Polynomial Homotopy Continuation

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Plan of the Lecture

1. Homotopies and Path Tracking

   the theorem of Bézout, *predictor-corrector methods*, *some complexity issues*

2. Coefficient-Parameter Continuation

   a case study: families of Stewart-Gough platforms

3. Exploiting Product Structures

   multi-homogenization, *linear-product start systems*

4. Software and Applications

   the software PHCpack, *illustrations of application fields*

Recommended Background Literature

E.L. Allgower and K. Georg: **Numerical Continuation Methods, an Introduction.** Springer 1990. To appear in the SIAM Classics in Applied Mathematics Series.

E.L. Allgower and K. Georg: **Numerical Path Following.** In *Techniques of Scientific Computing (Part 2)*, edited by P.G. Ciarlet and J.L. Lions volume 5 of *Handbook of Numerical Analysis*, pages 3–203. North-Holland, 1997.

A. Morgan: **Solving polynomial systems using continuation for engineering and scientific problems**. Prentice-Hall, 1987.

T.Y. Li: **Solving polynomial systems.** *The Mathematical Intelligencer* 9(3):33–39, 1987.

T.Y. Li: **Numerical solution of multivariate polynomial systems by homotopy continuation methods.** *Acta Numerica* 6:399–436, 1997.

Numerical Homotopy Continuation Methods

If we wish to solve $f(\mathbf{x}) = \mathbf{0}$, then we construct a system $g(\mathbf{x}) = \mathbf{0}$ whose solutions are known. Consider the *homotopy*

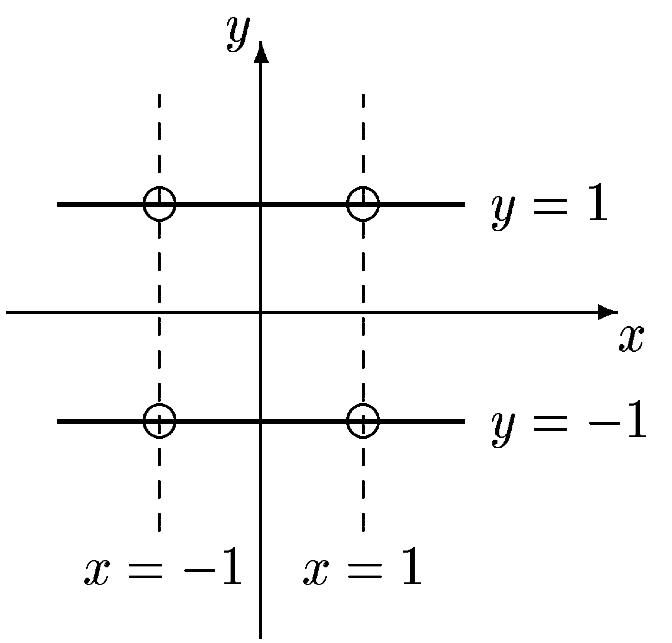
$$H(\mathbf{x}, t) := (1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}.$$

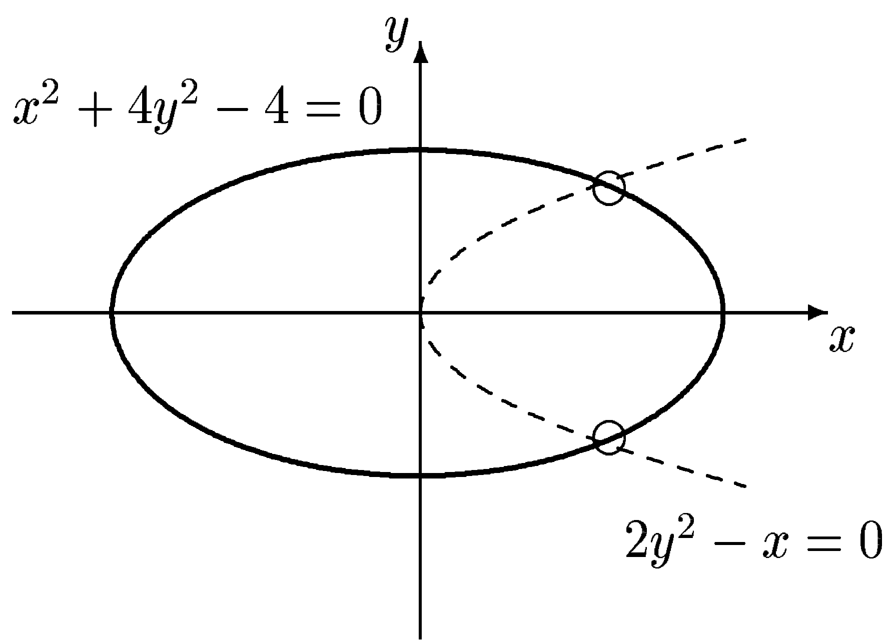
By *continuation*, we trace the paths starting at the known solutions of $g(\mathbf{x}) = \mathbf{0}$ to the desired solutions of $f(\mathbf{x}) = \mathbf{0}$, for t from 0 to 1.

