Polyhedral Homotopies

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Plan of the Lecture

1. Geometric Root Counting *why consider mixed volumes?*
2. The Theorems of Bernshteĭn *sharp root counts + deficiency criterion*
3. Mixed Volumes *mixed subdivisions visualize Minkowski's theorem*
4. Polyhedral End Games *finding certificates for divergence*
5. Polyhedral Continuation *solving sparse system in two stages*
6. Software and Applications *outline of blackbox solver*

Recommended Background Literature

I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky: **Discriminants, Resultants and Multidimensional Determinants**. Birkhäuser, 1994.

J.E. Goodman and J. O'Rourke (editors): **Handbook of Discrete and Computational Geometry**. CRC Press, 1997.

R. Schneider: **Convex Bodies: The Brunn-Minkowski Theory**. Cambridge University Press, 1993.

B. Sturmfels: **Gröbner Bases and Convex Polytopes.** AMS, 1996.

B. Sturmfels: **Polynomial equations and convex polytopes.** *Amer. Math. Monthly*, 105(10):907–922, 1998.

B. Sturmfels: **Solving Systems of Polynomial Equations.** AMS, 2002.

G.M. Ziegler: **Lectures on Polytopes.** Springer, 1995.

Solving Systems with Homotopies

Concerns *(of anyone who tries to use numerical homotopies)*

1. efficiency: #paths = bound on #solutions;

   how can we find good bounds on #solutions?

2. validation: how can we be sure to have **all** solutions?

Answers *(why we should consider polyhedral methods)*

1. generically sharp root counts,

   which can be computed by fully automatic blackboxes

2. certificates for diverging paths,

   which are cheap by-products of continuation

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	$V(\mathcal{P})$ mixed volume
$L(f) = L(f_2, f_1, \ldots, f_n)$	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

exploit sparsity $\boxed{L(f) = V(\mathcal{P})}$ 1st theorem of Bernshteĭn

The Theorems of Bernshteĭn

Theorem A: The number of roots of a generic system equals the mixed volume of its Newton polytopes.

Theorem B: Solutions at infinity are solutions of systems supported on faces of the Newton polytopes.

D.N. Bernshteĭn: **The number of roots of a system of equations.** *Functional Anal. Appl.*, 9(3):183–185, 1975.

Structure of proofs: First show Theorem B, looking at power series expansions of diverging paths defined by a linear homotopy starting at a generic system. Then show Theorem A, using Theorem B with a homotopy defined by *lifting* the polytopes.

Systems, Supports, and Newton Polytopes

$$f = (f_1, f_2) = \begin{cases} x_1^3 x_2 + x_1 x_2^2 + 1 = 0 \\ x_1^4 + x_1 x_2 + 1 = 0 \end{cases}$$

$$\mathcal{A} = (A_1, A_2)$$

$$A_1 = \{(3,1), (1,2), (0,0)\}$$

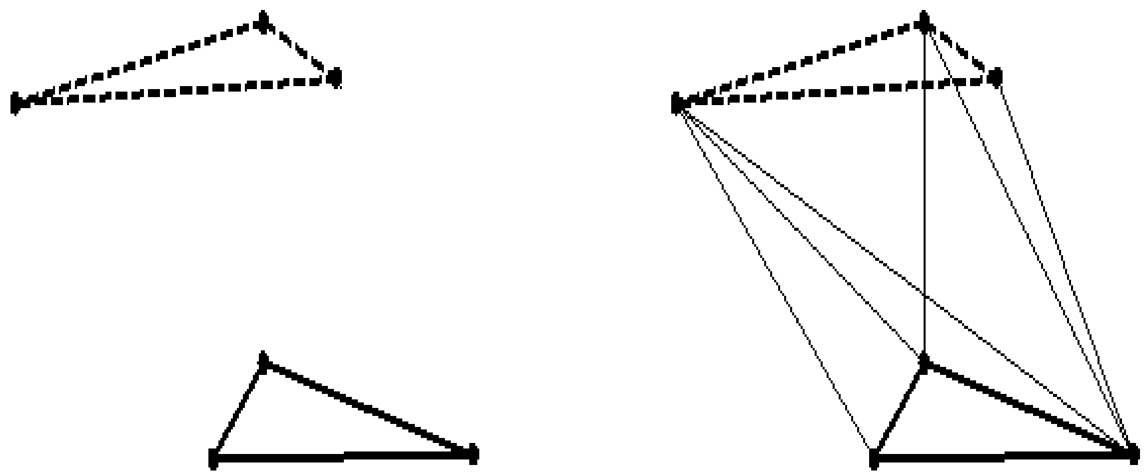
$$A_2 = \{(4,0), (1,1), (0,0)\}$$

The sparse structure of f is modeled by the tuple $\mathcal{A} = (A_1, A_2)$.

