Polynomial Homotopy Continuation

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Plan of the Lecture

1. Homotopies and Path Tracking
   *the theorem of Bézout, predictor-corrector methods, some complexity issues*

2. Coefficient-Parameter Continuation
   *a case study: families of Stewart-Gough platforms*

3. Exploiting Product Structures
   *multi-homogenization, linear-product start systems*

4. Software and Applications
   *the software PHCpack, illustrations of application fields*
Recommended Background Literature


If we wish to solve \( f(x) = 0 \), then we construct a system \( g(x) = 0 \) whose solutions are known. Consider the homotopy

\[
H(x, t) := (1 - t)g(x) + tf(x) = 0.
\]

By continuation, we trace the paths starting at the known solutions of \( g(x) = 0 \) to the desired solutions of \( f(x) = 0 \), for \( t \) from 0 to 1.

**homotopy continuation** methods are *symbolic-numeric*:

- homotopy methods treat polynomials as algebraic objects,
- continuation methods use polynomials as functions.

**geometric interpretation:** move from general to special,

solve special, and move solutions from special to general.
\[ \gamma \left( \begin{cases} x^2 - 1 = 0 \\ y^2 - 1 = 0 \end{cases} \right) (1-t) + \gamma \left( \begin{cases} x^2 + 4y^2 - 4 = 0 \\ 2y^2 - x = 0 \end{cases} \right) t, \quad \gamma \in \mathbb{C} \]
The theorem of Bézout

\[ f = (f_1, f_2, \ldots, f_n) \]
\[ d_i = \text{deg}(f_i) \]
\[ \text{total degree } D : \quad g(x) = \left\{ \begin{array}{l}
\alpha_1 x_1^{d_1} - \beta_1 = 0 \\
\alpha_2 x_2^{d_2} - \beta_2 = 0 \\
\vdots \\
\alpha_n x_n^{d_n} - \beta_n = 0
\end{array} \right. \]
\[ D = \prod_{i=1}^{n} d_i \]

Theorem: \( f(x) = 0 \) has at most \( D \) isolated solutions in \( \mathbb{C}^n \), counted with multiplicities.

Sketch of Proof: \( V = \{ (f, x) \in \mathbb{P}(\mathcal{H}_D) \times \mathbb{P}(\mathbb{C}^n) \mid f(x) = 0 \} \)
\( \Sigma' = \{(f, x) \in V \mid \det(D_x f(x)) = 0\}, \Sigma = \pi_1(\Sigma'), \pi_1 : V \to \mathbb{P}(\mathcal{H}_D) \)
Elimination theory: \( \Sigma \) is variety \( \Rightarrow \mathbb{P}(\mathcal{H}_D) - \Sigma \) is connected.
Thus \( h(x, t) = (1 - t)g(x) + tf(x) = 0 \) avoids \( \Sigma, \forall t \in [0, 1) \).
Implicitly defined curves

Consider a homotopy \( h_k(x(t), y(t), t) = 0, \ k = 1, 2. \)

By \( \frac{\partial}{\partial t} \) on homotopy:
\[
\frac{\partial h_k}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial h_k}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial h_k}{\partial t} \frac{\partial t}{\partial t} = 0, \ k = 1, 2.
\]

Set \( \Delta x := \frac{\partial x}{\partial t}, \Delta y := \frac{\partial y}{\partial t}, \) and \( \frac{\partial t}{\partial t} = 1. \)

Increment \( t := t + \Delta t \)

Solve
\[
\begin{bmatrix}
\frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\
\frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y}
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial h_1}{\partial t} \\
\frac{\partial h_2}{\partial t}
\end{bmatrix}
\]

(\textit{Newton})

Update
\[
\begin{cases}
x := x + \Delta x \\
y := y + \Delta y
\end{cases}
\]
Predictor-Corrector Methods

\[ \text{loop} \]

1. predict \[ \begin{align*}
    t_{k+1} &:= t_k + \Delta t \\
    x^{(k+1)} &:= x^{(k)} + \Delta x
\end{align*} \]

2. correct with Newton

3. if convergence
   then enlarge $\Delta t$
   continue with $k + 1$
   else reduce $\Delta t$
   back up and restart at $k$
until $t = 1$.

\[ x^{(k+1)} := x^{(k)} + \lambda (x^{(k)} - x^{(k-1)}) \]
Robustness of Continuation Methods

sure to find all roots at the end of the paths?