homotopy continuation methods are *symbolic-numeric*:
homotopy methods treat polynomials as algebraic objects,
continuation methods use polynomials as functions.

geometric interpretation: move from general to special,
solve special, and move solutions from special to general.

Product Deformations





$$\gamma \left(\left\{ \begin{array}{r} x^2 - 1 = 0 \\ y^2 - 1 = 0 \end{array} \right. \right) (1-t) \quad + \quad \left(\left\{ \begin{array}{r} x^2 + 4y^2 - 4 = 0 \\ 2y^2 - x = 0 \end{array} \right. \right) t, \quad \gamma \in \mathbb{C}$$

The theorem of Bézout

$f = (f_1, f_2, \ldots, f_n)$

$d_i = \deg(f_i)$

total degree D :

$$D = \prod_{i=1}^{n} d_i$$

$$g(\mathbf{x}) = \begin{cases} \alpha_1 x_1^{d_1} - \beta_1 = 0 \\ \alpha_2 x_2^{d_2} - \beta_2 = 0 \\ \vdots \\ \alpha_n x_n^{d_n} - \beta_n = 0 \end{cases}$$

start

system

$\alpha_i, \beta_i \in \mathbb{C}$

random

Theorem: $f(\mathbf{x}) = \mathbf{0}$ *has at most D isolated solutions in $\mathbb{C}^n$, counted with multiplicities.*

Sketch of Proof: $V = \{\ (f, \mathbf{x}) \in \mathbb{P}(\mathcal{H}_D) \times \mathbb{P}(\mathbb{C}^n) \mid f(\mathbf{x}) = \mathbf{0}\ \}$

$\Sigma' = \{(f, \mathbf{x}) \in V \mid \det(D_{\mathbf{x}} f(\mathbf{x})) = 0\}$, $\Sigma = \pi_1(\Sigma'), \pi_1 : V \to \mathbb{P}(\mathcal{H}_D)$

Elimination theory: Σ is variety $\Rightarrow \mathbb{P}(\mathcal{H}_D) - \Sigma$ is connected.

Thus $h(\mathbf{x}, t) = (1 - t) g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}$ avoids $\Sigma, \forall t \in [0, 1)$.

Implicitly defined curves

Consider a homotopy $h_k(x(t), y(t), t) = 0$, $k = 1, 2$.

By $\frac{\partial}{\partial t}$ on homotopy: $\frac{\partial h_k}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h_k}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial h_k}{\partial t}\frac{\partial t}{\partial t} = 0$, $k = 1, 2$.

Set $\Delta x := \frac{\partial x}{\partial t}$, $\Delta y := \frac{\partial y}{\partial t}$, and $\frac{\partial t}{\partial t} = 1$.

Increment $\quad t := t + \Delta t$

Solve
$$\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial t} \end{bmatrix} (Newton)$$

Update
$$\begin{cases} x := x + \Delta x \\ y := y + \Delta y \end{cases}$$

Predictor-Corrector Methods

loop

1. predict $\begin{cases} t_{k+1} := t_k + \Delta t \\ \mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + \Delta \mathbf{x} \end{cases}$

