithm. With *better*, we mean a set structure that yields a lower upper bound. The last example of the fourth section is an illustration of this. If we speak of *the* set structure $\mathcal{S}$, we mean *this* set structure $\mathcal{S}$ leading to the lowest upper bound. However, one may not conclude that such a set structure is unique.

2.3 The upper bound based on the set structure

This section explains the computation of a new upper bound for the number of finite solutions of a polynomial system based on its supporting set structure.

Definition 2.7 Let $\mathcal{S} = (S_1, S_2, \ldots, S_n)^T$ be a set structure. An *acceptable class of* S, denoted by $\mathcal{C}_{\mathcal{S}}$, is an n-tuple of subsets of X such that for $k = 1, 2, \ldots, n$ the following holds:

1. The k-th subset of $\mathcal{C}_{\mathcal{S}}$ belongs to S_k.
2. Any union of k subsets of $\mathcal{C}_{\mathcal{S}}$ contains at least k elements of X.

If an n-tuple of subsets of X satisfies the first condition, the second one can be checked by generating all possible unions U of k sets in the tuple and checking if $\#U \geq k$, for all $k = 1, 2, \ldots, n$. This is done in the algorithm shown in Figure 4.
The following definition characterizes the number $B^*_{\mathcal{S}}$:

Definition 2.8 Let F be a polynomial system and $\mathcal{S}$ a supporting set structure for F. Then $B^*_{\mathcal{S}}$ is defined as the number of all acceptable classes of $\mathcal{S}$.

The characterization of $B^*_{\mathcal{S}}$ in Definition 2.8 enables the calculation. By generating all n-tuples of the set structure $\mathcal{S}$ that satisfy the first condition of Definition 2.7, the algorithm shown in Figure 4 can be used for checking if the n-tuple is an acceptable class.

function BUILD_SET_OF_SETS (f : polynomial) return Set_of_Sets is

$--$ ON ENTRY : $f(\vec{x}) = \sum_{i=1}^{N} c_i \ x_1^{d_{i1}} x_2^{d_{i2}} \cdots x_n^{d_{in}}$.

$--$ ON RETURN : $T = \{T_1, T_2, \ldots, T_d\}$, with $d = \deg(f)$.

d : natural := $\deg(f)$;
m : natural;

begin

  for k in $1, 2, \ldots, d$ loop
    $T_k := \emptyset$;
  end loop;

  for i in $1, 2, \ldots, N$ loop
    if $\not\exists k < i$: $d_{kj} \geq d_{ij}$, $j = 1, 2, \ldots, n$
     then $m := 1$;
          for k in $1, 2, \ldots, n$ loop
            for l in $1, 2, \ldots, d_{ik}$ loop
              $T_m := T_m \cup \{x_k\}$;
              $m := m + 1$;
            end loop;
          end loop;
    end if;
  end loop;

  return T;

end BUILD_SET_OF_SETS;

Figure 2: Algorithm for the heuristic construction of a set of sets.

The polynomial f:

$$
\begin{array}{ccccccccc}
f & = & x_1^2 & + & x_1x_2 & + & 3x_1 & - & 1 \\
\\
 & & i=1 & & i=2 & & i=3 & & i=4 \\
 & & d_{11}=2 & & d_{21}=1 & & d_{31}=1 & & d_{41}=0 \\
 & & d_{12}=0 & & d_{22}=1 & & d_{32}=0 & & d_{42}=0
\end{array}
$$

Initalization:

$$
\begin{array}{l}
d := \deg(f) = 2; \\
T \;:=\; \{ \underset{T_1}{\emptyset} \;,\; \underset{T_2}{\emptyset} \};
\end{array}
$$

The execution of the main loop:

$$
\begin{array}{lllll}
i=1 & \not\exists k<1 & & & \\
 & m:=1; & & & \\
 & & k:=1; & d_{11}=2 & \\
 & & & l:=1; & T_1 := T_1 \cup \{x_1\}; \quad m:=m+1; \\
 & & & l:=2; & T_2 := T_2 \cup \{x_1\}; \quad m:=m+1; \\
 & & k:=2; & d_{12}=0 & \\
\\
i=2 & \not\exists k<2, \text{ because } d_{22}=1>d_{12}=0 & & & \\
 & m:=1; & & & \\
 & & k:=1; & d_{21}=1 & \\
 & & & l:=1; & T_1 := T_1 \cup \{x_1\}; \quad m:=m+1; \\
 & & k:=2; & d_{22}=1 & \\
 & & & l:=1; & T_2 := T_2 \cup \{x_2\}; \quad m:=m+1; \\
\\
i=3 & \exists k=1, \; d_{11}=2 \geq d_{31}=1, \; d_{12}=0 \geq d_{32}=0 & & & \\
\\
i=4 & \exists k=1, \; d_{11}=2 \geq d_{41}=0, \; d_{12}=0 \geq d_{42}=0 & & &
\end{array}
$$

