Searching for Solution Curves of Polynomial Systems (preliminary report)

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Problem Statement

Given is $f(\mathbf{x}) = \mathbf{0}$, a polynomial system

$$f(x_1, x_2, \dots, x_n) = \begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ f_N(x_1, x_2, \dots, x_n) = 0 \end{cases} \qquad \begin{array}{c} f_i \in C[\mathbf{x}] \\ \\ N \geq n. \end{array}$$

The coefficients are in C: “computer numbers”.

Does $f(\mathbf{x}) = \mathbf{0}$ have solution curves?

References

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- J. McDonald: *Fractional power series solutions for systems of equations*. Discrete Comput. Geom. 27(4): 501–529, 2002.
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Tropical Algebraic Geometry

an asymptotic view on varieties

Consider an ideal I in $K\{\{t\}\}[x_1, x_2, \ldots, x_n]$.

$K\{\{t\}\}$ is algebraically closed if K is algebraically closed,
by the theorem of Puiseux.

$\mathrm{Trop}(I)$, the tropicalization of I is defined as $\mathrm{Trop}(I) = \{ \omega \in \mathbb{Q}^n \mid \ldots$
either
(1) $\ldots$ the ideal of initial forms defined by ω is monomial free $\}$.
or
(2) $\ldots$ ω collects leading powers of series vanishing for $f \in I$ $\}$.

Fundamental Theorem of Tropical Algebraic Geometry: (1) $\Leftrightarrow$ (2).

Implemented in `tropical.lib`, a SINGULAR library, using Gfan, by Anders Jensen, Hannah Markwig, and Thomas Markwig.

Refined problem statement: numerical implementation?

Solving the cyclic 4-roots System

$$f(\mathbf{x}) = \left\{ \begin{array}{c} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{array} \right.$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \left\{ \begin{array}{c} x_2 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_2x_3x_4 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} x_1 = y_1^{+1} \\ x_2 = y_1^{-1}y_2 \\ x_3 = y_1^{+1}y_3 \\ x_4 = y_1^{-1}y_4 \end{array} \right.$$

The system $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{y}) = \mathbf{0}$ has two solutions.
We find two solution curves: $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$.

Sparse Polynomial Systems have Sparse Solutions

the cyclic 12-roots problem

J. Backelin: "Square multiples n give infinitely many cyclic n-roots". Reports, Matematiska Institutionen, Stockholms Universitet, 1989.

Mixed volume is 500,352 and increases to 983,952 by adding one random hyperplane and slack variable.

Like for cyclic 4, $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$ is a tropism. Mixed volume of $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}, s) = \mathbf{0}$ is 49,816.
One of the solutions is

$$
\begin{array}{ll}
x_0 = t & x_1 = 0.5 - 0.866025403784439i \\
x_2 = -1.0 & x_3 = -0.5 - 0.866025403784439i \\
x_4 = -0.5 + 0.866025403784439i & x_5 = 0.5 + 0.866025403784439i \\
x_6 = -1.0 & x_7 = -0.5 + 0.866025403784438i \\
x_8 = 1.0 & x_9 = 0.5 + 0.866025403784438i \\
x_{10} = 0.5 - 0.866025403784439i & x_{11} = -0.5 - 0.866025403784439i
\end{array}
$$

It satisfies not only $\mathrm{in}_{\mathbf{v}}(f)$, but also f itself.

An Exact Solution for cyclic 12-roots

For the tropism $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$:

$$
\begin{array}{ll}
z_0 = t^{-1} & z_1 = t\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_2 = -t^{-1} & z_3 = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) \\
z_4 = t^{-1}\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) & z_5 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_6 = -t^{-1} & z_7 = t\left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_8 = t^{-1} & z_9 = t\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right) \\
z_{10} = t^{-1}\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) & z_{11} = t\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)
\end{array}
$$

makes the system entirely and exactly equal to zero.

An Illustrative Example

for a numerical irreducible decomposition

$$f(x_1,x_2,x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

$$f^{-1}(\mathbf{0}) = Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$$

1. Z_{21} is the sphere $x_1^2 + x_2^2 + x_3^2 - 1 = 0$,
2. Z_{11} is the line $(x_1 = 0.5, x_3 = 0.5^3)$,
3. Z_{12} is the line $(x_1 = \sqrt{0.5}, x_2 = 0.5)$,
4. Z_{13} is the line $(x_1 = -\sqrt{0.5}, x_2 = 0.5)$,
5. Z_{14} is the twisted cubic $(x_2 - x_1^2 = 0, x_3 - x_1^3 = 0)$,
6. Z_{01} is the point $(x_1 = 0.5, x_2 = 0.5, x_3 = 0.5)$.