A_1 and A_2 are the *supports* of f_1 and f_2 respectively.

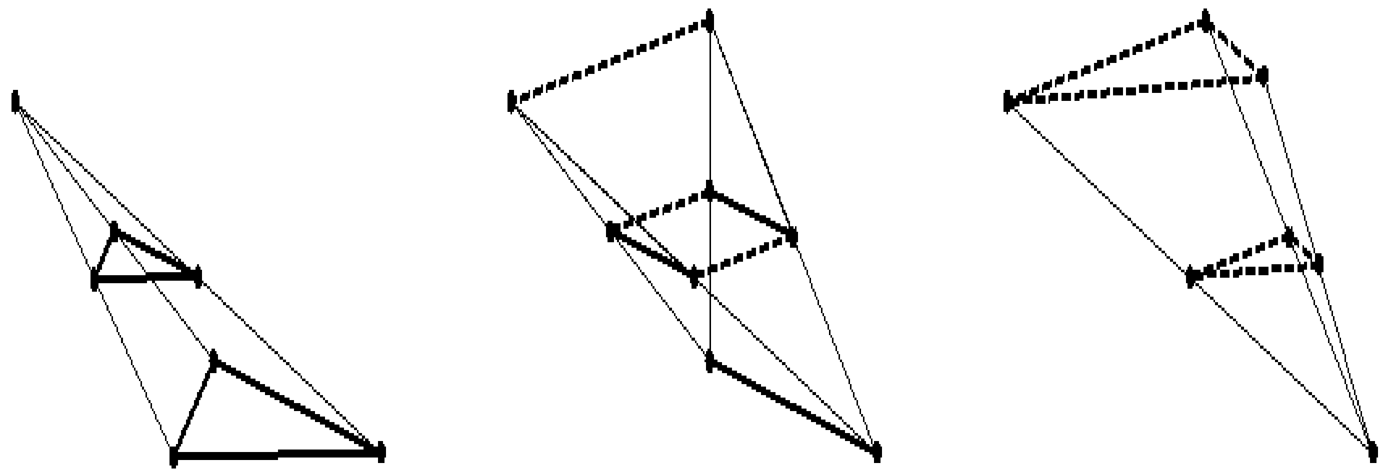
The Newton polytopes are the convex hulls of the supports.

The Cayley polytope

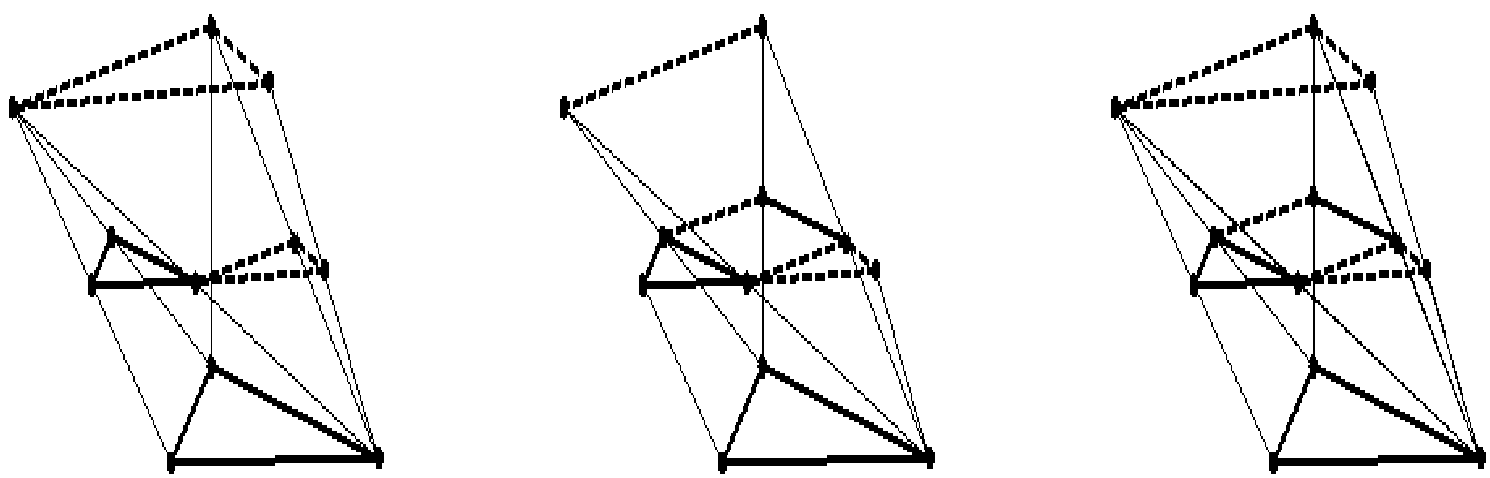


Place one polytope at level 0, the other at level 1.