2. correct with Newton

3. if convergence

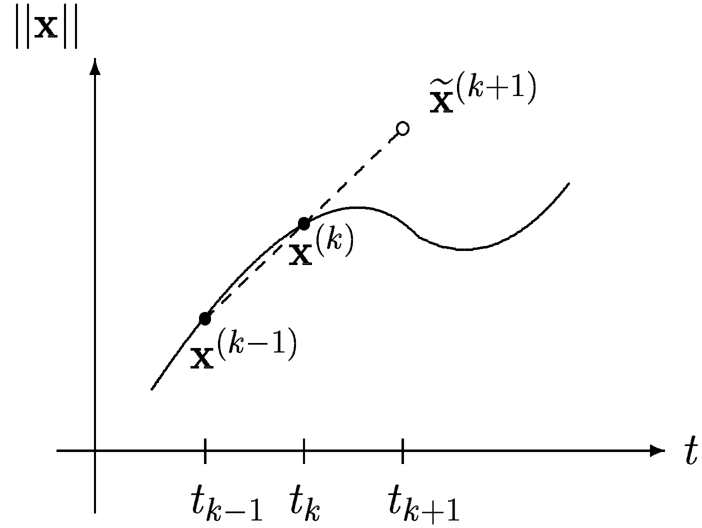
   then enlarge Δt

   continue with $k+1$

   else reduce Δt

   back up and restart at k

until $t = 1$.



$$\widetilde{\mathbf{x}}^{(k+1)} := \mathbf{x}^{(k)} + \lambda(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)})$$

Robustness of Continuation Methods

sure to find **all** *roots at the end of the paths?*

- dealing with curve jumping:
  1. fix #Newton steps to force quadratic convergence;
  2. rerun clustered paths with same discretization of t.
- Robust step control by interval methods, see

R.B. Kearfott and Z. Xing: **An interval step control for continuation methods.** *SIAM J. Numer. Anal.* 31(3): 892–914, 1994.

- Root of multiplicity μ will appear at the end of the paths as a cluster of μ roots.

  Use "endgames", eventually in multi-precision arithmetic.

Complexity Issues

The Problem: a hierarchy of complexity classes

P	:	evaluation of a system at a point
NP	:	find one root of a system
$\#P$	:	find **all** roots of a system (*intractable!*)

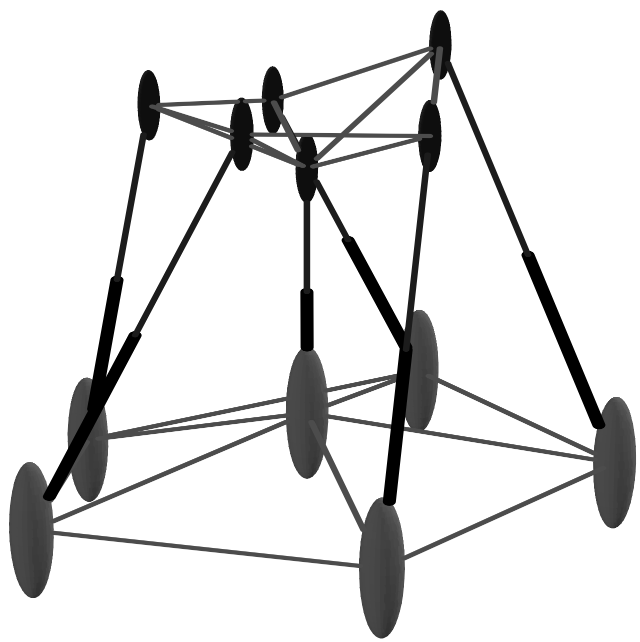
Complexity of Homotopies: for bounds on #Newton steps in a linear homotopy, see

L. Blum, F. Cucker, M. Shub, and S. Smale: **Complexity and Real Computation**. Springer 1998.

M. Shub and S. Smale: **Complexity of Bezout's theorem V: Polynomial Time**. *Theoretical Computer Science* 133(1):141–164, 1994.

On average, we can find an approximate zero in polynomial time.

A Case Study: Stewart-Gough Platforms



end plate, the platform

is connected by legs to

a stationary base

Forward Displacement Problem:

Given: position of base and leg lengths.

Wanted: position of end plate.

The Equations for the Platform Problem

workspace $\mathbb{R}^3 \times SO(3)$: position and orientation

$$SO(3) = \{ A \in \mathbb{C}^{3\times 3} \mid A^H A = I, \ \det(A) = 1 \}$$

more efficient to use Study (or soma) coordinates:

$[e : g] = [e_0 : e_1 : e_2 : e_3 : g_0 : g_1 : g_2 : g_3] \in \mathbb{P}^7$ quaternions on the Study quadric: $f_0(e, g) = e_0 g_0 + e_1 g_2 + e_2 g_2 + e_3 g_3 = 0$, excluding those e for which $ee' = 0$, $e' = (e_0, -e_1, -e_2, -e_3)$

given leg lengths L_i, find $[e : g]$ leads to

$$f_i(e, g) = gg' + (bb_i' + a_i a_i' - L_i^2) ee' + (gb_i' e' + eb_i g') - (ge' a_i' + a_i eg') - (eb_i e' a_i' + a_i eb_i' e') = 0, \quad i = 1, 2, \ldots 6$$

$\Rightarrow$ solve $f = (f_0, f_1, \ldots, f_6)$, 7 quadrics in $[e : g] \in \mathbb{P}^7$

expecting $2^7 = 128$ solutions...

