Homotopies for Solution Components

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Plan of the Lecture

1. Numerical Algebraic Geometry Dictionary

   witness points and membership tests

2. Homotopies to compute Witness Points

   a refined version of Bézout's theorem

3. Diagonal Homotopies

   to intersect positive dimensional components of solutions

4. Software and Applications

   numerical elimination methods

Recommended Background Literature

W. Fulton: **Introduction to Intersection Theory in Algebraic Geometry.** AMS 1984. Reprinted in 1996.

W. Fulton: **Intersection Theory.** Springer 1998, 2nd Edition.

J. Harris: **Algebraic Geometry. A First Course.** Springer 1992.

D. Mumford: **Algebraic Geometry I. Complex Projective Varieties.** Springer 1995. Reprint of the 1976 Edition.

Solution sets to polynomial systems

Polynomial in One Variable	System of Polynomials
one equation, one variable	n equations, N variables
solutions are points	points, lines, surfaces, ...
multiple roots	sets with multiplicity
Factorization: $\prod_i (x - a_i)^{\mu_i}$	**Irreducible Decomposition**

Numerical Representation	
set of points	set of witness sets

Joint Work with A.J. Sommese and C.W. Wampler

A.J. Sommese and C.W. Wampler: **Numerical algebraic geometry.** In *The Mathematics of Numerical Analysis*, ed. by J. Renegar et al., volume 32 of *Lectures in Applied Mathematics*, 749–763, AMS, 1996.

A.J. Sommese and JV: **Numerical homotopies to compute generic points on positive dimensional algebraic sets.** *Journal of Complexity* 16(3):572–602, 2000.

A.J. Sommese, JV and C.W. Wampler: **Numerical decomposition of the solution sets of polynomial systems into irreducible components.** *SIAM J. Numer. Anal.* 38(6):2022–2046, 2001.

A.J. Sommese, JV and C.W. Wampler: **Numerical irreducible decomposition using PHCpack.** In *Algebra, Geometry, and Software Systems*, edited by M. Joswig and N. Takayama, pages 109–130, Springer-Verlag, 2003.

A.J. Sommese, JV and C.W. Wampler: **Homotopies for Intersecting Solution Components of Polynomial Systems.** Manuscript, 2003.

An Illustrative Example

$$f(x,y,z) = \begin{cases} (y-x^2)(x^2+y^2+z^2-1)(x-0.5) = 0 \\ (z-x^3)(x^2+y^2+z^2-1)(y-0.5) = 0 \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-0.5) = 0 \end{cases}$$

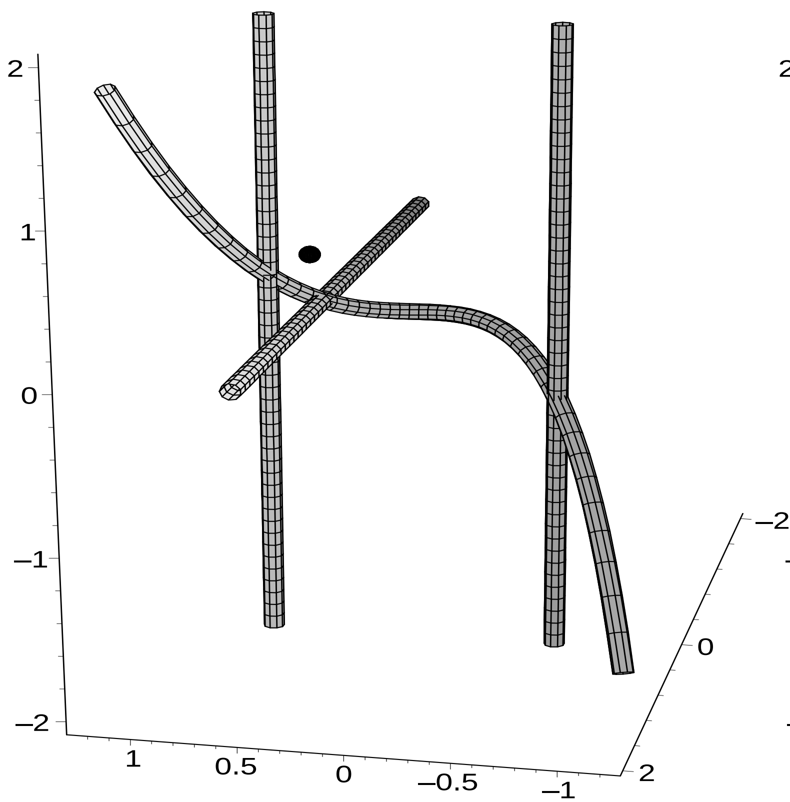
Irreducible decomposition of $Z = f^{-1}(\mathbf{0})$ is