A triangulation of the Cayley polytope



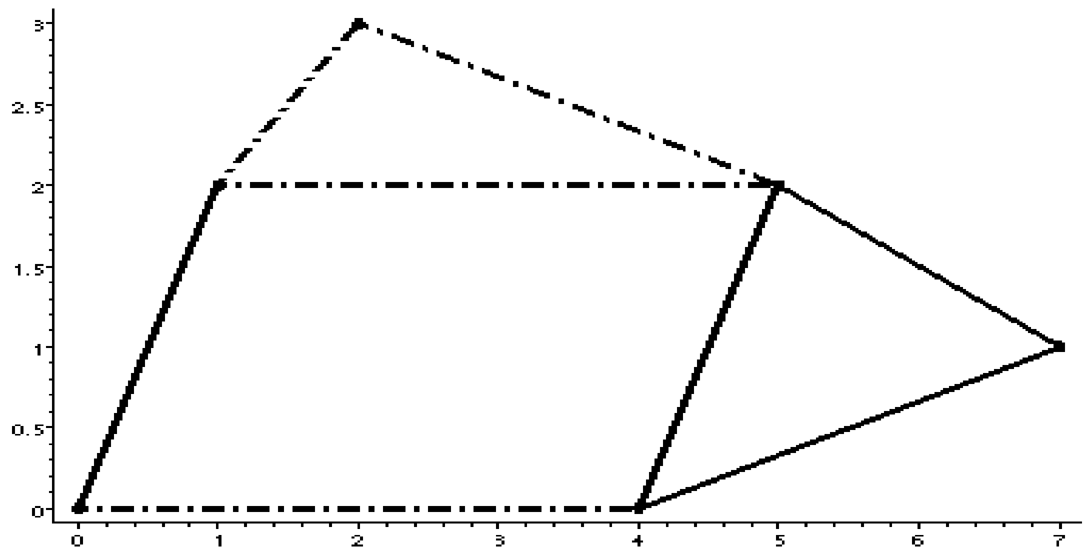
A mixed subdivision induced by a triangulation of the Cayley polytope



Mixed Volumes

Mixed subdivisions visualize Minkowski's theorem:

$$\begin{aligned}\mathrm{area}(\lambda_1 P_1 + \lambda_2 P_2) &= V(P_1,P_1)\lambda_1^2 + 2V(P_1,P_2)\lambda_1\lambda_2 + V(P_2,P_2)\lambda_2^2 \\ &= 5\lambda_1^2 + 2\times 8\lambda_1\lambda_2 + 5\lambda_2^2\end{aligned}$$



Newton Polytopes and Real Solutions

B. Sturmfels: **On the number of real roots of a sparse polynomial system.** In *Hamiltonian and Gradient Flows: Algorithms and Control*, ed. by A. Bloch, pages 137–143, AMS 1994.

B. Sturmfels: **Viro's theorem for complete intersections.** *Annali della Scuola Normale Superiore di Pisa* 21(3):377–386, 1994.

I. Itenberg and M.-F. Roy: **Multivariate Descartes' rule**. *Beiträge zur Algebra and Geometry* 37(2):337–346, 1996.

T.Y. Li and X. Wang: **On multivariate Descartes' rule – a counterexample**. *Beiträge zur Algebra and Geometry* 39(1):1–5, 1998.

I. Itenberg and E. Shustin: **Viro theorem and topology of real and complex combinatorial hypersurfaces.** *Israel Math. J.* 133: 189–238, 2003. `math.AG/0105198`

Bernshteĭn's second theorem

- Face $\partial_\omega f = (\partial_\omega f_1, \partial_\omega f_2, \ldots, \partial_\omega f_n)$ of system $f = (f_1, f_2, \ldots, f_n)$ with Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ and mixed volume $V(\mathcal{P})$.

$$\partial_\omega f_i(\mathbf{x}) = \sum_{\mathbf{a} \in \partial_\omega A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad \begin{array}{c} \partial_\omega P_i = \mathrm{conv}(\partial_\omega A_i) \\ \text{face of Newton polytope} \end{array}$$

<u>Theorem:</u> If $\forall \omega \neq \mathbf{0}$, $\partial_\omega f(\mathbf{x}) = \mathbf{0}$ has no solutions in $(\mathbb{C}^*)^n$,

then $V(\mathcal{P})$ is exact and all solutions are isolated.

Otherwise, for $V(\mathcal{P}) \neq 0$: $V(\mathcal{P}) >$ #isolated solutions.