- dealing with curve jumping:
  1. fix \#Newton steps to force quadratic convergence;
  2. rerun clustered paths with same discretization of $t$.

- Robust step control by interval methods, see

- Root of multiplicity $\mu$ will appear at the end of the paths as a cluster of $\mu$ roots.
  Use “endgames”, eventually in multi-precision arithmetic.
Complexity Issues

The Problem: a hierarchy of complexity classes
- \( P \): evaluation of a system at a point
- \( NP \): find one root of a system
- \( \#P \): find all roots of a system \((intractable!)\)

Complexity of Homotopies: for bounds on \( \# \)Newton steps in a linear homotopy, see


On average, we can find an approximate zero in polynomial time.
A Case Study: Stewart-Gough Platforms

end plate, the platform is connected by legs to a stationary base

Forward Displacement Problem:
Given: position of base and leg lengths.
Wanted: position of end plate.
The Equations for the Platform Problem

workspace $\mathbb{R}^3 \times SO(3)$: position and orientation

$$SO(3) = \{ A \in \mathbb{C}^{3 \times 3} \mid A^H A = I, \det(A) = 1 \}$$

more efficient to use Study (or soma) coordinates:

$[e : g] = [e_0 : e_1 : e_2 : e_3 : g_0 : g_1 : g_2 : g_3] \in \mathbb{P}^7$ quaternions on
the Study quadric: $f_0(e, g) = e_0g_0 + e_1g_2 + e_2g_2 + e_3g_3 = 0,$
excluding those $e$ for which $ee' = 0$, $e' = (e_0, -e_1, -e_2, -e_3)$

given leg lengths $L_i$, find $[e : g]$ leads to

$$f_i(e, g) = gg' + (bb_i' + a_ia_i' - L_i^2)ee' + (gb_i'e' + eb_ig') - (ge'a_i' + a_ie'g')$$

$$- (eb_ie'a_i' + a_ieb_i'e') = 0, \quad i = 1, 2, \ldots 6$$

$\Rightarrow$ solve $f = (f_0, f_1, \ldots, f_6)$, 7 quadrics in $[e : g] \in \mathbb{P}^7$

expecting $2^7 = 128$ solutions...
Literature on Stewart-Gough platforms


Coefficient-Parameter Homotopies

- Study how solutions change when parameters vary.

- Key Idea:
  1. solve system once for a generic choice of the parameters;
  2. use homotopy to move from generic to specific instance.

- Works for nested parameter spaces (Charles Wampler).

For the theory, see

A family of Stewart-Gough platforms

6-6, 40 solutions
4-6, 32 solutions
4-4a, 16 solutions
3-3, 16 solutions
4-4b, 24 solutions

thanks to Charles Wampler
The system $f$ in $[e : g]$ has parameters: $a_i, b_i, i = 1, 2, \ldots, 6$, the ball joints of the stationary base and the moving platform.

Solve $f$ once for generic choice of $a_i, b_i \in \mathbb{C}^3$, (e.g., tracing 128 paths with total degree homotopy) to find 40 isolated roots.

The parameter space
\[ SG = \{ (a_i, b_i, L_i), i = 1, 2, \ldots, 6 \} \subset (\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C})^6 = \mathbb{C}^{42}. \]

For $p_0, p_1 \in SG$, consider the coefficient-parameter parameter homotopy
\[ h_{SG}([e : g], t) = f([e : g], t p_0 + (1 - t) p_1) = 0, \]
where $t$ goes from the generic to the special parameters.
Multihomogeneous version of Bézout’s theorem

Consider the eigenvalue problem $A \mathbf{x} = \lambda \mathbf{x}$, $A \in \mathbb{C}^{n \times n}$.

Add one general hyperplane $\sum_{i=1}^{n} c_i x_i + c_0 = 0$ for unique $\mathbf{x}$.

Bézout’s theorem: $D = 2^n \leftrightarrow$ at most $n$ solutions

Embed in multi-projective space: $\mathbb{P} \times \mathbb{P}^n$, separating $\lambda$ from $\mathbf{x}$.