Literature on Stewart-Gough platforms

M. Raghavan: **The Stewart platform of general geometry has 40 configurations.** *ASME J. Mech. Design* 115:277–282, 1993.

J.C. Faugère and D. Lazard: **Combinatorial classes of parallel manipulators.** *Mech. Mach. Theory* 30(6):765–776, 1995.

M.L. Husty: **An algorithm for solving the direct kinematics of general Stewart-Gough Platforms.** *Mech. Mach. Theory,* 31(4):365–380, 1996.

C.W. Wampler: **Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates.** *Mech. Mach. Theory* 31(3): 331–337, 1996.

P. Dietmaier: **The Stewart-Gough platform of general geometry can have 40 real postures.** In *Advances in Robot Kinematics: Analysis and Control*, ed. by J. Lenarcic and M.L. Husty, pages 1–10. Kluwer 1998.

J.P. Merlet: **Parallel Robots**. Kluwer Academic Publishers, 2000.

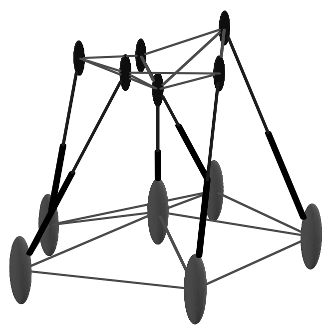
Coefficient-Parameter Homotopies

- Study how solutions change when parameters vary.
- Key Idea:
  1. solve system once for a generic choice of the parameters;
  2. use homotopy to move from generic to specific instance.
- Works for nested parameter spaces (Charles Wampler).

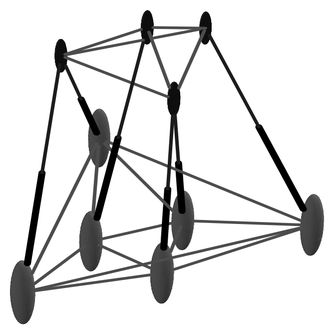
For the theory, see

A.P. Morgan and A.J. Sommese: **Coefficient-parameter polynomial continuation.** *Appl. Math. Comput.*, 29(2):123–160, 1989.

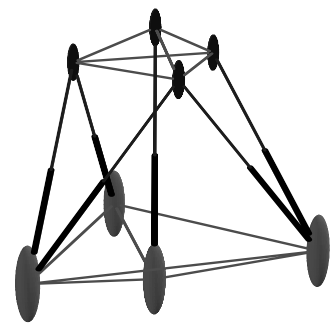
A family of Stewart-Gough platforms



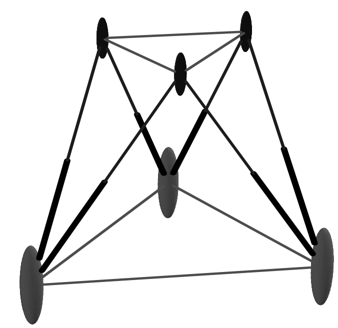
6-6, 40 solutions



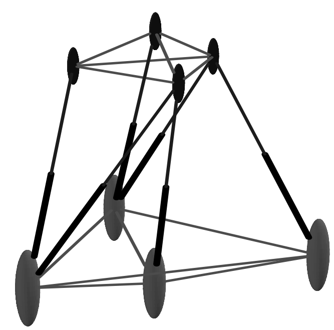
4-6, 32 solutions



4-4a, 16 solutions



3-3, 16 solutions



4-4b, 24 solutions

thanks to Charles Wampler

Coefficient-Parameter Continuation on Case Study

The system f in $[e : g]$ has parameters: a_i, b_i, $i = 1, 2, \ldots, 6$, the ball joints of the stationary base and the moving platform.