- Newton polytopes in general position: $V(\mathcal{P})$ is exact for every nonzero choice of the coefficients.

Newton polytopes in general position

Consider $f(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

The Newton polytopes:

$$\forall \omega \neq \mathbf{0} : \partial_\omega A_1 + \partial_\omega A_2 \leq 3 \quad \Rightarrow \quad V(P_1, P_2) = 4 \text{ always exact}$$

for all nonzero coefficients

Power Series

Theorem: $\forall \mathbf{x}(t)$, $h(\mathbf{x}(t), t) = (1-t)g(\mathbf{x}(t)) + tf(\mathbf{x}(t)) = \mathbf{0}$,

$\exists s > 0$, $m \in \mathbb{N} \setminus \{0\}$, $\omega \in \mathbb{Z}^n$:

$$\begin{cases} x_i(s) = b_i s^{\omega_i}(1 + O(s)), & i = 1, 2, \ldots, n \\ t(s) = 1 - s^m & \text{for } t \approx 1, s \approx 0 \end{cases}$$

$$\lim_{t \to 1} x_i(t) \in \mathbb{C}^*? \qquad x_i(t) \begin{cases} \to \infty \\ \in \mathbb{C}^* \\ \to 0 \end{cases} \quad \Leftrightarrow \quad \omega_i \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

m is the *winding number*, i.e. the smallest number so that

$$\mathbf{z}(2\pi m) = \mathbf{z}(0), \quad h(\mathbf{z}(\theta), t(\theta)) = \mathbf{0}, \quad t = 1 + (t_0 - 1)e^{i\theta}, \quad t_0 \approx 1.$$

Face Systems and Power Series

assume $\lim\limits_{t \to 1} x_i(t) \not\in \mathbb{C}^*$, thus $\omega_i \neq 0$, a diverging path

- $h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}$ substitute power series $\left\{ \begin{array}{c} x_i(s) = b_i s^{\omega_i}(1 + O(s)) \\ t(s) = 1 - s^m, s \approx 0 \end{array} \right.$

$$h(\mathbf{x}(s), t(s)) = \underbrace{f(\mathbf{x}(s))}_{\text{dominant as } s \to 0} + s^m(g(\mathbf{x}(s)) - f(\mathbf{x}(s))) = \mathbf{0}$$

- $f_i(\mathbf{x}) = \sum\limits_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \to f_i(\mathbf{x}(s)) = \underbrace{\sum\limits_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \prod\limits_{i=1}^{n} b_i^{a_i} s^{\langle \mathbf{a}, \omega \rangle}(1 + O(s))}_{\partial_\omega f_i(\mathbf{x}(s)) \text{ dominant}}$

face $\partial_\omega A_i := \{ \ \mathbf{a} \in A_i \mid \langle \mathbf{a}, \omega \rangle = \min\limits_{\mathbf{a}' \in A_i} \langle \mathbf{a}', \omega \rangle \ \}$

$\Rightarrow \partial_\omega f(\mathbf{b}) = \mathbf{0}, \mathbf{b} \in (\mathbb{C}^*)^n$

key idea in proof of Bernshteǐn's second theorem

Richardson Extrapolation for ω and m

$$\begin{cases} x_i(s) &= b_i s^{\omega_i}(1+O(s)) \\ t(s) &= 1-s^m \end{cases}$$

$$x_i(s_k) = b_i h^{k\omega_i/m} s_0(1+O(h^{k/m}s_0))$$

Geometric sampling $0 < h < 1$

$$1-t_k = h(1-t_k) = \cdots = h^k(1-t_0)$$

$$s_k = h^{1/m} s_{k-1} = \cdots = h^{k/m} s_0$$

- $\log|x_i(s_k)| = \log|b_i| + \frac{k\omega_i}{m}\log(h) + \omega_i \log(s_0)$

$$+\log(1+\textstyle\sum_{j=0}^{\infty} b'_j (h^{k/m}s_0)^j)$$

$$v_{kk+1} := \log|x_i(s_k+1)| - \log|x_i(s_k)|$$

Extrapolation on samples

$$v_{k..l} = v_{k..l-1} + \frac{v_{k+1..l} - v_{k..l-1}}{1-h}$$

$$\omega_i = m\frac{v_{0..r}}{\log(h)} + O(s_0^r)$$

- $e_i^{(k)} = (\log|x_i(s_k)| - \log|x_i(s_{k+1})|)$

$$-(\log|x_i(s_{k+1})| - \log|x_i(s_{k+2})|)$$

$$= c_1 h^{k/m} s_0(1+0(h^{k/m}))$$

$$e_i^{(kk+1)} := \log(e_i^{(k+1)}) - \log(e_i^{(k)})$$

Extrapolation on errors

$$e_i^{(k..l)} = e_i^{(k+1..l)} + \frac{e_i^{(k..l-1)} - e_i^{(k+1..l)}}{1-h_{k..l}}$$