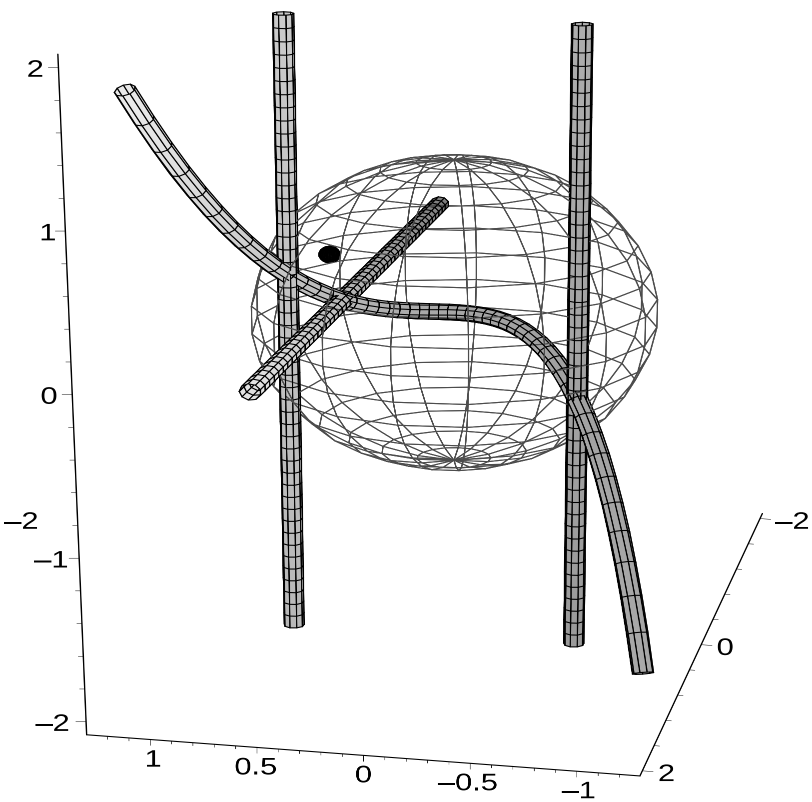
$Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$

with

1. Z_{21} is the sphere $x^2 + y^2 + z^2 - 1 = 0$,
2. Z_{11} is the line $(x = 0.5, z = 0.5^3)$,
3. Z_{12} is the line $(x = \sqrt{0.5}, y = 0.5)$,
4. Z_{13} is the line $(x = -\sqrt{0.5}, y = 0.5)$,
5. Z_{14} is the twisted cubic $(y - x^2 = 0, z - x^3 = 0)$,
6. Z_{01} is the point $(x = 0.5, y = 0.5, z = 0.5)$.

An Illustrative Example - the plots





Witness Sets

A witness point is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component

  equals the **dimension** of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes

  equals the **degree** of the solution component.

A witness set for a k-dimensional solution component consists of k random hyperplanes and a set of isolated solutions of the system cut with those hyperplanes.

Membership Test

Does the point $\mathbf{z}$ *belong to a component?*

Given: a point in space $\mathbf{z} \in \mathbb{C}^N$; a system $f(\mathbf{x}) = \mathbf{0}$;

and a witness set W, $W = (Z, L)$:

for all $\mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0}$ and $L(\mathbf{w}) = \mathbf{0}$.