Returning the result :

$$\text{return } T = \{\{x_1\}, \{x_1, x_2\}\};$$

Figure 3: An example illustrating the construction of a set of sets.

```
function IS_ACCEPTABLE ( T : n-tuple ) return boolean is

  -- ON ENTRY : T = (T_1, T_2, ..., T_n)^T, an n-tuple of subsets of X.
                T satisfies the first condition of Definition 2.7.

  -- ON RETURN : true if T is an acceptable class, false otherwise.

begin

  for k in 2, 3, ..., n loop
    -- (T_1, ..., T_{k-1})^T is acceptable
    for l in 1, 2, ..., k - 1 loop
      for all possible unions U of l sets out of (T_1, ..., T_{k-1})^T loop
        if #(U ∪ T_k) < k
         then return false;
        end if;
      end loop;
    end loop;
  end loop;
  return true;

end IS_ACCEPTABLE;
```

Figure 4: Algorithm for checking if an n-tuple is an acceptable class.

The following gives the meaning of the defined number $B^*_{\mathcal{S}}$:

Proposition 2.1 *Let F be a polynomial system with supporting set structure $\mathcal{S}$. If F has a finite number of solutions in $\mathbb{C}^n$, counted with multiplicities, then this number is lower than or equal to $B^*_{\mathcal{S}}$.*

It will be proved in the next section.

Example 2.3 For the system of Example 2.1, the upper bound $B^*_{\mathcal{S}}$, based on the set structure proposed in Example 2.2, will be calculated as follows

$$B^*_{\mathcal{S}} = \underset{\{\{x_1\},\{x_1,x_2\}\}}{1} + \underset{\{\{x_1,x_2\},\{x_1\}\}}{1} + \underset{\{\{x_1,x_2\},\{x_1,x_2\}\}}{1} = 3 \ . \tag{5}$$

Underneath the formula (5), the acceptable classes are indicated. This yields an upper bound for the number of finite solutions of the system presented in Example 2.1, which is better than the total degree.

3 Homotopy Construction

In this section, the algorithm for the construction of a random product system G will be explained. Theoretical results follow.

3.1 Random Product Start Systems

Definition 3.1 Let $S = \{T_1, T_2, \ldots, T_m\}$ be a set of subsets of X. A *random product start polynomial g based on S* is defined as

$$g = \prod_{k=1}^{m} \left(\alpha_0^{(k)} + \sum_{x_i \in T_k} \alpha_i^{(k)} x_i \right) \tag{6}$$

where all $\alpha_i^{(k)}$ and $\alpha_0^{(k)}$ are randomly chosen complex numbers, different from zero.

Definition 3.2 Let $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ be a set structure. A *random product start system G based on $\mathcal{S}$* is defined as the polynomial system $G = (g_1, g_2, \ldots, g_n)$, where each g_k is a random product start polynomial based on S_k, for $k = 1, 2, \ldots, n$.

Example 3.1 For the system of Example 2.1, based on the supporting set structure, see Example 2.2, the following *random product start system G* can be constructed:

$$G(\vec{x}) = \begin{cases} (x_1 + \alpha_1)(x_1 + \alpha_2 x_2 + \alpha_3) = 0 \\ (x_1 + \beta_1)(x_1 + \beta_2 x_2 + \beta_3) = 0 \end{cases} \tag{7}$$

where α_1, α_2, α_3, β_1, β_2 and β_3 are randomly chosen numbers. Thus, applying this start system, only 3 paths remain to be traced. Note that the classical and the 2-homogeneous Bézout numbers all equal 4.