<table>
<thead>
<tr>
<th>${\lambda}$</th>
<th>${x_1, x_2}$</th>
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<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

degree table

<table>
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<tr>
<th>${\lambda}$</th>
<th>${x_1, x_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda + \gamma_1$</td>
<td>$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$</td>
</tr>
<tr>
<td>$\lambda + \gamma_2$</td>
<td>$\beta_0 + \beta_1 x_1 + \beta_2 x_2$</td>
</tr>
<tr>
<td>1</td>
<td>$c_0 + c_1 x_1 + c_2 x_2$</td>
</tr>
</tbody>
</table>

linear-product start system

The root count $B = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 1$ is a permanent.
How to find the best partition?

A multi-homogeneous Bézout number depends on the choice of a partition of the set of unknowns. So, how to choose?

- Knowledge of the application, e.g.: eigenvalue problem.
- Enumerate all partitions and retain the partition with the smallest Bézout number.

<table>
<thead>
<tr>
<th>#unknowns</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>⋮</th>
</tr>
</thead>
<tbody>
<tr>
<td>#partitions</td>
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<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
<td>4140</td>
<td>21147</td>
<td>⋮</td>
</tr>
</tbody>
</table>


- Heuristics based on structures of the monomials are effective in most of the practical cases.
linear-product start systems

\[ f(x) = \begin{cases} 
  x_1 x_2^2 + x_1 x_3^3 - cx_1 + 1 = 0 & c \in \mathbb{C} \\
  x_2 x_1^2 + x_2 x_3^2 - cx_2 + 1 = 0 \\
  x_3 x_1^2 + x_3 x_2^2 - cx_3 + 1 = 0 & D = 27 
\end{cases} \]

\{x_1\} \quad \{x_2, x_3\} \quad \{x_2, x_3\} \quad \text{symmetric} \\
\{x_2\} \quad \{x_1, x_3\} \quad \{x_1, x_3\} \quad \text{supporting} \quad B = 21 \\
\{x_3\} \quad \{x_1, x_2\} \quad \{x_1, x_2\} \quad \text{set structure}

Choose 7 random complex numbers \( c_1, c_2, \ldots, c_7 \) and create

\[ g(x) = \begin{cases} 
  (x_1 + c_1)(c_2 x_2 + c_3 x_3 + c_4)(c_5 x_2 + c_6 x_3 + c_7) = 0 \\
  (x_2 + c_1)(c_2 x_1 + c_3 x_3 + c_4)(c_5 x_1 + c_6 x_3 + c_7) = 0 \\
  (x_3 + c_1)(c_2 x_1 + c_3 x_2 + c_4)(c_5 x_1 + c_6 x_2 + c_7) = 0 
\end{cases} \]

8 generating solutions
Below line A: solving start systems is done automatically.
Above line A: special ad-hoc methods must be designed.
A. Morgan and A. Sommese: A homotopy for solving general polynomial systems that respects m-homogeneous structures. 


The software PHCpack


Modes of operation:

1. As a blackbox: `phc -b input output`.
2. In toolbox mode (call `phc` with other options).
3. The library PHCpack, in Ada with C interface.
Welcome to PHC (Polynomial Homotopy Continuation) Version 2.1(beta).

Running in full mode. Note also the following options:

- **phc -s**: Equation and variable Scaling on system and solutions
- **phc -d**: Linear and nonlinear Reduction w.r.t. the total degree
- **phc -r**: Root counting and Construction of start systems
- **phc -m**: Mixed-Volume Computation by four lifting strategies
- **phc -p**: Polynomial Continuation by a homotopy in one parameter
- **phc -v**: Validation, refinement and purification of solutions
- **phc -e**: SAGBI/Pieri homotopies to intersect linear subspaces
- **phc -c**: Irreducible decomposition for solution components
- **phc -f**: Factor pure dimensional solution set into irreducibles
- **phc -b**: Batch or black-box processing
- **phc -z**: strip phc output solution lists into Maple format

Is the system on a file? (y/n/i=info)
Papers documenting the usefulness of PHCpack


More papers documenting the usefulness of PHCpack


Exercises

• Download executable `phc` (currently available for SUN Solaris workstations, Windows and Linux PCs).

• Solve one small system with blackbox solver

  ```
  phc -b input output
  ```

  Interpret the (numerical) output.

• Find more interesting examples and explore the `phc -r` (root counting) and `phc -p` (polynomial continuation) menus.