PHCpack in Macaulay2

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Abstract

The MACAULAY2 package `PHCpack` provides an interface to PHCPACK, a general-purpose polynomial system solver that uses homotopy continuation. The main method is a numerical blackbox solver which is implemented for all Laurent systems. The package also provides a fast mixed volume computation, the ability to filter solutions, homotopy path tracking, and a numerical irreducible decomposition method. As the size of many problems in applied algebraic geometry often surpasses the capabilities of symbolic software, this package will be of interest to those working on problems involving large polynomial systems.

1 Numerical homotopy continuation and PHCPACK

Many problems in applied algebraic geometry require solving, or counting the solutions of, a large polynomial or rational system. `PHCpack` is an interface to the program PHCPACK, one of several efficient polynomial system solvers that use numerical homotopy continuation methods [5].

The basic idea behind homotopy continuation is simple: to solve a polynomial system $f(\mathbf{x}) = 0$, one first constructs a system $g(\mathbf{x}) = 0$ that is easy to solve and then constructs a homotopy, $H(\mathbf{x}(t)) = (1-t)g(\mathbf{x}) + tf(\mathbf{x})$, in order to numerically track paths from known solutions of g (with $t = 0$) to the solutions of the target system f (with $t = 1$).

Available since release 1.4 of MACAULAY2 [3], this package is motivated by [4] and uses the data types defined by Leykin in `NAGtypes.m2`. The main function of the package allows a MACAULAY2 user to solve a system numerically through a blackbox solver, where the creation of the start system and homotopy continuation is done behind the scenes. The package also provides a fast mixed volume computation and allows the user to filter solutions, to track solution paths explicitly, and to perform numerical irreducible decompositions.

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In fact, the interface PHCpack offers access to most of the functionality of the software PHCPACK, which has been serving as a development platform for many of the algorithms in numerical algebraic geometry [7]. Computations in this paper were done with phc version 2.3.61 (version 1.0 was archived in [8]). Since version 2.3.13, PHCPACK contains MIXEDVOL [2], and more recently added features are described in [9]. Note that PHCPACK can solve Laurent systems, so the package includes a method to convert a rational system to a Laurent polynomial system. The underlying polyhedral methods perform well on benchmark problems; in many of those, the mixed volume is computed essentially instantaneously.

Although PHCPACK is open source, we follow the idea of OpenXM [6] and require only that the executable phc is available in the execution path of the computer.

2 Numerical solutions of a polynomial system

The main function, solveSystem, returns solutions of a system of polynomial or rational equations. Solutions are returned using data types from NAGtypes: a collection of Points which are approximations to all complex isolated solutions, or a WitnessSet for positive-dimensional components. The following system consists of 21 polynomial equations in 21 unknowns, related to a Gaussian cycle conjecture [1, §7.4, page 159] in algebraic statistics. The corresponding variety is zero-dimensional of degree 67.

```
Macaulay2, version 1.4
i1 : CC[x_11,x_12,x_16,x_22,x_23,x_33,x_34,x_44,x_45,x_55,x_56,x_66,
      y_13,y_14,y_15,y_24,y_25,y_26,y_35,y_36,y_46];
i2 : system = {x_11+2*x_12+2/3*x_16-1,       x_23*y_13+5/2*x_12+11/2*x_22,
     x_33*y_13+x_34*y_14+11/2*x_23, x_34*y_13+x_44*y_14+x_45*y_15,
     x_45*y_14+x_55*y_15+(45/2)*x_56, x_56*y_15+(22/3)*x_16+(45/2)*x_66,
     82/7*x_12+17/2*x_22+12/5*x_23-1, x_34*y_24+(14/11)*x_23+(12/5)*x_33,
     x_44*y_24+x_45*y_25+(12/5)*x_34, x_45*y_24+x_55*y_25+x_56*y_26,
     x_56*y_25+x_66*y_26+(82/7)*x_16, (12/5)*x_23+(282/5)*x_33+(102/14)*x_34-1,
     x_45*y_35+(282/5)*x_34+(102/14)*x_44, x_55*y_35+x_56*y_36+(102/14)*x_45,
     x_16*y_13+x_56*y_35+x_66*y_36, 10/1*x_34+205/16*x_44+(30/2)*x_45-1,
     x_56*y_46+(205/16)*x_45+(30/2)*x_55, x_16*y_14+x_66*y_46+(305/25)*x_56,
     305/25*x_45+517/7*x_55+(89/3)*x_56-1, x_16*y_15+517/78*x_56+(89/3)*x_66,
     (450/21)*x_16+(89/3)*x_56+(293/19)*x_66-1};
i3 : time solutions = solveSystem system ;
     using temporary files /tmp/M2-5331-2PHCinput and /tmp/M2-5331-2PHCoutput
     -- used 0.056381 seconds
i4 : # solutions
o4 = 67
```