Observe the duality between the computation of the upper bound $B^*_{\mathcal{S}}$ and the solution of the associated start system G. More precisely, for each acceptable class of the supporting set structure $\mathcal{S}$, there corresponds one linear system, yielding a regular solution of the start system. For example, for the first acceptable class in the formula (5) for calculating $B^*_{\mathcal{S}}$ in Example 2.3, the following correspondence holds:

$$\left\{ \begin{array}{c} \{x_1\} \\ \{x_1, x_2\} \end{array} \right\} \iff \left\{ \begin{array}{c} x_1 + \alpha_1 = 0 \\ x_1 + \beta_2 x_2 + \beta_3 = 0 \end{array} \right. \tag{8}$$

Definition 3.3 A solution to a polynomial system is *nonsingular* if the Jacobian matrix has full rank.

Theorem 3.1 *Let $\mathcal{S} = (S_1, S_2, \ldots, S_n)$ be a given set structure. Then for every random choice of the coefficients of the start polynomials, except for a set of measure zero, the random product start system G has exactly $B^*_{\mathcal{S}}$ finite nonsingular solutions, where $B^*_{\mathcal{S}}$ equals the BKK bound of G.*

Proof. The BKK bound of the system G will be computed, by considering all linear systems that come out of the random product system G. First some notations are needed. Let $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ be the n-tuple of Newton polytopes of G. Because each equation g_i of G is the product of linear equations, each polytope P_i can be written as $P_i = L_{i1} + L_{i2} + \ldots + L_{im_i}$, where $m_i = \#S_i$ and where each L_{ij} is the Newton polytope of a linear equation. Because of the fact that the mixed volume $V(\mathcal{P})$ is multilinear, see [3] [15, appendix A.4], $V(\mathcal{P})$ is the sum of all mixed volumes $V(L_{1j_1}, L_{2j_2}, \ldots, L_{nj_n})$, with $1 \leq j_i \leq m_i$, for all $i = 1, 2, \ldots, n$. Denote then the corresponding linear systems by $A^{(J)}\vec{x} = b^{(J)}$, using a multi-index notation, $J = (j_1, j_2, \ldots, j_n)$, where for each linear system M_J nonzero coefficients are involved. Let N denote the total number of nonzero coefficients which can be chosen freely in the start polynomials, then $M_J \leq N$.
For each linear system $A^{(J)}\vec{x} = b^{(J)}$, there are two possibilities:

1. If the linear system corresponds to an acceptable class, then the system has one finite solution and $V(L_{1j_1}, L_{2j_2}, \ldots, L_{nj_n}) = 1$. Except for the case where $\det(A^{(J)}) = 0$, which can be expressed by a polynomial equation in M_J unknowns determining a space of dimension $M_J - 1$, a set of measure zero in $\mathbb{C}^N$.

2. If the linear system does not correspond to an acceptable class, then there is no finite solution and $V(L_{1j_1}, L_{2j_2}, \ldots, L_{nj_n}) = 0$. In this case, the second condition of Definition 2.7 is violated. This means that there are k sets, whose union contains less than k unknowns. Denote this union by the set S, with $s = \#S$ and let $k = s+r$, with $r \geq 1$. So the linear system contains $s + r$ equations in the unknowns of the set S, which has in general no finite solution. The exceptional case where there is a finite solution corresponds to the case where all possible choices of $s + 1$ equations out of these $s + r$ equations are linearly dependent. Denote the number of all possible choices by c and denote all choices of $s + 1$ equations by $A^{(Jl)}\vec{x} = b^{(Jl)}$, for $l = 1, 2, \ldots, c$. The exceptional case can then be expressed by c polynomial equations, defined by $\det(A^{(Jl)}|b^{(Jl)}) = 0$, yielding spaces of dimension $M_J - k_l$, with all $k_l \geq 1$, for $l = 1, 2, \ldots, c$. Hence, in order to have a finite solution, the coefficients of these $s + r$ equations must belong to the intersection of these spaces of measure zero, which is again a space of measure zero in $\mathbb{C}^N$.