$$h_{k..l} = h^{(l-k-1)/m_{k..l}}$$

$$m_{k..l} = \frac{\log(h)}{e_i^{(k..l)}} + O(h^{(l-k)k/m})$$

the system of Cassou-Noguès

$$f(b,c,d,e) = \left\{ \begin{array}{rcl} 15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e & & \\ -28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120 & = & 0 \\ 30c^3b^4d - 32de^2c - 720db^2c - 24c^3b^2e - 432c^2b^2 + 576ec & & \\ -576de + 16cb^2d^2e + 16d^2e^2 + 16e^2c^2 + 9c^4b^4 + 5184 & & \\ +39d^2b^4c^2 + 18d^3b^4c - 432d^2b^2 + 24d^3b^2e - 16c^2b^2de - 240c & = & 0 \\ 216db^2c - 162d^2b^2 - 81c^2b^2 + 5184 + 1008ec - 1008de & & \\ +15c^2b^2de - 15c^3b^2e - 80de^2c + 40d^2e^2 + 40e^2c^2 & = & 0 \\ 261 + 4db^2c - 3d^2b^2 - 4c^2b^2 + 22ec - 22de & = & 0 \end{array} \right.$$

Root counts: $D = 1344$, $B = 312$, $V(\mathcal{P}) = 24 > 16$ finite roots.

$$\begin{array}{l} \partial_{(0,0,0,-1)} f(b,c,d,e) = \\ m = 2 \end{array} \left\{ \begin{array}{r} -8b^2c^2e - 28b^2cde + 36b^2d^2e = 0 \\ -32de^2c + 16d^2e^2 + 16e^2c^2 = 0 \\ -80de^2c + 40d^2e^2 + 40e^2c^2 = 0 \\ 22ec - 22de = 0 \end{array} \right.$$

Some further recommended reading

B. Huber and J. Verschelde: **Polyhedral end games for polynomial continuation.**
Numerical Algorithms 18(1):91–108, 1998.

J. Verschelde: **Toric Newton Method for Polynomial Homotopies.**
J. Symbolic Computation 29(4-5): 777–793, 2000.

Sparsity and Unimodular Transformations

$$f(\mathbf{x}) = \left\{ \begin{array}{r} x_1^3 x_2^{-1} + c_1 = 0 \\ x_1 x_2^2 + c_2 = 0 \end{array} \right. \qquad f(\mathbf{x} = \mathbf{y}^U) = \left\{ \begin{array}{r} y_1 + c_1 = 0 \\ y_1^{-2} y_2^7 + c_2 = 0 \end{array} \right.$$

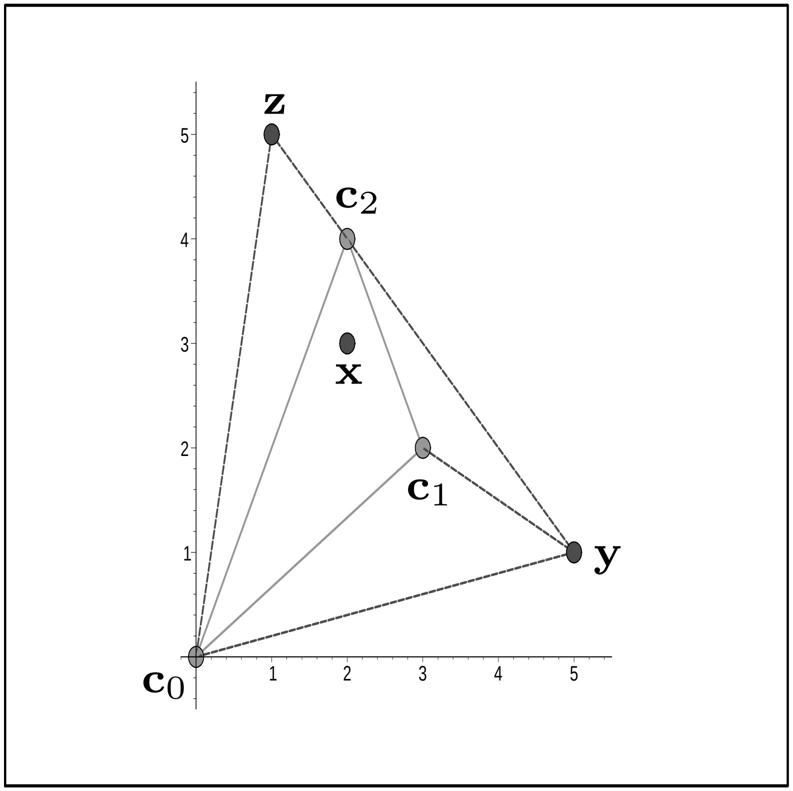
The substitution $\mathbf{x}^V = (\mathbf{y}^U)^V = \mathbf{y}^{VU} = \mathbf{y}^L$ is elaborated as