Solve f once for generic choice of $a_i, b_i \in \mathbb{C}^3$, (e.g., tracing 128 paths with total degree homotopy) to find 40 isolated roots.

The parameter space

$\text{SG} = \{ \ (a_i, b_i, L_i), i = 1, 2, \ldots, 6 \ \} \subset (\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C})^6 = \mathbb{C}^{42}$.

For $p_0, p_1 \in \text{SG}$, consider the coefficient-parameter homotopy

$$h_{\text{SG}}([e : g], t) = f([e : g], tp_0 + (1 - t)p_1) = 0,$$

where t goes from the generic to the special parameters.

Multihomogeous version of Bézout's theorem

Consider the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$, $A \in \mathbb{C}^{n\times n}$.

Add one general hyperplane $\displaystyle\sum_{i=1}^{n} c_i x_i + c_0 = 0$ for unique $\mathbf{x}$.

Bézout's theorem: $D = 2^n \leftrightarrow$ at most n solutions

Embed in multi-projective space: $\mathbb{P} \times \mathbb{P}^n$, separating λ from $\mathbf{x}$.

$\{\lambda\}$	$\{x_1, x_2\}$
1	1
1	1
0	1

degree table

$\Longleftrightarrow$

$\{\lambda\}$	$\{x_1, x_2\}$
$\lambda + \gamma_1$	$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$
$\lambda + \gamma_2$	$\beta_0 + \beta_1 x_1 + \beta_2 x_2$
1	$c_0 + c_1 x_1 + c_2 x_2$

linear-product start system

The root count $B = 1\cdot 1\cdot 1 + 1\cdot 1\cdot 1 + 0\cdot 1\cdot 1$ is a permanent.

How to find the best partition?

A multi-homogeneous Bézout number depends on the choice of a partition of the set of unknowns. So, how to choose?

- Knowledge of the application, e.g.: eigenvalue problem.
- Enumerate all partitions and retain the partition with the smallest Bézout number.

#unknowns	1	2	3	4	5	6	7	8	9	···
#partitions	1	2	5	15	52	203	877	4140	21147	···

C.W. Wampler: **Bezout number calculations for multi-homogeneous polynomial systems.** *Appl. Math. Comput.* 51(2–3):143–157, 1992.

- Heuristics based on structures of the monomials are effective in most of the practical cases.

linear-product start systems

$$f(\mathbf{x}) = \begin{cases} x_1 x_2^2 + x_1 x_3^3 - c x_1 + 1 = 0 & c \in \mathbb{C} \\ x_2 x_1^2 + x_2 x_3^2 - c x_2 + 1 = 0 & \\ x_3 x_1^2 + x_3 x_2^2 - c x_3 + 1 = 0 & D = 27 \end{cases}$$

