The Euler-Maclaurin Summation Formula

To understand Romberg integration, we must know that the error expansion of the composite trapezoidal rule has only even powers of h. We can see this by the Euler-Maclaurin summation formula.

Theorem 1.1 *For $g \in C^{2m+2}[0,N]$ (g is sufficiently many times continuously differentiable over $[0,N]$):*

$$\frac{1}{2}g(0) + g(1) + \cdots + g(N-1) + \frac{1}{2}g(N) = \int_0^N g(t)dt + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}\left(g^{(2l-1)}(N) - g^{(2l-1)}(0)\right) + \frac{B_{2m+2}}{(2m+2)!}Ng^{(2m+2)}(\alpha), \quad \alpha \in [0,N], \tag{1}$$

where B_k are the Bernoulli numbers.

To see the connection with the composite trapezoidal rule, we make a change of coordinates:

$$[0,N] \rightarrow [a,b] : t \mapsto x = a + ht, \quad h = \frac{b-a}{N}, \quad dx = hdt, \quad \text{so} \int_0^N g(t)dt = \int_a^b f(x)\frac{1}{h}dx. \tag{2}$$

To replace the derivatives of $g(t)$ in (1), we observe $g(t) = f(a+ht) = f(x)$ and apply the chain rule:

$$g'(t) = f'(x)h \quad \text{and for any } l : g^{(l)}(t) = f^{(l)}(x)h^l. \tag{3}$$

After executing the coordinate change (2), formula (1) turns into

$$\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b) = \int_a^b f(x)\frac{1}{h}dx + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}h^{2l-1}\left(f^{(2l-1)}(b) - f^{(2l-1)}(a)\right) + \frac{B_{2m+2}}{(2m+2)!}Nh^{2m+2}f^{(2m+2)}(\beta), \quad \beta \in [a,b]. \tag{4}$$

Multiplying (4) by h, we obtain the error formula for the composite trapezoidal rule $T(h)$:

$$T(h) = \int_a^b f(x)dx + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!}h^{2l}\left(f^{(2l-1)}(b) - f^{(2l-1)}(a)\right) + \frac{B_{2m+2}}{(2m+2)!}(Nh)h^{2m+2}f^{(2m+2)}(\beta). \tag{5}$$

What matters most to us are the powers of the h in the error series: we observe that only even powers of h occur. This justifies the extrapolation formula used in the so-called Romberg integration.

The Bernoulli numbers B_k are defined as the values of the Bernoulli polynomials $B_k(t)$ at $t = 0$: $B_k = B_k(0)$. The Bernoulli polynomials satisfy the following differential equation:

$$B'_{k+1}(t) = (k+1)B_k(t), \quad \text{with } B_{2k+1}(0) = 0, B_{2k+1}(1) = 0, \text{for all } k > 0. \tag{6}$$

The last conditions imply that all Bernoulli numbers with an odd index are zero. The equations (6) define a recursion to compute $B_k(t)$, starting at $B_0(t) = 1$ and $B_1(t) = t - 1/2$. By taking anti-derivatives we solve the recursion (6), for example:

$$B'_2(t) = 2B_1(t) \Rightarrow B_2(t) = 2\int B_1(t)dt + C_1 = 2\int\left(t - \frac{1}{2}\right)dt + C_1 = 2\left(\frac{t^2}{2} - \frac{t}{2}\right) + C_1; \quad \text{and} \tag{7}$$

$$B'_3(t) = 3B_2(t) \Rightarrow B_3(t) = 3\int B_2(t)dt + C_2 = 3\int\left(t^2 - t + C_1\right) + C_2 = 3\left(\frac{t^3}{3} - \frac{t^2}{2} + C_1 t\right) + C_2. \tag{8}$$

The condition $B_3(0) = 0$ implies $C_2 = 0$ and $B_3(1) = 0$ leads to $C_1 = 1/6$, thus $B_2 = B_2(0) = 1/6$.

The Bernoulli polynomials appear naturally when integrating by parts. Recall this rule:

$$D(f \cdot g) = (Df) \cdot g + f \cdot (Dg) \quad \Rightarrow \quad \int D(f \cdot g) = \int (Df) \cdot g + \int f \cdot (Dg) \tag{9}$$

$$\Rightarrow \quad \int (Df) \cdot g = f \cdot g - \int f \cdot (Dg) \tag{10}$$

Let us apply this rule to our integral:

$$\int_0^1 1 \cdot g(t)dt = \int_0^1 B_1'(t) \cdot g(t)dt \tag{11}$$

$$= [B_1(t) \cdot g(t)]_0^1 - \int_0^1 B_1(t) \cdot g'(t)dt, \quad B_1(t) = t - \frac{1}{2}, B_2'(t) = 2B_1(t) \tag{12}$$

$$= \frac{g(0) + g(1)}{2} - \int_0^1 \frac{1}{2} B_2'(t) \cdot g'(t)dt \tag{13}$$

$$= \frac{g(0) + g(1)}{2} - \left[\frac{1}{2} B_2(t) \cdot g'(t)\right]_0^1 + \int_0^1 \frac{1}{2} B_2(t) \cdot g''(t)dt, \quad B_3'(t) = 3B_2(t) \tag{14}$$

$$= \frac{g(0) + g(1)}{2} - \left[\frac{1}{2!} B_2(t) \cdot g'(t)\right]_0^1 + \left[\frac{1}{3!} B_3(t) \cdot g''(t)\right]_0^1 - \int_0^1 \frac{1}{3!} B_3(t) \cdot g'''(t)dt \tag{15}$$

$$= \frac{g(0) + g(1)}{2} - \left[\frac{1}{2!} B_2(t) \cdot g'(t)\right]_0^1 + \left[\frac{1}{3!} B_3(t) \cdot g''(t)\right]_0^1 - \left[\frac{1}{4!} B_4(t) \cdot g'''(t)\right]_0^1 \tag{16}$$

$$+ \int_0^1 \frac{1}{4!} B_4(t) \cdot g^{(4)}(t)dt \tag{17}$$

We see how the Bernoulli numbers come in when evaluating the definite integrals. But, notice, we need to evaluate the Bernoulli polynomials at $t = 1$. Fortunately, the Bernoulli polynomials satisfy

$$(-1)^k B_k(1 - t) = B_k(t), \tag{18}$$

because the polynomials $(-1)^k B_k(1 - t)$ satisfy the same recursion (6) as $B_k(t)$, and since the recursion starts off with the same polynomials, all polynomials must be the same. Thus we know $B_k(1)$ from B_k. With the mean value theorem for integrals, there exists some $\xi \in [0, 1]$, so we have

$$\int_0^1 g(t)dt = \frac{g(0) + g(1)}{2} + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!} \left(g^{(2l-1)}(0) - g^{(2l-1)}(1)\right) - \frac{B_{2m+2}}{(2m + 2)!} g^{(2m+2)}(\xi) \tag{19}$$

This shows the Euler-Maclaurin summation formula for one interval. To show (1), we apply (19) N times:

$$\int_0^N g(t)dt = \int_0^1 g(t)dt + \int_1^2 g(t)dt + \cdots + \int_{N-1}^N g(t)dt. \tag{20}$$

References

[1] J. Stoer and R. Bulirsch *Introduction to Numerical Analysis.* Second Edition, Springer–Verlag, 1993.