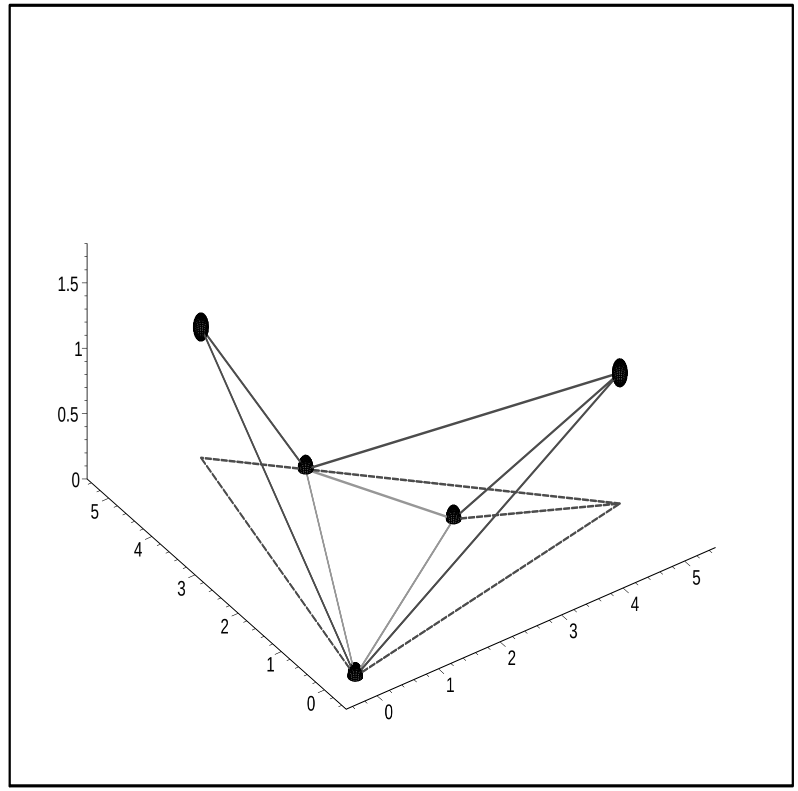
$$\begin{aligned} \left(\begin{array}{c} x_1^3 \cdot x_2^{-1} \\ x_1^1 \cdot x_2^2 \end{array} \right) & = \left(\begin{array}{c} (y_1^0 y_2^1)^3 \cdot (y_1^{-1} y_2^3)^{-1} \\ (y_1^0 y_2^1)^1 \cdot (y_1^{-1} y_2^3)^2 \end{array} \right) \\ & = \left(\begin{array}{c} y_1^{3\cdot 0 - 1\cdot(-1)} \cdot y_2^{3\cdot 1 - 1\cdot 3} \\ y_1^{1\cdot 0 + 2\cdot(-1)} \cdot y_2^{1\cdot 1 + 2\cdot 3} \end{array} \right) = \left(\begin{array}{c} y_1^1 \cdot y_2^0 \\ y_1^{-2} \cdot y_2^7 \end{array} \right). \end{aligned}$$

factorization $VU = L$: $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 7 \end{bmatrix}$

(U unimodular, $\det(U)$=1)

Pivoting to update a triangulation





$\mathbf{c}_0 = (0,0)$	$\mathbf{x} = (2,3)$	$\mathbf{x} = +\frac{1}{8}\mathbf{c}_0 + \frac{1}{4}\mathbf{c}_1 + \frac{5}{8}\mathbf{c}_2$
$\mathbf{c}_1 = (3,2)$	$\mathbf{y} = (5,1)$	$\mathbf{y} = -\frac{1}{3}\mathbf{c}_0 + \frac{9}{4}\mathbf{c}_1 - \frac{7}{8}\mathbf{c}_2$
$\mathbf{c}_2 = (2,4)$	$\mathbf{z} = (1,5)$	$\mathbf{z} = +\frac{1}{8}\mathbf{c}_0 - \frac{3}{4}\mathbf{c}_1 + \frac{13}{8}\mathbf{c}_2$

barycentric coordinates

Incremental Polyhedral Continuation

$$g(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{211}x_1x_2 + c_{210}x_1 + c_{201}x_2 + c_{200} = 0 \end{cases}$$