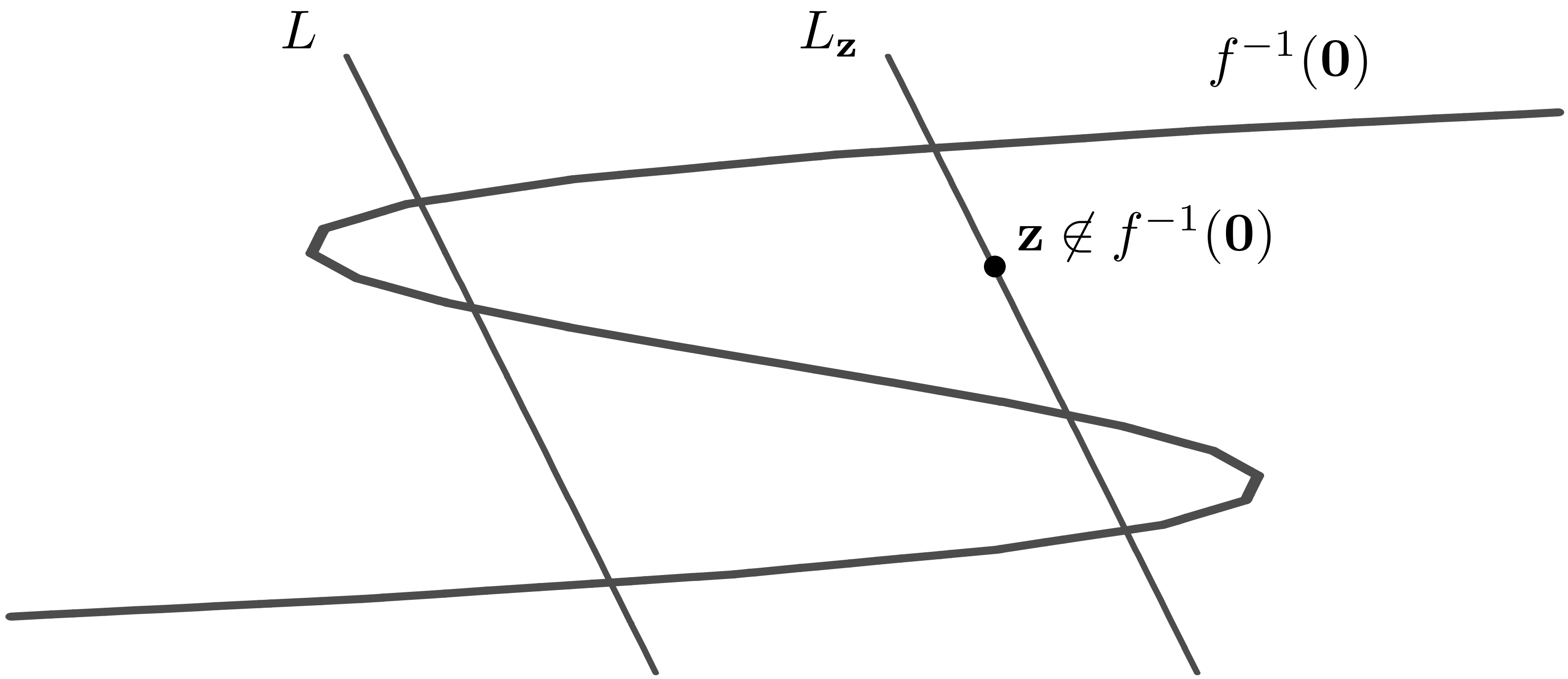
1. Let $L_{\mathbf{z}}$ be a set of hyperplanes through $\mathbf{z}$, and define

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{z}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

2. Trace all paths starting at $\mathbf{w} \in Z$, for t from 0 to 1.

3. The test $(\mathbf{z}, 1) \in h^{-1}(\mathbf{0})$? answers the question above.

Membership Test – an example



$$h(\mathbf{x},t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{z}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

Numerical Algebraic Geometry Dictionary

Algebraic Geometry	example in 3-space	Numerical Analysis
variety	collection of points, algebraic curves, and algebraic surfaces	polynomial system + union of witness sets, see below for the definition of a witness set
irreducible variety	a single point, or a single curve, or a single surface	polynomial system + witness set + probability-one membership test
generic point on an irreducible variety	random point on an algebraic curve or surface	point in witness set; a witness point is a solution of polynomial system on the variety and on a random slice whose codimension is the dimension of the variety
pure dimensional variety	one or more points, or one or more curves, or one or more surfaces	polynomial system + set of witness sets of same dimension + probability-one membership tests
irreducible decomposition of a variety	several pieces of different dimensions	polynomial system + array of sets of witness sets and probability-one membership tests

Randomization and Embedding

Overconstrained systems, e.g.: $f = (f_1, f_2, \ldots, f_5)$, with $\mathbf{x} = (x_1, x_2, x_3)$.