The Illustrative Example

numerically computing positive dimensional solution sets

Used in two papers in numerical algebraic geometry:

- first cascade of homotopies: 197 paths

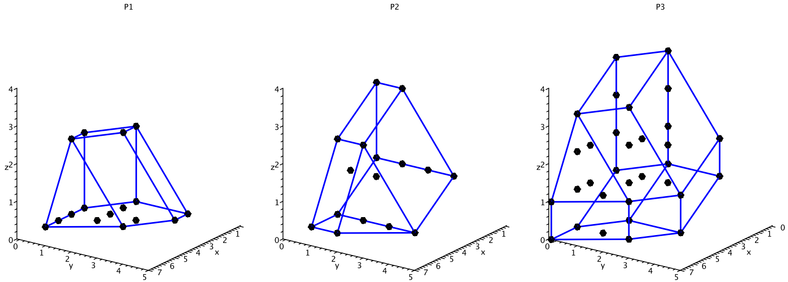
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Numerical decomposition of the solution sets of polynomial systems into irreducible components*. SIAM J. Numer. Anal. 38(6):2022–2046, 2001.

- equation-by-equation solver: 13 paths

  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Solving polynomial systems equation by equation*. In Algorithms in Algebraic Geometry, Volume 146 of The IMA Volumes in Mathematics and Its Applications, pages 133–152, Springer-Verlag, 2008.

The mixed volume of the Newton polytopes of this system is 124.
By theorem A of Bernshteĭn, the mixed volume is an upper bound on the number of isolated solutions.

Three Newton Polytopes



$$f(x_1, x_2, x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

Looking for Solution Curves

The twisted cubic is $(x_1 = t, x_2 = t^2, x_3 = t^3)$.

We look for solutions of the form

$$\begin{cases} x_1 = t^{v_1}, & v_1 > 0, \\ x_2 = c_2 t^{v_2}, & c_2 \in \mathbb{C}^*, \\ x_3 = c_3 t^{v_3}, & c_3 \in \mathbb{C}^*. \end{cases}$$

Substitute $x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3$ into f

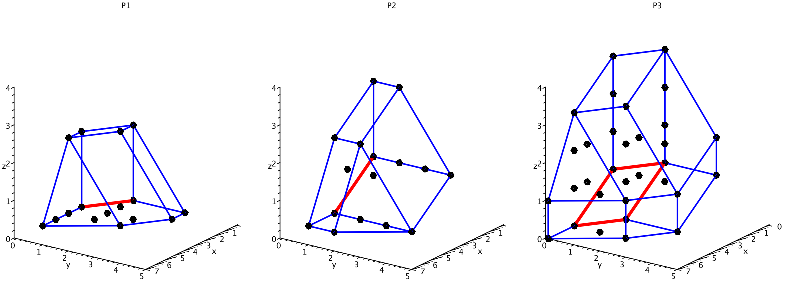
$$f(x_1 = t, x_2 = c_2 t^2, x_3 = c_3 t^3) = \begin{cases} (0.5c_2 - 0.5)t^2 + O(t^3) = 0 \\ (0.5c_3 - 0.5)t^3 + O(t^5) = 0 \\ 0.5(c_2 - 1.0)(c_3 - 1.0)t^5 + O(t^7) \end{cases}$$

$\longrightarrow$ conditions on c_2 and c_3.

How to find $(v_1, v_2, v_3) = (1, 2, 3)$?

Faces of Newton Polytopes

Looking at the Newton polytopes in the direction $\mathbf{v} = (1,2,3)$:



Selecting those monomials supported on the faces

$$\partial_{\mathbf{v}} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} 0.5x_2 - 0.5x_1^2 = 0 \\ 0.5x_3 - 0.5x_1^3 = 0 \\ -0.5x_2x_1^3 - 0.5x_3x_1^2 + 0.5x_3x_2 + 0.5x_1^5 = 0 \end{array} \right.$$

Degenerating the Sphere

$$f(x_1,x_2,x_3) = \begin{cases} (x_2 - x_1^2)(x_1^2 + x_2^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ (x_2 - x_1^2)(x_3 - x_1^3)(x_1^2 + x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_1 = t \to 0$:

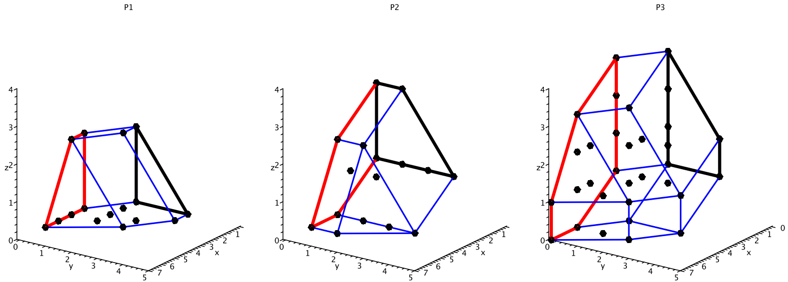
$$\partial_{(1,0,0)} f(x_1,x_2,x_3) \begin{cases} x_2(x_2^2 + x_3^2 - 1)(-0.5) = 0 \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) = 0 \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

As $x_2 = s \to 0$:

$$\partial_{(0,1,0)} f(x_1,x_2,x_3) \begin{cases} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) = 0 \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) = 0 \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) = 0 \end{cases}$$