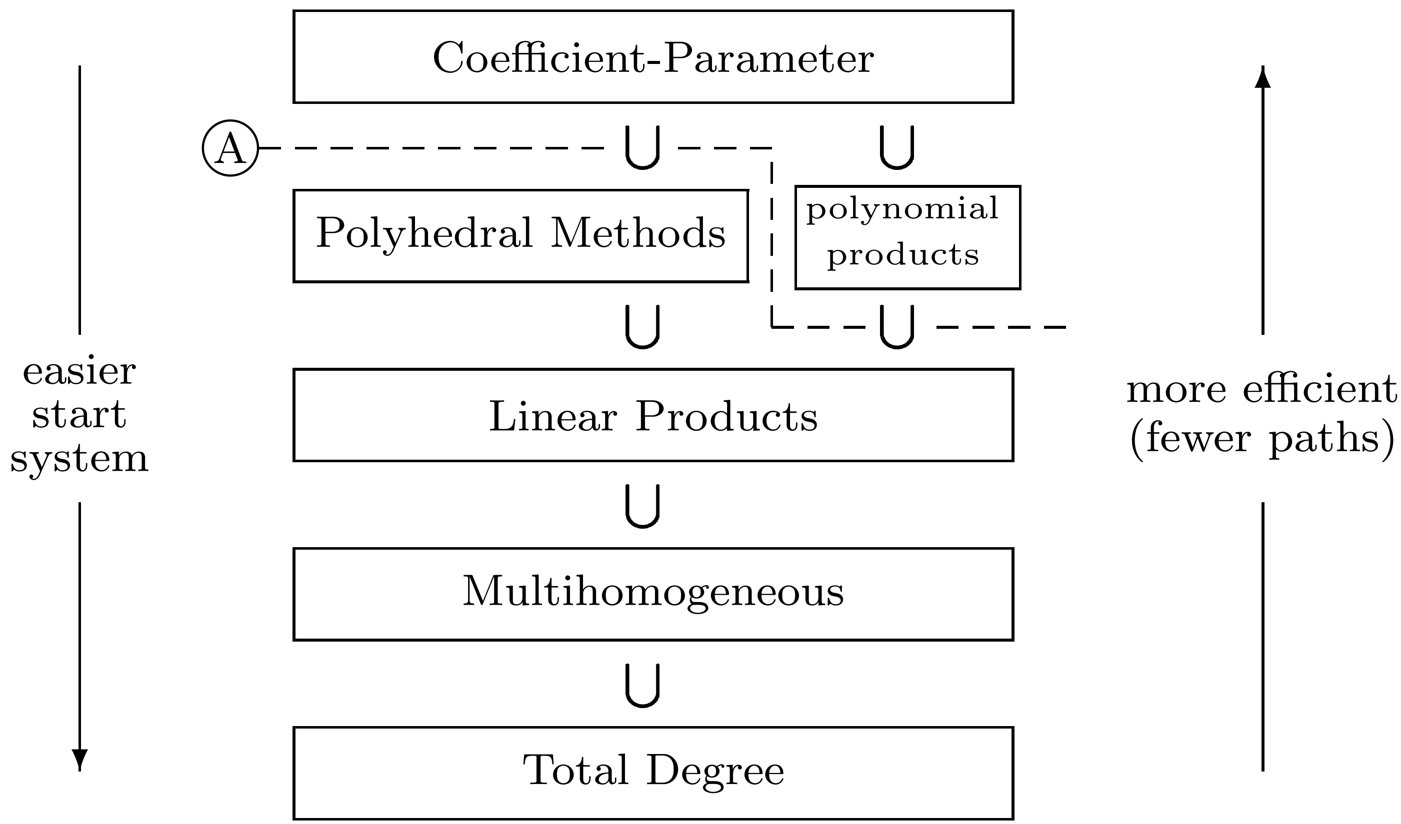
$\{x_1\}$	$\{x_2, x_3\}$	$\{x_2, x_3\}$	symmetric	
$\{x_2\}$	$\{x_1, x_3\}$	$\{x_1, x_3\}$	supporting	$B = 21$
$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_2\}$	set structure	

Choose 7 random complex numbers $c_1, c_2, \ldots, c_7$ and create

$$g(\mathbf{x}) = \begin{cases} (x_1 + c_1)(c_2 x_2 + c_3 x_3 + c_4)(c_5 x_2 + c_6 x_3 + c_7) = 0 \\ (x_2 + c_1)(c_2 x_1 + c_3 x_3 + c_4)(c_5 x_1 + c_6 x_3 + c_7) = 0 \\ (x_3 + c_1)(c_2 x_1 + c_3 x_2 + c_4)(c_5 x_1 + c_6 x_2 + c_7) = 0 \end{cases}$$

8 generating solutions

A Hierarchy of Structures



Below line A: solving start systems is done automatically.

Above line A: special ad-hoc methods must be designed.

Papers on Exploiting Product Structures

A. Morgan and A. Sommese: **A homotopy for solving general polynomial systems that respects m-homogeneous structures.** *Appl. Math. Comput.* 24(2):101–113, 1987.

T.Y. Li, T. Sauer, and J.A. Yorke: **The random product homotopy and deficient polynomial systems.** *Numer. Math.* 51(5):481–500, 1987.

J. Verschelde and A. Haegemans: **The *GBQ*-Algorithm for constructing start systems of homotopies for polynomial systems.** *SIAM J. Numer. Anal.* 30(2):583–594, 1993.

C.W. Wampler: **An efficient start system for multi-homogeneous polynomial continuation.** *Numer. Math.* 66(4):517–523, 1994.

A.P. Morgan, A.J. Sommese, and C.W. Wampler: **A product-decomposition theorem for bounding Bézout numbers.** *SIAM J. Numer. Anal.* 32(4):1308–1325, 1995.

T.Y. Li, T. Wang, and X. Wang: **Random product homotopy with minimal BKK bound.** In *The Mathematics of Numerical Analysis*, ed. by J. Renegar, M. Shub, and S. Smale, pages 503–512, AMS, 1996.