randomization: choose random complex numbers a_{ij}:

$$\begin{cases} f_1(\mathbf{x}) + a_{11} f_4(\mathbf{x}) + a_{12} f_5(\mathbf{x}) = 0 \\ f_2(\mathbf{x}) + a_{21} f_4(\mathbf{x}) + a_{22} f_5(\mathbf{x}) = 0 \\ f_3(\mathbf{x}) + a_{31} f_4(\mathbf{x}) + a_{32} f_5(\mathbf{x}) = 0 \end{cases}$$

embedding: z_1 and z_2 are slack variables ($a_{ij} \in \mathbb{C}$ again at random):

$$\begin{cases} f_1(\mathbf{x}) + a_{11} z_1 + a_{12} z_2 = 0 \\ f_2(\mathbf{x}) + a_{21} z_1 + a_{22} z_2 = 0 \\ f_3(\mathbf{x}) + a_{31} z_1 + a_{32} z_2 = 0 \\ f_4(\mathbf{x}) + a_{41} z_1 + a_{42} z_2 = 0 \\ f_5(\mathbf{x}) + a_{51} z_1 + a_{52} z_2 = 0 \end{cases}$$

Embedding with Slack Variables

The cyclic 4-roots system defines 2 quadrics in $\mathbb{C}^4$:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{lllllll} x_1 & + x_2 & + x_3 & + x_4 & + & \gamma_1 z & = & 0 \\ x_1 x_2 & + x_2 x_3 & + x_3 x_4 & + x_4 x_1 & + & \gamma_2 z & = & 0 \\ x_1 x_2 x_3 & + x_2 x_3 x_4 & + x_3 x_4 x_1 & + x_4 x_1 x_2 & + & \gamma_3 z & = & 0 \\ x_1 x_2 x_3 x_4 & - \quad 1 & & & + & \gamma_4 z & = & 0 \end{array} \right. \\ \quad a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + z = 0 \end{array} \right.$$

Original system : 4 equations in x_1, x_2, x_3, and x_4.

Cut with random hyperplane to find isolated points.

Slack variable z with random γ_i, $i = 1, 2, 3, 4$: square system.

Solve embedded system to find $4 = 2{+}2$ witness points as isolated solutions with $z = 0$.

A cascade of embeddings

dimension two:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}z_1 + a_{12}z_2 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 + a_{22}z_2 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 + a_{32}z_2 = 0 \\ L_1(\mathbf{x}) + z_1 = 0 \\ L_2(\mathbf{x}) + z_2 = 0 \end{cases}$$

$\longrightarrow$

dimension one:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}z_1 + a_{12}z_2 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 + a_{22}z_2 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 + a_{32}z_2 = 0 \\ L_1(\mathbf{x}) + z_1 = 0 \\ z_2 = 0 \end{cases}$$

$\swarrow$

dimension zero:

$$\begin{cases} f_1(\mathbf{x}) + a_{11}z_1 = 0 \\ f_2(\mathbf{x}) + a_{21}z_1 = 0 \\ f_3(\mathbf{x}) + a_{31}z_1 = 0 \\ z_1 = 0 \\ z_2 = 0 \end{cases}$$

A cascade of homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde):

1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

solution paths defined by

$$h_i(\mathbf{x}, \mathbf{z}, t) = (1-t)\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + t \left(\begin{array}{c} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{array} \right) = \mathbf{0}$$

starting at $t = 0$ with all solutions with $z_i \neq 0$
reach at $t = 1$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

A refined version of Bézout's theorem

The linear equations added to $f(\mathbf{x}) = \mathbf{0}$ in the cascade of homotopies do not increase the total degree.