Solutions are returned as a list, each entry being of type Point, which includes diagnostic information such as the condition number and the value of the path-tracking variable t. This allows one to decide if a solution is "good" by using peek, suppressed here in the interest of space. Note also that the names of temporary input/output files allow the user to access the details of the entire phc computation, if desired.

```
i5 : solutions_0
```

```
o5 = {-3.34446-1.36293*ii, 2.1944+.682742*ii, -.0664982-.0038353*ii,
-2.90447-1.02757*ii, -.0074284+.306877*ii, .018283-.00390056*ii,
-.0018297-.0708941*ii, .23468+.062941*ii, -.13257-.0064993*ii,
.00187754+.000036983*ii, .083551+.0025807*ii, -.00348342+.000364664*ii,
12.0202-34.4693*ii, 14.2637-.25899*ii, 6.77557+.029139*ii,
5.38612-.346571*ii, 9.67526+.18591*ii, 8.34346-.39709*ii,
10.7841-27.2306*ii, 11.3312+.8239*ii, 20.0039+.37216*ii}
o5 : Point
```

The solutions can be further refined as necessary. To best illustrate refinement, consider the same system as above, but where the rational coefficients have been changed to larger rational numbers. Let `newSolutions` be the solutions of this modified system `newSystem`, which can be found in the Appendix (suppressed here in the interest of space).

Solutions with coordinates below or above a given tolerance can be extracted by `zeroFilter` and `nonZeroFilter`, respectively. In the following example, we ask for solutions whose 12th coordinate is effectively zero (i.e., smaller than 10^{-19}). Then, we confirm this by refining the answer to precision 64; notice that the 12th coordinate is now on the order of 10^{-67}.

```
i9 : smallSolution = zeroFilter(newSolutions,11,1.0e-19)
o9 = {{.0677823, -.386278, .0204925, -1.44743, .982877, -.366596, -.435274,
       .725281, -.422346, .0841728, .0218581, 2.23306e-20, 46.7882, -12.922,
       -70.411, 8.2731, -10.9958, 202.197, -43.8649, 306.199, 198.688}}
i10 : time smallerSolution=refineSolutions(newSystem,smallSolution,64)
     -- used 0.008859 seconds
o10 = {{.0677823, -.386278, .0204925, -1.44743, .982877, -.366596, -.435274,
       .725281, -.422346, .0841728, .0218581, -1.4308e-67, 46.7882, -12.922,
       -70.411, 8.2731, -10.9958, 202.197, -43.8649, 306.199, 198.688}}
```

Note that when refining solutions, `phc` also recomputes input coefficients to a higher precision, since rational coefficients may not always have an exact floating-point representation when the precision is limited.

3 Mixed volume

If the system has as many equations as unknowns, the mixed volume gives an upper bound on the number of isolated solutions with nonzero coordinates. For sufficiently generic coefficients, this bound is sharp. The function `mixedVolume` is illustrated below.

```
i11 : time mixedVolume (system )
using temporary files /tmp/M2-4281-6PHCinput and /tmp/M2-4281-6PHCoutput
     -- used 0.011375 seconds
o11 = 75
```

This polyhedral computation is faster than solving the system and provides an upper bound on the number of complex isolated roots in the torus. Computing the degree is much slower (and we note that it takes just as long to verify that the variety is zero-dimensional):

```
i12 : time degree ideal(system)
  -- used 767.432 seconds
o12 = 67
```

While mixed volume counts solutions on the torus, one can also compute the stable mixed volume, which counts solutions with zero components as well, by using optional inputs to the method `mixedVolume`. `phc` offers additional functionality and flexibility, not all of which we can illustrate in this short note. Most interestingly, `mixedVolume` offers an option to use a start system, and creates a polyhedral homotopy from a random start system to the given system. The interested reader is referred to the documentation of the package for more information.