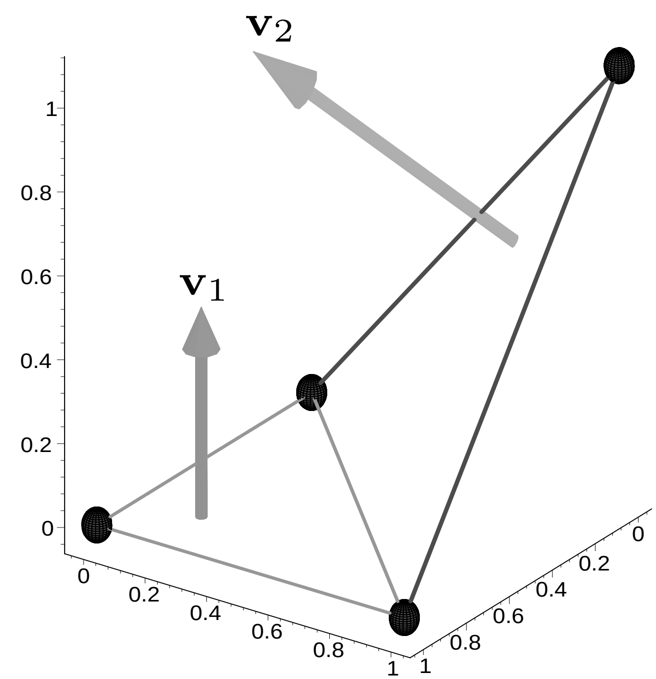
$$\mathbf{v}_1 = (0,0,1)$$

$$g_1(\mathbf{x},t) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2t + c_{100} = 0 \\ c_{211}x_1x_2 + c_{210}x_1 + c_{201}x_2t + c_{200} = 0 \end{cases}$$

$$\boxed{x_1 = \widetilde{x}_1 t \quad x_2 = \widetilde{x}_2 t^{-1}}$$

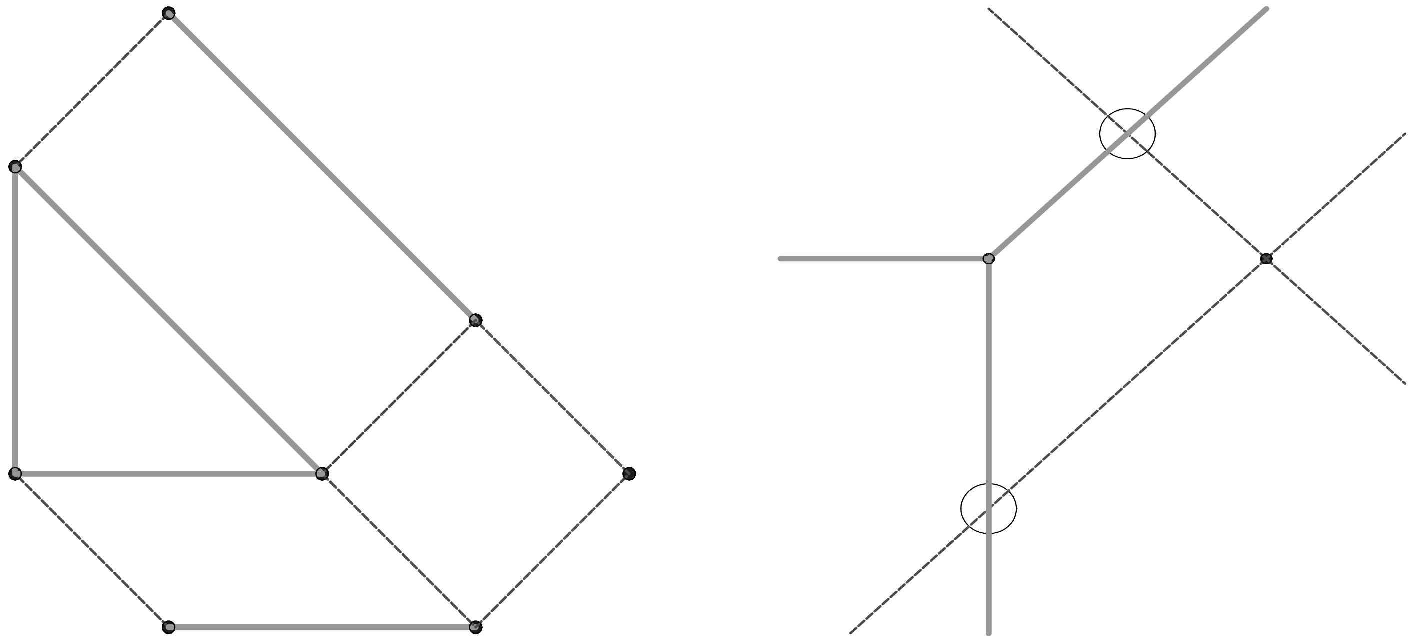
$$\mathbf{v}_2 = (1,-1,1)$$

$$g_2(\widetilde{\mathbf{x}},t) = \begin{cases} c_{111}\widetilde{x}_1\widetilde{x}_2 + c_{110}\widetilde{x}_1t + c_{101}\widetilde{x}_2 + c_{100} = 0 \\ c_{211}\widetilde{x}_1\widetilde{x}_2 + c_{210}\widetilde{x}_1t + c_{201}\widetilde{x}_2 + c_{200} = 0 \end{cases}$$