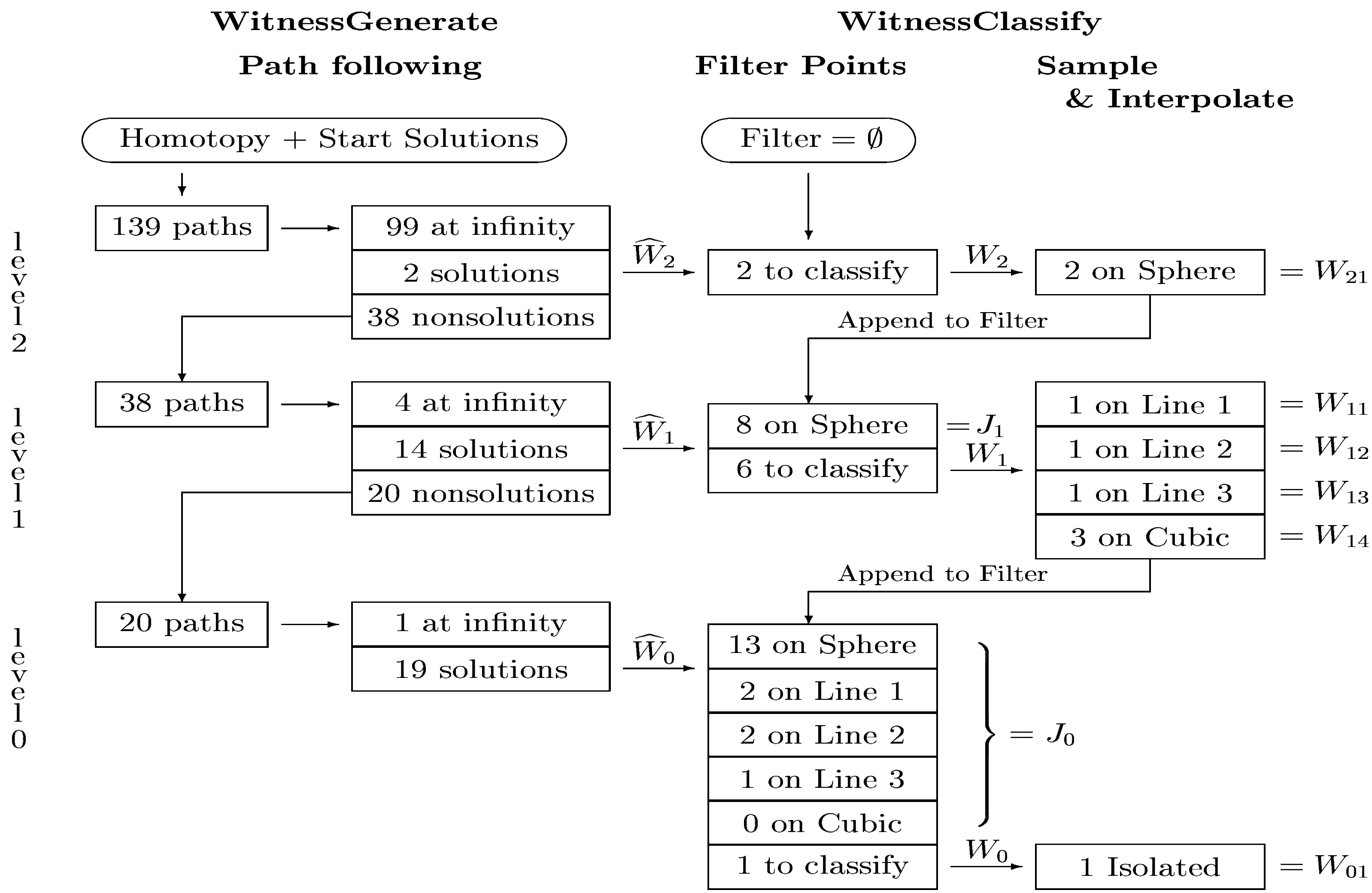
Let $f = (f_1, f_2, \ldots, f_n)$ be a system of n polynomial equations in N variables, $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

$$\text{Bézout bound:} \quad \prod_{i=1}^{n} \deg(f_i) \geq \sum_{j=0}^{N} \mu_j \deg(W_j),$$

where W_j is a j-dimensional solution component of $f(\mathbf{x}) = \mathbf{0}$ of multiplicity μ_j.

Note: $j = 0$ gives the "classical" theorem of Bézout.

A Numerical Irreducible Decomposition of the Illustrative Example



Solving Systems Incrementally

- Extrinsic and Intrinsic Deformations

  extrinsic : defined by explicit equations

  intrinsic : following the actual geometry

- Diagonal Homotopies

  → to intersect pure dimensional solution sets

- Intersecting with Hypersurfaces

  adding the polynomial equations one after the other we arrive at an incremental polynomial system solver.

Extrinsic Homotopy Deformations

$f(\mathbf{x}) = \mathbf{0}$ has k-dimensional solution components. We cut with k hyperplanes to find isolated solutions = *witness sets*:

$$a_{i0} + \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, 2, \ldots, k, \quad a_{ij} \in \mathbb{C} \text{ random}$$

$$\text{Sample} \quad \left\{ \begin{array}{cc} f(\mathbf{x}) + \gamma \mathbf{z} = 0 & \mathbf{z} = \textit{slack} \\ a_{i0}(t) + \sum_{j=1}^{n} a_{ij}(t) x_j = 0 & \textit{moving} \end{array} \right.$$

$$\#\text{witness points} = \sum_{\substack{C \subseteq f^{-1}(0) \\ \dim(C) = k}} \deg(C)$$

Intrinsic Homotopy Deformations

$f(\mathbf{x}) = \mathbf{0}$ has k-dimensional solution components. We cut with a random affine $(n-k)$-plane to find witness points :

$$\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i \in \mathbb{C}^n$$

The vectors $\mathbf{b}$ and $\mathbf{v}_i$ are choosen at random.

$$\text{Sample} \quad f\left(\mathbf{x}(\lambda, t) = \mathbf{b}(t) + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i(t) \right) = \mathbf{0}$$

Points on the moving $(n-k)$-plane are determined by $n-k$ independent variables λ_i, $i = 1, 2, \ldots, n-k$.