The finite union of sets of measure zero is also a set of measure zero in $\mathbb{C}^N$. Except for this set of measure zero, there are exactly $B^*_{\mathcal{S}}$ linear systems whose matrices are nonsingular. Multiple solutions can only occur when two linear systems are identical, which is again a choice of the coefficients belonging to a set of measure zero. Hence, except for some set of measure zero, G has exactly $B^*_{\mathcal{S}}$ finite nonsingular solutions and $B^*_{\mathcal{S}}$ equals the BKK bound $V(\mathcal{P})$. □

The start system can be solved by computing all solutions to the linear systems, but one has only to solve *these* linear systems that correspond to acceptable classes. There is a one-to-one correspondence between the set structure $\mathcal{S}$ and the start system G. Positions within the set structure $\mathcal{S}$ determine linear systems to be solved. Thus, the algorithm for computing $B^*_{\mathcal{S}}$ should also give the positions corresponding to the acceptable classes in order to solve the start system G more efficiently. For the solution of the start system in Example 3.1, only 3 linear systems must be solved, instead of 4.

3.2 Theoretical results

Lemma 3.1 *Let F be a polynomial system with supporting set structure $\mathcal{S}$ and G the random product start system based on $\mathcal{S}$. Define the homotopy $\mathcal{R}$ by*

$$\mathcal{R}(\vec{x}, t) = G(\vec{x}) + tF(\vec{x}). \tag{9}$$

Then for all t, the system $\mathcal{R}(\vec{x}, t) = \vec{0}$ has not more than $B^*_{\mathcal{S}}$ finite nonsingular solutions.

Proof. By Theorem 3.1, the system G has exactly $B^*_{\mathcal{S}}$ finite nonsingular solutions. By definition of the random product system G, the Newton polytopes of G contain those of F. Therefore, the Newton polytopes remain invariant, for all t. Hence, the BKK bound for all systems $\mathcal{R}(\vec{x}, t) = \vec{0}$ equals $B^*_{\mathcal{S}}$. □

Definition 3.4 A solution to a polynomial system is called *geometrically isolated* if there exists a neighborhood of the solution that contains no other solution.

Theorem 3.2 allows the usage of the random product start system G in a homotopy continuation environment.

Theorem 3.2 *Let F be a polynomial system with supporting set structure $\mathcal{S}$. Let G be the start system based on the set structure $\mathcal{S}$ with exactly $B^*_{\mathcal{S}}$ nonsingular solutions. Consider the following homotopy:*

$$\mathcal{H}(\vec{x}, t) = \gamma(1-t)^k G(\vec{x}) + t^k F(\vec{x}) = \vec{0}, \ \gamma \in \mathbb{C}, \ t \in [0, 1], \ k \in \mathbb{N}_0. \tag{10}$$

Then for all, but a finite number of angles θ, $\gamma = re^{i\theta}$, $r \in \mathbb{R}^+_0$, the following holds:

1. *$\mathcal{H}^{-1}(0)$ consists of smooth paths over $[0, 1)$ and every geometrically isolated solution of $F(\vec{x}) = \vec{0}$ has a path converging to it;*
2. *if m_0 is the multiplicity of a geometrically isolated solution $\vec{z}_0$, then $\vec{z}_0$ has exactly m_0 paths converging to it;*
3. *the paths are strictly increasing in t, $\frac{dt}{ds} > 0$, for $t \in [0, 1)$ where s is the arc length parameter.*

Proof. First a homogenization of the homotopy will be described. To the k-th equation of F and G corresponds the supporting set S_k. If x_j occurs in j_l sets of S_k, then, for the s-th occurrence of x_j, x_j will be replaced by x_{js}. As this introduces new unknowns, the following linear equations will be added in order to keep the same solutions:

$$x_{j1} - x_{js} = 0 \quad \text{for } s = 2, 3, \ldots, j_l. \tag{11}$$

By replacing x_{js} in the k-th equation by x_{kjs} and adding the following linear equations

$$x_{1j1} - x_{kj1} = 0 \quad \text{for } k = 2, 3, \ldots, n \quad \text{for } j = 1, 2, \ldots, n \tag{12}$$

the solutions remain unchanged and all sets belonging to the set structure $\mathcal{S}$ can be linearized into one partition Z. With respect to this partition Z, both systems have the same multi-homogeneous structure. Denote the classical projective space by $\mathbb{P}^1$. The unknowns belonging to the i-th set S_i of the partition Z will be embedded in an m_i-dimensional projective space $\mathbb{P}^{m_i}$, where $m_i = \#S_i$. The direct product of all projective spaces $\mathbb{P}^{m_i}$ will be denoted by $\mathbb{P}$.