4 Numerical irreducible decomposition

Given a list of generators of an ideal I, the package can also compute a `NumericalVariety` with a `WitnessSet` for each irreducible component of $V(I)$. The example below appears in [1].

```
i13 : CC[x11,x22,x21,x12,x23,x13,x14,x24];
i14 : system={x11*x22-x21*x12,x12*x23-x22*x13,x13*x24-x23*x14};
i15 : V=numericalIrreducibleDecomposition(system)
writing output to file /tmp/M2-5241-2PHCoutput
calling phc -c < /tmp/M2-5241-3PHCbatch > /tmp/M2-5241-5PHCsession
output of phc -c is in file /tmp/M2-5241-2PHCoutput
... constructing witness sets ...
preparing input file to /tmp/M2-5241-7PHCinput
preparing batch file to /tmp/M2-5241-9PHCbatch
... calling monodromy breakup ...
session information of phc -f is in /tmp/M2-5241-10PHCsession
output of phc -f is in file /tmp/M2-5241-8PHCoutput
found 3 irreducible factors
o15 = A variety of dimension 5 with components in
     dim 5:  (dim=5,deg=4) (dim=5,deg=2) (dim=5,deg=2)
o15 : NumericalVariety
```

Witness sets are accessed by dimension:

```
i16 : WitSets=V#5;
i17 : w=first WitSets;
i18 : w#IsIrreducible
o18 = true
```

In the above example we found three components of dimension five. Let's verify the solutions.

```
i19 : R=QQ[x11,x22,x21,x12,x23,x13,x14,x24];
i20 : system={x11*x22-x21*x12,x12*x23-x22*x13,x13*x24-x23*x14};
i21 : PD=primaryDecomposition(ideal(system));
i22: PD/print
ideal (x23*x14 - x13*x24, x21*x14 - x11*x24, x22*x14 - x12*x24,
      x12*x23 - x22*x13, x11*x23 - x21*x13, x11*x22 - x21*x12)
ideal (x13, x23, x11*x22 - x21*x12)
ideal (x12, x22, x23*x14 - x13*x24)
```

As we see, the dimension and degree of each component agree with the numerical calculation:

```
i23 : PD/dim
o23 = {5, 5, 5}
i24 : PD/degree
o24 = {4, 2, 2}
```

References

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Appendix

The polynomial system used in some of the examples in Section 2. It has the same support as the example system `system`, but the rational coefficients have been changed. It also has 67 solutions.

```
i6 : newSystem = {(22531/300)*x_11+(821/70)*x_12+(4507/210)*x_16-1,
       x_23*y_13+(22531/300)*x_12+(821/70)*x_22,
       x_33*y_13+x_34*y_14+(821/70)*x_23, x_34*y_13+x_44*y_14+x_45*y_15,
       x_45*y_14+x_55*y_15+(4507/210)*x_56,
```

```
      x_56*y_15+(22531/300)*x_16+(4507/210)*x_66,
      (821/70)*x_12+(140953/11025)*x_22+(12325/504)*x_23-1,
      x_34*y_24+(140953/11025)*x_23+(12325/504)*x_33,
      x_44*y_24+x_45*y_25+(12325/504)*x_34, x_45*y_24+x_55*y_25+x_56*y_26,
      x_56*y_25+x_66*y_26+(821/70)*x_16,
      (12325/504)*x_23+(282013/5184)*x_33+(10231/1440)*x_34-1,
      x_45*y_35+(282013/5184)*x_34+(10231/1440)*x_44,
      x_55*y_35+x_56*y_36+(10231/1440)*x_45, x_16*y_13+x_56*y_35+x_66*y_36,
      (10231/1440)*x_34+(205697/16200)*x_44+(30529/2520)*x_45-1,
      x_56*y_46+(205697/16200)*x_45+(30529/2520)*x_55,
      x_16*y_14+x_66*y_46+(30529/2520)*x_56,
      (30529/2520)*x_45+(5175321/78400)*x_55+(897/35)*x_56-1,
      x_16*y_15+(5175321/78400)*x_56+(897/35)*x_66,
      (4507/210)*x_16+(897/35)*x_56+(293581/19600)*x_66-1};
i7 : time newSolutions=phcSolve(newSystem);
using temporary files /tmp/M2-3582-2PHCinput and /tmp/M2-3582-2PHCoutput
     -- used 0.054471 seconds
o7 : List
i8 : # newSolutions
o8 = 67
```