The software PHCpack

J. Verschelde: **Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.** *ACM Transactions on Mathematical Software* 25(2): 251-276, 1999.

Available via `http://www.math.uic.edu/~jan/download.html`.

Modes of operation:

1. As a blackbox: `phc -b input output`.
2. In toolbox mode (call `phc` with other options).
3. The library PHCpack, in Ada with C interface.

PHCpack is menu-driven and file oriented

```
Welcome to PHC (Polynomial Homotopy Continuation) Version 2.1(beta).

Running in full mode.  Note also the following options:
  phc -s : Equation and variable Scaling on system and solutions
  phc -d : Linear and nonlinear Reduction w.r.t. the total degree
  phc -r : Root counting and Construction of start systems
  phc -m : Mixed-Volume Computation by four lifting strategies
  phc -p : Polynomial Continuation by a homotopy in one parameter
  phc -v : Validation, refinement and purification of solutions
  phc -e : SAGBI/Pieri homotopies to intersect linear subspaces
  phc -c : Irreducible decomposition for solution components
  phc -f : Factor pure dimensional solution set into irreducibles
  phc -b : Batch or black-box processing
  phc -z : strip phc output solution lists into Maple format

Is the system on a file ? (y/n/i=info)
```

Papers documenting the usefulness of PHCpack

R.S. Datta: **Using Computer Algebra To Compute Nash Equilbria**. To be presented at ISSAC 2003.

C. Durand and C.M. Hoffmann: **Variational Constraints in 3D**. In Proceedings of the International Conference on Shape Modeling and Applications, Aizu-Wakamatsu, Japan, pages 90-98, IEEE Computer Society, 1999.

C. Durand and C.M. Hoffmann: **A systematic framework for solving geometric constraints analytically**. *J. Symbolic Computation* 30(5):493-520, 2000.

B. Haas: **A Simple Counterexample to Kouchnirenko's Conjecture**. *Beitraege zur Algebra und Geometrie/Contributions to Algebra and Geometry* 43(1):1-8, 2002.

E. Lee and C. Mavroidis: **Solving the Geometric Design Problem of Spatial 3R Robot Manipulators Using Polynomial Continuation.** *Journal of Mechanical Design, Transactions of the ASME* 124(4):652-661, 2002.

E. Lee, C. Mavroidis, and J. Morman: **Geometric Design of Spatial 3R Manipulators**. *Proceedings of the 2002 NSF Design, Service, and Manufactoring Grantees and Research Conference*, San Juan, Puerto Rico, January 7-10, 2002.

More papers documenting the usefulness of PHCpack

M. Oskarsson, A. Zisserman and K. Astrom: **Minimal Projective Reconstruction for combinations of Points and Lines in Three Views**. *Electronic Proceedings of BMVC2002 - The 13th British Machine Vision Conference 2002*, pages 63 - 72.

P.A. Parillo and B. Sturmfels: **Minimizing Polynomial Functions**. presented at the *Workshop on Algorithmic and Quantitative Aspects of Real Algebraic Geometry in Mathematics and Computer Science,* held at DIMACS, Rutgers University, March 12-16, 2001.

H. Schreiber, K. Meer, and B.J. Schmitt: **Dimensional synthesis of planar Stephenson mechanisms for motion generation using circlepoint search and homotopy methods.** *Mechanism and Machine Theory* 37(7):717-737, 2002.

F. Sottile: **Real Schubert Calculus: Polynomial systems and a conjecture of Shapiro and Shapiro**. *Experimental Mathematics* 9(2): 161-182, 2000.

C.W. Wampler: **Isotropic coordinates, circularity and Bezout numbers: planar kinematics from a new perspective**. *Proceedings of the 1996 ASME Design Engineering Technical Conference.* Irvine, CA, Aug 18–22, 1996. (CD-ROM).

F. Xie, G. Reid, and S. Valluri: **A numerical method for the one dimensional action functional for FBG structures**. *Can J. Phys.* 76: 1-21, 2002.

Exercises

- Download executable `phc` (currently available for SUN Solaris workstations, Windows and Linux PCs).
- Solve one small system with blackbox solver

  ```
  phc -b input output
  ```

  Interpret the (numerical) output.
- Find more interesting examples and explore the `phc -r` (root counting) and `phc -p` (polynomial continuation) menus.