Consider the multi-homogeneous homotopy

$$\tilde{\mathcal{H}} = \mu_0 \tilde{G}(\vec{z}) + \mu_1 \tilde{F}(\vec{z}), \qquad (\mu_0, \mu_1) \in \mathbb{P}^1, \tag{13}$$

where $\vec{z}$ belongs to the multi-projective space $\mathbb{P}$. Let $\tilde{Y}$ be the union of the irreducible components of $\tilde{\mathcal{H}}^{-1}(\vec{0})$ in $\mathbb{P}$ which contain at least one of the $B^*_{\mathcal{S}}$ nonsingular finite solutions of $\tilde{G}$, $\tilde{Y}$ is an algebraic set in $\mathbb{P} \times \mathbb{P}^1$. By Theorem 3.1, for $(1, 0) \in \mathbb{P}^1$, $\tilde{Y}$ contains exactly $B^*_{\mathcal{S}}$ nonsingular finite solutions. Denote the natural projection on $\mathbb{P}^1$ by

$$\pi_2 : \mathbb{P} \times \mathbb{P}^1 \to \mathbb{P}^1. \tag{14}$$

Let $U \subset \tilde{Y}$ be the set of points where singularities occur. By [6, Lemma, p. 97], U is an analytic set, and by Chow's Theorem [7, p. 167], U is an algebraic set. By the Main Theorem of elimination theory [14, p. 33], the projection of U, $\pi_2(U)$ is an algebraic set. $\pi_2(U)$ is a proper subset of $\mathbb{P}^1$, because for $(1, 0)$ all solutions are nonsingular. Hence, $\pi_2(U)$ is finite.

Let $V \subset \tilde{Y}$ be the set of points were solutions at infinity occur. V is an algebraic set and so is its projection $\pi_2(V)$. Because for $(1, 0)$ all $B^*_{\mathcal{S}}$ solutions are finite, $\pi_2(V)$ is a proper algebraic set in $\mathbb{P}^1$. Hence, $\pi_2(V)$ is finite. Let $W = \pi_2(U) \cup \pi_2(V)$. Because $W \subset \mathbb{P}^1$ is finite, only a finite number of rays $re^{i\theta}$ can intersect W. Since then no singularities occur for the interval $[0, 1)$, $\frac{dt}{ds} > 0$. Hence, the smoothness property is proved.

Consider the homotopy $\mathcal{H}$ in affine space, with set of paths Y. Let $\vec{z}_0$ be a nonsingular isolated finite solution of F. By the Implicit Function Theorem [14, p. 10–11], there are unique convergent power series in t to denote the solutions in the neighborhood of $\vec{z}_0$. So the solution $\vec{z}_0$ can be extended for $t < 1$. Because the solution is finite, for $t < 1$, the extended solution is also finite. By the smoothness property, there exists a path, parameterized by $t \in (0, 1)$. By Lemma 3.1, the path that ends at $\vec{z}_0$, belongs to Y.

Let $\vec{z}_0$ be an isolation solution of F with multiplicity m_0. By a slight perturbation of F, for $t < 1$, m_0 isolated regular solutions lie in the neighborhood of $\vec{z}_0$. According to previous reasoning, every isolated regular solution is reached by a path starting at a solution of G. Hence, for $t \to 1$, every isolated solution $\vec{z}_0$ with multiplicity m_0, has m_0 paths converging to it. □

In the proof, a transformation has been made into a higher dimensional space. Because of practical considerations, continuation happens in the n-dimensional space. Otherwise, the computational advantage of this approach would be destroyed.

The following can be considered as a generalization of Bézout's theorem.

Corollary 3.1 *If F has a finite number of solutions in $\mathbb{C}^n$, counted with multiplicities, then this number is lower than or equal to $B^*_{\mathcal{S}}$.*

The following illustrates the usefulness of the upper bound $B^*_{\mathcal{S}}$ w.r.t. the total degree d of the polynomial system.