Intersecting Hypersurfaces Extrinsicially

$$\begin{cases} f_1(\mathbf{x}) = 0 & \mathbf{x} \in \mathbb{C}^n \\ L_1(\mathbf{x}) = \mathbf{0} & {\scriptstyle n-1 \text{ hyperplanes}} \end{cases} \qquad \begin{cases} f_2(\mathbf{y}) = 0 & \mathbf{y} \in \mathbb{C}^n \\ L_2(\mathbf{y}) = \mathbf{0} & {\scriptstyle n-1 \text{ hyperplanes}} \end{cases}$$

diagonal homotopy *extrinsic version*

$$\left(\begin{cases} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{y}) = 0 \\ L_1(\mathbf{x}) = \mathbf{0} \\ L_2(\mathbf{y}) = \mathbf{0} \end{cases} \right) t + \left(\begin{cases} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{y}) = 0 \\ \mathbf{x} - \mathbf{y} = \mathbf{0} \\ M(\mathbf{y}) = \mathbf{0} \end{cases} \right) (1-t) = \mathbf{0}$$

At $t = 1$ **:** $\deg(f_1) \times \deg(f_2)$ solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n \times n}$.

At $t = 0$ **:** witness points $(\mathbf{x} = \mathbf{y} \in \mathbb{C}^n)$ on $f_1^{-1}(0) \cap f_2^{-1}(0)$ cut out by $n-2$ hyperplanes M.

Intersecting Hypersurfaces Intrinsically

Consider a general affine line $\mathbf{x}(\lambda) = \mathbf{b} + \lambda\mathbf{v} \in \mathbb{C}^n$.

$$f_1(\mathbf{x}(\lambda) = \mathbf{b} + \lambda\mathbf{v}) \quad \bigcap \quad f_2(\mathbf{y}(\mu) = \mathbf{b} + \mu\mathbf{v})$$

deg(f_1) values for λ deg(f_2) values for μ

diagonal homotopy

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \left(\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

intrinsic version

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} + \lambda \left(\begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} t + \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_1 \end{bmatrix} (1-t) \right) + \mu \left(\begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix} t + \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{u}_2 \end{bmatrix} (1-t) \right)$$

At $t = 1$ **:** $\deg(f_1) \times \deg(f_2)$ solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n \times n}$.

At $t = 0$ **:** witness points on $\mathbf{x} = \mathbf{b} + \lambda\mathbf{u}_1 + \mu\mathbf{u}_2$, a general 2-plane defined by a random point $\mathbf{b}$ and 2 random vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

Intersecting with Hypersurfaces

Let $f(\mathbf{x}) = \mathbf{0}$ have k-dimensional solution components described by witness points on a general $(n-k)$-dimensional affine plane, i.e.:

$$f\left(\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i\right) = \mathbf{0}.$$

Let $g(\mathbf{x}) = 0$ be a hypersurface with witness points on a general affine line, i.e.:

$$g(\mathbf{x}(\mu) = \mathbf{b} + \mu\mathbf{w}) = 0.$$

Assuming $g(\mathbf{x}) = 0$ properly cuts one degree of freedom from $f^{-1}(\mathbf{0})$, we want to find witness points on all $(k-1)$-dimensional components of $f^{-1}(\mathbf{0}) \cap g^{-1}(0)$.

Computing Nonsingular Solutions Incrementally

Suppose $(f_1, f_2, \ldots, f_k)$ defines the system $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \mathbb{C}^n$, whose solution set is pure dimensional of multiplicity one for all $k = 1, 2, \ldots, N \leq n$, i.e.: we find only nonsingular roots if we slice the solution set of $f(\mathbf{x}) = \mathbf{0}$ with a generic linear space of dimension $n - k$.