More Faces of Newton Polytopes

Looking at the Newton polytopes along $\mathbf{v} = \mathbf{(1,0,0)}$ and $\mathbf{v} = (0,1,0)$:



$$\partial_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

$$\partial_{(0,1,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

Faces of Faces

The sphere degenerates to circles at the coordinate planes.

$$\partial_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_2^2 + x_3^2 - 1)(-0.5) \\ x_3(x_2^2 + x_3^2 - 1)(x_2 - 0.5) \\ x_2 x_3(x_2^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

$$\partial_{(0,1,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} -x_1^2(x_1^2 + x_3^2 - 1)(x_1 - 0.5) \\ (x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(-0.5) \\ -x_1^2(x_3 - x_1^3)(x_1^2 + x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

Degenerating even more:

$$\partial_{(0,1,0)} \partial_{(1,0,0)} f(x_1, x_2, x_3) = \left\{ \begin{array}{r} x_2(x_3^2 - 1)(-0.5) \\ x_3(x_3^2 - 1)(-0.5) \\ x_2 x_3(x_3^2 - 1)(x_3 - 0.5) \end{array} \right.$$

The factor $x_3^2 - 1$ is shared with $\partial_{(1,0,0)} \partial_{(0,1,0)} f(x_1, x_2, x_3)$.

Representing a Solution Surface

The sphere is two dimensional, x_1 and x_2 are free:

$$\begin{cases} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = 1 + c_1 t_1^2 + c_2 t_2^2. \end{cases}$$

For $t_1 = 0$ and $t_2 = 0$, $x_3 = 1$ is a solution of $x^3 - 1 = 0$.

Substituting $(x_1 = t_1, x_2 = t_2, x_3 = 1 + c_1 t_1^2 + c_2 t_2^2)$
into the original system gives linear conditions on the coefficients of the second term: $c_1 = -0.5$ and $c_2 = -0.5$.

Asymptotics of Witness Sets

Getting generic points on a two dimensional surface:

$$\left\{ \begin{array}{r} f(\mathbf{x}) = 0 \\ c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 = 0 \\ c_{20} + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 = 0 \end{array} \right. \rightarrow \left\{ \begin{array}{r} f(\mathbf{x}) = 0 \\ c_{10} + c_{11}x_1 = 0 \\ c_{20} + c_{22}x_2 = 0 \end{array} \right.$$

Specializing the two planes more:

$$\left\{ \begin{array}{r} f(\mathbf{x}) = 0 \\ x_1 = t_1 \\ x_2 = t_2 \end{array} \right.$$

As $t_1 \rightarrow 0$ and $t_2 \rightarrow 0$, the leading powers of the Puiseux series solution define a tropism.

If the solution after specialization is regular,
then we can extend to compute witness sets.

Computing a Series Expansion

a staggered approach to find a certificate for a regular solution curve

"tropicalization"

compute tropisms

no tropism

$\Rightarrow$ no root at ∞

solve initial forms

no root at ∞

$\Rightarrow$ no series

compute 2nd term

no series

$\Rightarrow$ no curve

series

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$\downarrow$

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$\downarrow$

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$\downarrow$

compute 2nd term $\bigcirc$ $\searrow$ no series $\Rightarrow$ no curve

$\downarrow$

series

Three Separate Stages

1. compute candidate tropisms
   $\rightarrow$ a tropism is perpendicular to a facet that is a sum of edges of the Newton polytopes
2. find leading coefficient of Puiseux series:
   1. change coordinates so one variable cancels
   2. apply a solver to a much sparser system
3. get the second term of the Puiseux series
   symbolic substitution and cancellation of lowest terms

The second Term of a Puiseux Expansion

for a component of the cyclic 8-roots system

Because we find a nonzero solution for the y_k coefficients, we use it as the second term of a Puiseux expansion:

$$\left\{ \begin{array}{lll} x_0 = t^1 & & \\ x_1 = (\ 0.5 + 0.5i\ )\ t^0 & + (\ -0.5i\ )\ t & \\ x_2 = (\ 1 + i\ )\ t^0 & + (\ -i\ )\ t & \\ x_3 = (\ -i\ )\ t^0 & + (\ 1 - i\ )\ t & \\ x_4 = (\ -0.5 - 0.5i\ )\ t^0 & + (\ 0.5i\ )\ t & \\ x_5 = (\ -1\ )\ t^0 & + (\ 0\ )\ t & \\ x_6 = (\ i\ )\ t^0 & + (\ -1 + i\ )\ t & \\ x_7 = (\ -1 - i\ )\ t^0 & + (\ i\ )\ t & \end{array} \right. \qquad i = \sqrt{-1}.$$

Substitute series in $f(\mathbf{x})$: result is $O(t^2)$.

Note: exploitation of symmetry is immediate.

Conclusions

An apriori certificate for a solution component consists of

1. a tropism: leading powers of a Puiseux series,
2. a root at infinity: leading coefficients of the Puiseux series,
3. the next term in the Puiseux series.

The certificate is compact and easy to verify with substitution.

For more, see `http://www.math.uic.edu/~jan`:
Polyhedral methods in numerical algebraic geometry.
To appear in Contemporary Mathematics.