Proposition 3.1 *Let F be a polynomial system with supporting set structure $\mathcal{S}$, where $\mathcal{S} = (S_1, S_2, \ldots, S_n)^T$. If the number of sets in S_k, for all $k = 1, 2, \ldots, n$, does not exceed d_k, the degree of the k-th equation of F, then $B^*_{\mathcal{S}} \leq d$, where d is the total degree of F.*

Proof. Based on the set structure $\mathcal{S}$, a random product start system G can be constructed. G has exactly $B^*_{\mathcal{S}}$ finite nonsingular solutions. While the number of sets in S_k does not exceed d_k, $\deg(g_k) \leq d_k$, where g_k is the k-th equation of G. Hence, $B^*_{\mathcal{S}} \leq d$. $\square$

4 Applications

4.1 Polynomial systems

All systems presented, occur in the literature [4, 11, 12, 19] and are coming from practical applications. We focus on a class of systems for which $B^*_{\mathcal{S}}$ yields a sharper upper bound than the Bézout number obtained by m-homogenization and for which the Random Product Homotopy cannot be applied. Together with the system, the supporting set structure will be written. For the first three systems, the set structure has been generated by the algorithm shown in Figure 2. But for the fourth example, a better supporting set structure exists, yielding a lower upper bound. Also the partition Z of the set of unknowns will be given, yielding the lowest m-homogeneous Bézout number, denoted by B_Z. In [13], one can find a combinatorial definition of B_Z.

1. This system is derived from optimizing the Wood function [12]:

$$F_A(\vec{x}) = \begin{cases} 200x_1^3 - 200x_1x_2 + x_1 - 1 = 0 \\ -100x_1^2 + 110.1x_2 + 9.9x_4 - 20 = 0 \\ 180x_3^3 - 180x_3x_4 + x_3 - 1 = 0 \\ -90x_3^2 + 9.9x_2 + 100.1x_4 - 20 = 0 \end{cases}$$

The total degree of this system equals 36, while there is only one real solution and 8 complex conjugate solutions.
Table 2 shows the supporting set structure $\mathcal{S}$, which yields $B^*_{\mathcal{S}} = 16$.

1	$\{x_1\}$	$\{x_1\}$	$\{x_1, x_2\}$
2		$\{x_1\}$	$\{x_1, x_2, x_4\}$
3	$\{x_3\}$	$\{x_3\}$	$\{x_3, x_4\}$
4		$\{x_3\}$	$\{x_2, x_3, x_4\}$

Table 2: The supporting set structure $\mathcal{S}$ for F_A.

Taking $Z = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}\}$, $B_Z = 25$.

2. The following chemical equilibrium problem has been stated in [11]:

$$F_B(\vec{x}) = \begin{cases} x_1x_2 + x_1 - 3x_5 = 0 \\ 2x_1x_2 + x_1 + 2R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 \\ \quad +R_9x_2x_4 + R_8x_2 - Rx_5 = 0 \\ 2x_2x_3^2 + R_7x_2x_3 + 2R_5x_3^2 + R_6x_3 - 8x_5 = 0 \\ R_9x_2x_4 + 2x_4^2 + 4Rx_5 = 0 \\ x_1x_2 + x_1 + R_{10}x_2^2 + x_2x_3^2 + R_7x_2x_3 + R_9x_2x_4 \\ \quad +R_8x_2 + R_5x_3^2 + R_6x_3 + x_4^2 - 1 = 0 \end{cases}$$

The total degree equals 108, but there are only 4 real and 12 complex solutions. The constants R and R_j can be found in [11].

The supporting set structure $\mathcal{S}$ is listed in Table 3, yielding $B^*_{\mathcal{S}} = 44$.

1		$\{x_1, x_5\}$	$\{x_2\}$
2	$\{x_3\}$	$\{x_2, x_3, x_4\}$	$\{x_1, x_2, x_5\}$
3	$\{x_3\}$	$\{x_3\}$	$\{x_2, x_5\}$
4		$\{x_4\}$	$\{x_2, x_4, x_5\}$
5	$\{x_3\}$	$\{x_2, x_3, x_4\}$	$\{x_1, x_2, x_4\}$

Table 3: The supporting set structure for F_B.

The lowest m-homogeneous Bézout number $B_Z = 56$, with $Z = \{\{x_1\}, \{x_2, x_4, x_5\}, \{x_3\}\}$.