Mixed Cell Configurations and Normal Fans

normals to mixed cells are in the intersections of normal cones to the edges



Find all $\mathbf{v}$ satisfying
$$\begin{cases} \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle = \langle \widehat{\mathbf{b}}, \mathbf{v} \rangle & \forall \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \in \partial_{\mathbf{v}} \widehat{A}_i \\ \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle \geq \langle \widehat{\mathbf{b}}, \mathbf{v} \rangle & \forall \widehat{\mathbf{a}} \in \widehat{A}_i \setminus \partial_{\mathbf{v}} \widehat{A}_i, \forall \widehat{\mathbf{b}} \in \partial_{\mathbf{v}} \widehat{A}_i \\ \quad v_{n+1} = 1 & \qquad\qquad i=1,2,\ldots,n \end{cases}$$

Polyhedral Homotopies

Let $g(\mathbf{x}) = \mathbf{0}$ have the same Newton polytopes $\mathcal{P}$ as $f(\mathbf{x}) = \mathbf{0}$, but with randomly choosen complex coefficients.

I. Compute $V_n(\mathcal{P})$:			II. Solve $g(\mathbf{x}) = \mathbf{0}$:	
I.1	lift polytopes		II.1	introduce parameter t
I.2	mixed cells	$\Leftrightarrow$	II.2	start systems
I.3	volume of mixed cell		II.3	path following

III. Solve the specific system $f(\mathbf{x}) = \mathbf{0}$:

$$h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}, \quad \text{for } t \text{ from } 0 \text{ to } 1.$$

coefficient-parameter continuation

Some references on polyhedral methods

B. Huber and B. Sturmfels: **A polyhedral method for solving sparse polynomial systems.** *Math. Comp.* 64(212):1541–1555, 1995.

I.Z. Emiris and J.F. Canny: **Efficient incremental algorithms for the sparse resultant and the mixed volume.** *J. Symbolic Computation* 20(2):117–149, 1995.

I.Z. Emiris: **Sparse Elimination and Applications in Kinematics**. *PhD thesis*, UC Berkeley, 1994.

J. Verschelde: **Homotopy Continuation Methods for Solving Polynomial Systems**. *PhD thesis*, KU Leuven, 1996.

B. Sturmfels: **Polynomial equations and convex polytopes.** *Amer. Math. Monthly* 105(10):907–922, 1998.

Recent computational advances

more efficient use of linear programming:

T.Y. Li and X. Li: **Finding mixed cells in the mixed volume computation.** *Found. Comput. Math.* 1(2): 161–181, 2001. Software available at `http://www.math.msu.edu/~li`.

T. Gao and T.Y. Li: **Mixed volume computation for semi-mixed systems.** *Discrete Comput. Geom.* 29(2):257-277, 2003.

and parallel mixed-volume computations:

A. Takeda, M. Kojima and K. Fujisawa: **Enumeration of all solutions of a combinatorial linear inequality system arising from the polyhedral homotopy continuation Method.** *Journal of the Operations Research Society of Japan* 45(1): 64–82, 2002.

Y. Dai, S. Kim and M. Kojima: **Computing all nonsingular solutions of cyclic-n polynomial using polyhedral homotopy continuation methods.** *J. Comput. Appl. Math.* 152(1-2): 83–97, 2003.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani: **PHoM – a polyhedral homotopy continuation method for polynomial systems.** `http://www.is.titech.ac.jp/~kojima/sdp.html`.

Blackbox Solving and Benchmarking

Building a simple blackbox solver:

`phc -b` first computes various root counts based on versions of Bézout's theorem and mixed volumes. The start system is based on the smallest root count, and in case of equal counts, using the least complicated method.

The collection of test systems:

available at `http://www.math.uic.edu/ jan/demo.html`

blackbox strategy opened up a wide range of applications

Exercises

- Take any polynomial system, solve it with the blackbox solver of PHCpack (as `phc -b input output`), and see what root count was used to build the start system.
- Explore the options of `phc -m`. In particular, the Cayley trick is efficient when there are only few different Newton polytopes. Find such an example where dynamic lifting with the Cayley trick outperforms the static lifting techniques.