The Euler-Maclaurin Summation Formula

To understand Romberg integration, we must know that the error expansion of the composite trapezoidal rule has only even powers of \( h \). We can see this by the Euler-Maclaurin summation formula.

**Theorem 1.1** For \( g \in C^{2m+2}[0,N] \) (\( g \) is sufficiently many times continuously differentiable over \([0,N]\)):

\[
\frac{1}{2}g(0) + g(1) + \cdots + g(N-1) + \frac{1}{2}g(N) = \int_0^N g(t)dt + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!} \left( g^{(2l-1)}(N) - g^{(2l-1)}(0) \right) + \frac{B_{2m+2}}{(2m+2)!} Nh^{2m+2}(\alpha), \quad \alpha \in [0,N],
\]

where \( B_k \) are the Bernoulli numbers.

To see the connection with the composite trapezoidal rule, we make a change of coordinates:

\[
[0,N] \rightarrow [a,b]: t \mapsto x = a + ht, \quad h = \frac{b-a}{N}, \quad dx = hdt, \quad \text{so} \quad \int_0^N g(t)dt = \int_a^b f(x)\frac{1}{h}dx.
\]

To replace the derivatives of \( g(t) \) in (1), we observe \( g(t) = f(a + ht) = f(x) \) and apply the chain rule:

\[
g'(t) = f'(x)h \quad \text{and for any } l: g^{(l)}(t) = f^{(l)}(x)h^l.
\]

After executing the coordinate change (2), formula (1) turns into

\[
\frac{1}{2}f(a) + f(a+h) + \cdots + f(b-h) + \frac{1}{2}f(b) = \int_a^b f(x)\frac{1}{h}dx + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!} h^{2l-1} \left( f^{(2l-1)}(b) - f^{(2l-1)}(a) \right) + \frac{B_{2m+2}}{(2m+2)!} Nh^{2m+2}f^{(2m+2)}(\beta), \quad \beta \in [a,b].
\]

Multiplying (4) by \( h \), we obtain the error formula for the composite trapezoidal rule \( T(h) \):

\[
T(h) = \int_a^b f(x)dx + \sum_{l=1}^{m} \frac{B_{2l}}{(2l)!} h^{2l} \left( f^{(2l-1)}(b) - f^{(2l-1)}(a) \right) + \frac{B_{2m+2}}{(2m+2)!} Nh^{2m+2}f^{(2m+2)}(\beta).
\]

What matters most to us are the powers of the \( h \) in the error series: we observe that only even powers of \( h \) occur. This justifies the extrapolation formula used in the so-called Romberg integration.

**The Bernoulli numbers** \( B_k \) are defined as the values of the Bernoulli polynomials \( B_k(t) \) at \( t = 0 \): \( B_k = B_k(0) \). The Bernoulli polynomials satisfy the following differential equation:

\[
B'_{k+1}(t) = (k+1)B_k(t), \quad \text{with } B_{2k+1}(0) = 0, B_{2k+1}(1) = 0, \text{for all } k > 0.
\]

The last conditions imply that all Bernoulli numbers with an odd index are zero. The equations (6) define a recursion to compute \( B_k(t) \), starting at \( B_0(t) = 1 \) and \( B_1(t) = t - \frac{1}{2} \). By taking anti-derivatives we solve the recursion (6), for example:

\[
B'_2(t) = 2B_1(t) \Rightarrow B_2(t) = 2 \int B_1(t)dt + C_1 = 2 \int \left( t - \frac{1}{2} \right) dt + C_1 = 2 \left( \frac{t^2}{2} - \frac{t}{2} \right) + C_1; \quad \text{and}
\]

\[
B'_3(t) = 3B_2(t) \Rightarrow B_3(t) = 3 \int B_2(t)dt + C_2 = 3 \int \left( t^2 - t + C_1 \right) dt + C_2 = 3 \left( \frac{t^3}{3} - \frac{t^2}{2} + C_1 t \right) + C_2.
\]

The condition \( B_3(0) = 0 \) implies \( C_2 = 0 \) and \( B_3(1) = 0 \) leads to \( C_1 = 1/6 \), thus \( B_2 = B_2(0) = 1/6 \).
The Bernoulli polynomials appear naturally when integrating by parts. Recall this rule:

\[ D(f \cdot g) = (Df) \cdot g + f \cdot (Dg) \Rightarrow \int D(f \cdot g) = \int (Df) \cdot g + \int f \cdot (Dg) \]

(9)

\[ \Rightarrow \int (Df) \cdot g = f \cdot g - \int f \cdot (Dg) \]

(10)

Let us apply this rule to our integral:

\[
\int_0^1 1 \cdot g(t)dt = \int_0^1 B'_1(t) \cdot g(t)dt
\]

(11)

\[ = [B_1(t) \cdot g(t)]_0^1 - \int_0^1 B_1(t) \cdot g'(t)dt, \quad B_1(t) = t - \frac{1}{2}, B'_1(t) = 2B_1(t) \]

(12)

\[ = \frac{g(0) + g(1)}{2} - \int_0^1 \frac{1}{2} B'_2(t) \cdot g'(t)dt \]

(13)

\[ = \frac{g(0) + g(1)}{2} - \left[ \frac{1}{2!} B_2(t) \cdot g'(t) \right]_0^1 + \int_0^1 \frac{1}{2} B_2(t) \cdot g''(t)dt, \quad B'_3(t) = 3B_2(t) \]

(14)

\[ = \frac{g(0) + g(1)}{2} - \left[ \frac{1}{2!} B_2(t) \cdot g'(t) \right]_0^1 + \left[ \frac{1}{3!} B_3(t) \cdot g''(t) \right]_0^1 - \int_0^1 \frac{1}{3!} B_3(t) \cdot g'''(t)dt \]

(15)

\[ = \frac{g(0) + g(1)}{2} - \left[ \frac{1}{2!} B_2(t) \cdot g'(t) \right]_0^1 + \left[ \frac{1}{3!} B_3(t) \cdot g''(t) \right]_0^1 - \left[ \frac{1}{4!} B_4(t) \cdot g'''(t) \right]_0^1 + \int_0^1 \frac{1}{4!} B_4(t) \cdot g^{(4)}(t)dt \]

(17)

We see how the Bernoulli numbers come in when evaluating the definite integrals. But, notice, we need to evaluate the Bernoulli polynomials at \( t = 1 \). Fortunately, the Bernoulli polynomials satisfy

\[ (-1)^kB_k(1 - t) = B_k(t), \]

(18)

because the polynomials \((-1)^kB_k(1 - t)\) satisfy the same recursion (6) as \( B_k(t) \), and since the recursion starts off with the same polynomials, all polynomials must be the same. Thus we know \( B_k(1) \) from \( B_k \).

With the mean value theorem for integrals, there exists some \( \xi \in [0, 1] \), so we have

\[
\int_0^1 g(t)dt = \frac{g(0) + g(1)}{2} + \sum_{l=1}^{m} \frac{B_2l}{(2l)!} \left( g^{(2l-1)}(0) - g^{(2l-1)}(1) \right) - \frac{B_{2m+2}}{(2m+2)!} g^{(2m+2)}(\xi)
\]

(19)

This shows the Euler-Maclaurin summation formula for one interval. To show (1), we apply (19) \( N \) times:

\[
\int_0^N g(t)dt = \int_0^1 g(t)dt + \int_1^2 g(t)dt + \cdots + \int_{N-1}^N g(t)dt.
\]

(20)

References