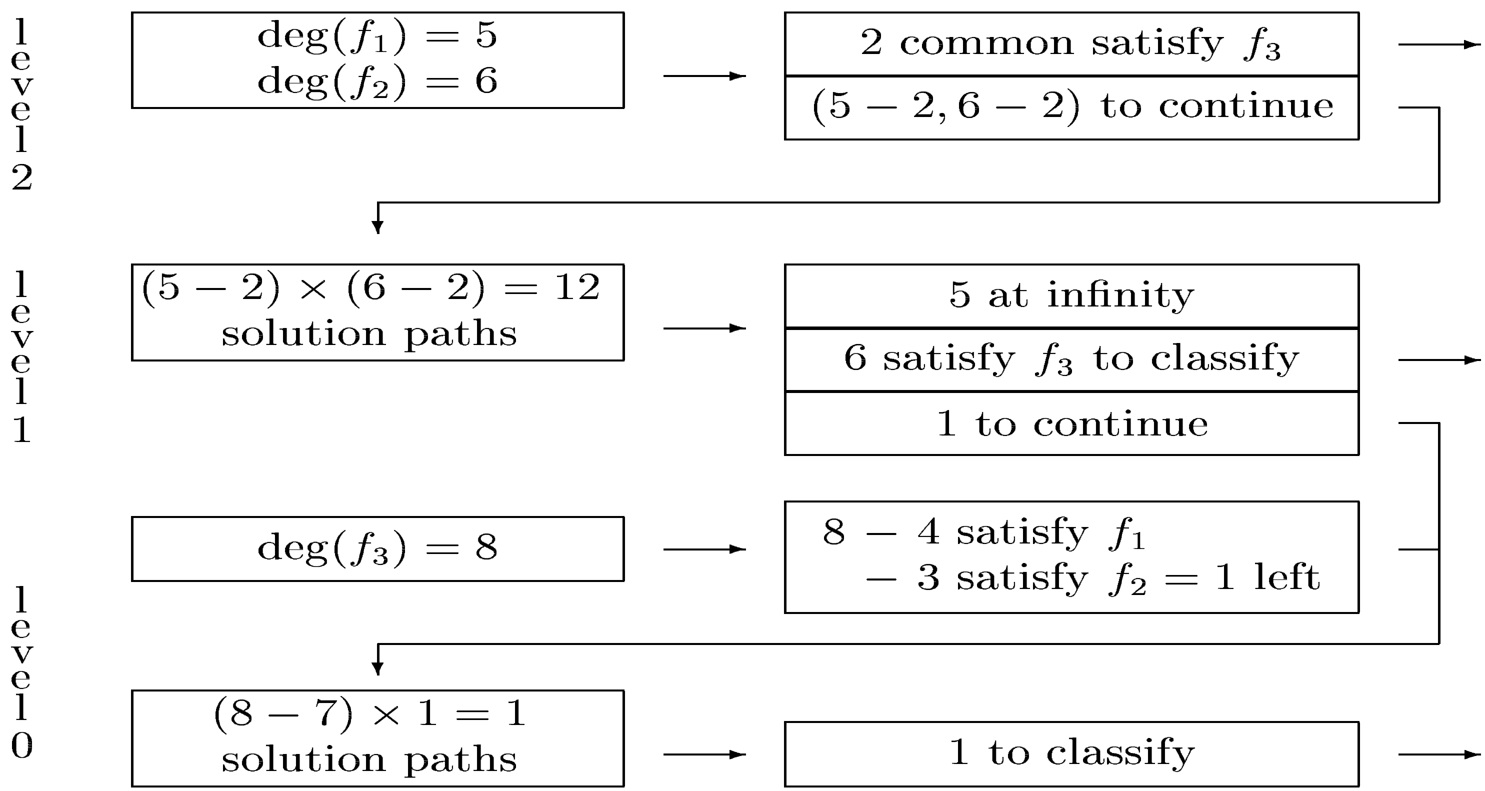
Main loop in the solver :

for $k = 2, 3, \ldots, N-1$ do
  use a diagonal homotopy to intersect

$$(f_1, f_2, \ldots, f_k)^{-1}(\mathbf{0}) \text{ with } f_{k+1}(\mathbf{x}) = 0,$$

  to find witness points on all $(n-k-1)$-dimensional solution components.

New WitnessGenerate for the Illustrative Example



Software Tools in PHCpack

In computing a numerical irreducible decomposition of a given polynomial system, we typically run through the following steps:

1.	**Embed** (phc -c)	add #random hyperplanes = top dimension, add slack variables to make the system square
2.	**Solve** (phc -b)	solve the system constructed above
3.	**WitnessGenerate** (phc -c)	apply a sequence of homotopies to compute witness point sets on all solution components
4.	**WitnessClassify** (phc -f)	filter junk from witness point sets; factor components into irreducible components

Especially step 2 is a computational bottleneck...