3. The third example is a system coming out of an application in the field of electrochemistry. It is known as problem 601 in [19].

$$F_C(\vec{x}) = \begin{cases} a_1 x_2^6 + a_2 x_2^5 + a_3 x_2^4 + a_4 x_1^2 x_3 + a_5 x_2^3 + a_6 x_2^2 + a_7 x_2 + a_8 & = 0 \\ \qquad a_9 x_2^5 + a_{10} x_2^4 + a_{11} x_1^2 x_2 + a_{12} x_1^2 x_3 + a_{13} x_2^3 & \\ \qquad\qquad +a_{14} x_1 x_2 + a_{15} x_2^2 + a_{16} x_2 + a_{17} & = 0 \\ \qquad\qquad a_{18} x_1^2 + a_{19} x_1 x_3 + a_{20} x_2 + a_{21} & = 0 \end{cases}$$

The total degree equals 60, while there are only 13 solutions. The coefficients a_j for this problem are available on request to the author of [19].
In Table 4 the supporting set structure $\mathcal{S}$ is displayed, yielding $B^*_{\mathcal{S}} = 34$.
With $Z = \{\{x_1\}, \{x_2\}, \{x_3\}\}$, the lowest m-homogeneous Bézout number $B_Z = 48$.

4. The last system belongs to a family of systems, given in [4]:

$$F_D(\vec{x}) = \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0 \\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0 \\ x_1x_2x_3x_4x_5 - 1 = 0 \end{cases}$$

The total degree equals 120, but there are only 70 finite solutions.
In Table 5 the supporting set structure is displayed, yielding $B^*_{\mathcal{S}} = 108$. Although this does not substantially improve the total degree, it is an interesting example, because the heuristic algorithm presented in Figure 2 fails to give a supporting set structure which leads to a lower upper bound than the total degree. It justifies the

1	$\{x_2\}$	$\{x_2\}$	$\{x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_2, x_3\}$
2		$\{x_2\}$	$\{x_2\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	$\{x_2, x_3\}$
3					$\{x_1, x_2\}$	$\{x_1, x_3\}$

Table 4: The supporting set structure for F_C.

1	$\{x_1, x_2, x_3, x_4, x_5\}$
2	$\{x_1, x_3\}$ $\{x_2, x_4\}$ $\{x_5\}$
3	$\{x_1\}$ $\{x_2\}$ $\{x_3\}$ $\{x_4\}$ $\{x_5\}$
4	$\{x_1\}$ $\{x_2\}$ $\{x_3\}$ $\{x_4\}$ $\{x_5\}$
5	$\{x_1\}$ $\{x_2\}$ $\{x_3\}$ $\{x_4\}$ $\{x_5\}$

Table 5: The supporting set structure for F_D.

generality of Definition 2.5. A consequence of this is the fact that the total degree of the start system G can now be larger than the total degree of the system F that has to be solved. Therefore, for solving the start system G, more computational time can be gained by making use of the positions corresponding to the acceptable classes, see Table 6.
By using m-homogenization, no better upper bound than the total degree can be found, so $Z = \{\{x_1, x_2, x_3, x_4, x_5\}\}$, with $B_Z = 120$.

4.2 Performance

Table 6 shows why it is better to use our method for the construction of a start system. For the computation of the N finite solutions, during continuation, d, B_Z and $B^*_{\mathcal{S}}$ solution

	d	B_Z	$B^*_{\mathcal{S}}$	N
P_A	36	25	16	9
P_B	108	56	44	16
P_C	60	48	34	13
P_D	120	120	108	70

Table 6: Performance of the homotopies.

paths must be traced, when the start system is based on the total degree d, on the m-homogeneous Bézout number B_Z or on the upper bound $B^*_{\mathcal{S}}$.

The algorithms for computing $B^*_{\mathcal{S}}$, given the set structure $\mathcal{S}$, and for constructing and solving the start system G have been implemented on a SUN 3/280. Execution times, measured in cpu seconds, described in Table 7 only have a relative meaning.
As demonstrated in Table 7, one sees that, with the effort of computing $B^*_{\mathcal{S}}$, the start system G can be solved more efficiently, because of the fact that the acceptable classes are retained. Otherwise, all possible linear systems must be solved, when the numerical calculations are based on the total degree d of the start system.

	Computing $B_{\mathcal{S}}^*$	Solving G	
		based on $B_{\mathcal{S}}^*$	based on d
P_A	0.040	0.460	0.920
P_B	0.240	1.720	3.660
P_C	0.060	0.460	1.000
P_D	0.520	4.100	12.580

Table 7: Performance of the algorithms.

5 Conclusions

As start systems must be trivial to solve, random product systems are useful to the homotopy continuation method to solve polynomial systems. This paper describes a condition upon random product start systems, together with an efficient algorithm to construct and to solve them. Due to symbolic preprocessing, the start system can be solved efficiently. Finally, an efficient homotopy has been constructed symbolically.

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