Numerical Elimination Methods

- Elimination = Projection
  1. slice component with hyperplanes
  2. drop coordinates from samples
  3. interpolate at projected samples

- An example: the twisted cubic $\begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \end{cases}$
  1. general slice $ax + by + cz + d = 0$, random $a, b, c, d \in \mathbb{C}$, twisted cubic projects to a cubic in the plane.
  2. slice restricted to $\mathbb{C}[x, y]$, set $c = 0$, find $y - x^2 = 0$
  3. slice restricted to $\mathbb{C}[x, z]$, set $b = 0$, find $z - x^3 = 0$

Application: Spatial Six Positions

Planar Body Guidance (Burmester 1874)

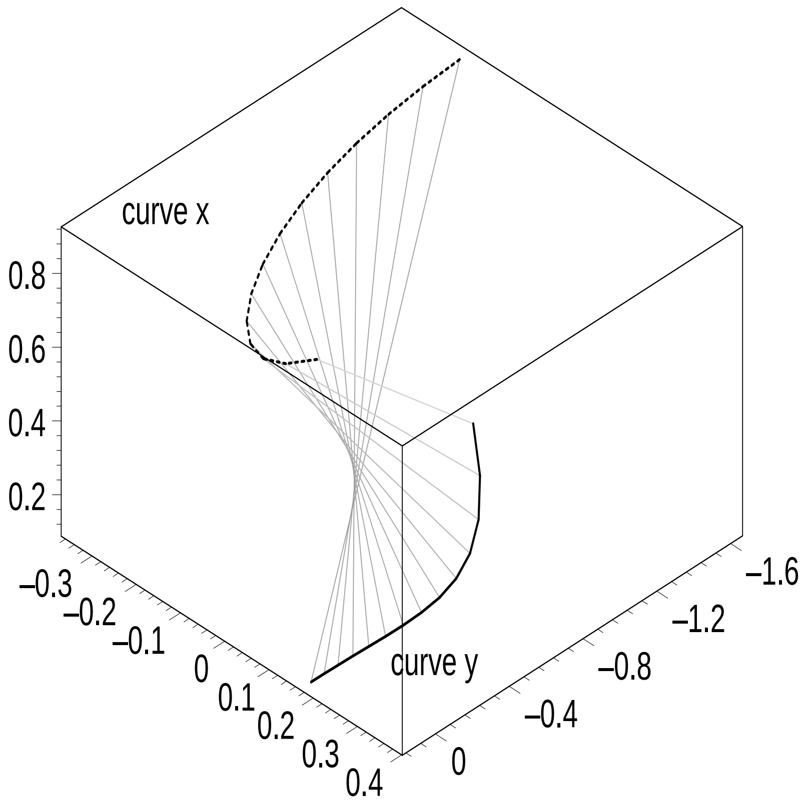
- 5 positions determine 6 circle-point/center-point pairs
- 4 positions give cubic circle-point & center-point curves

Spatial Body Guidance (Schoenflies 1886)

- 7 positions determine 20 sphere-point/center-point pairs
- 6 positions give 10^{th}-degree sphere-point & center-point curves

Question: *Can we confirm this result using continuation?*

Spatial Six Positions: Solution



Sphere-point/center-point curves are irreducible, degree 10.

An illustration of Numerical Elimination.

Witness Points

for the Spatial Burmester Problem

- The input polynomial system consists of five quadrics in six unknowns $(\mathbf{x}, \mathbf{y})$.
- The new incremental solver computes 20 witness points in 7s 181ms on Pentium III 1Ghz Windows 2000 PC.
- Projection onto $\mathbf{x}$ or $\mathbf{y}$ reduces the degree from 20 to 10.

Exercises

- Consider the adjacent minors of a general 2×4-matrix:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \qquad f(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{cases}$$

  Verify that $\dim(f^{-1}(\mathbf{0})) = 5$ and $\deg(f^{-1}(\mathbf{0})) = 8$.

- Consider $f(x,y) = \begin{cases} y - x^2 = 0 \\ z - x^3 = 0 \end{cases}$ (the twisted cubic).

  Use `phc -c` to generate a special slice in x and y only. Solve the embedded system with `phc -b` and use `phc -f` to find $y - x^2$.