Locating the Closest Singularity in a Polynomial Homotopy (*preliminary report*)

Jan Verschelde[†]

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
`http://www.math.uic.edu/~jan`
`https://github.com/janverschelde`
`janv@uic.edu`

AMS Special Session on
Optimization, Complexity, and Real Algebraic Geometry
26-27 March 2022

[†]Supported by the National Science Foundation, grant DMS 1854513.

Outline

1. Introduction
   - problem statement
   - the theorem of Fabry
2. Monomial Homotopies
   - Taylor series of roots of a polynomial homotopy
   - extrapolation to accelerate slow convergence
   - unimodular coordinate transformations
3. Software and Precision
   - the need for higher precision
   - setting the convergence radius to one

problem statement

A polynomial homotopy is a family of polynomial systems, where the systems in the family depend on one parameter.

Problem:

If for one parameter we know a regular solution,
then what is the nearest value of the parameter for which
the solution in the polynomial homotopy is singular?

detecting nearby singularities

Applying the ratio theorem of Fabry, we can detect singular points based on the coefficients of the Taylor series.

Theorem (the ratio theorem, Fabry 1896)

If for the series $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + c_{n+1} t^{n+1} + \cdots$,

we have $\lim_{n \rightarrow \infty} c_n / c_{n+1} = z$*, then*

- *z is a singular point of the series, and*
- *it lies on the boundary of the circle of convergence of the series.*

Then the radius of this circle is less than $|z|$.

The ratio c_n / c_{n+1} is the pole of Padé approximants of degrees $[n/1]$ (n is the degree of the numerator, with linear denominator).

the ratio theorem of Fabry and Padé approximants

Consider $n = 3$, $x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$.

$$[3/1] = \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3}{1 + b_1 t}$$

$$\begin{array}{rcl} (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4)(1 + b_1 t) & = & a_0 + a_1 t + a_2 t^2 + a_3 t^3 \\ c_0 + \quad c_1 t + \quad c_2 t^2 + \quad c_3 t^3 + \quad c_4 t^4 & & \\ + b_1 c_0 t + b_1 c_1 t^2 + b_1 c_2 t^3 + b_1 c_3 t^4 & = & a_0 + a_1 t + a_2 t^2 + a_3 t^3 \end{array}$$

We solve for b_1 in the term for t^4: $c_4 + b_1 c_3 = 0 \Rightarrow b_1 = -c_4/c_3$.

The denominator of $[3/1]$ is $1 - c_4/c_3 t$. The pole of $[3/1]$ is c_3/c_4.

prior work

- N. Bliss and J. Verschelde. **The method of Gauss–Newton to compute power series solutions of polynomial homotopies.** *Linear Algebra and its Applications*, 542:569–588, 2018.
- S. Telen, M. Van Barel, and J. Verschelde. **A Robust Numerical Path Tracking Algorithm for Polynomial Homotopy Continuation.** *SIAM Journal on Scientific Computing* 42(6):A3610–A3637, 2020.
- S. Telen, M. Van Barel, and J. Verschelde. **Robust numerical tracking of one path of a polynomial homotopy on parallel shared memory computers.** In the *Proceedings of the 22nd International Workshop on Computer Algebra in Scientific Computing (CASC 2020)*, pages 563–582. Springer-Verlag, 2020.

Extensive computational experiments demonstrated that eight terms in the Taylor series in the solutions are sufficient to avoid a singularity.

influences

1. computational algebraic geometry

   G. Jeronimo, G. Matera, P. Solernó, and A. Waissbein.
   Deformation techniques for sparse systems.
   Foundations of Computational Mathematics, 9:1–50, 2009.

2. the Chebfun project

   T. A. Driscoll, N. Hale, and L. N. Trefethen, editors.
   Chebfun Guide. Pafnuty Publications, Oxford, 2014.

3. parallel computers and multiple double precision

   Graphics processing units capable of teraflop performance can offset the overhead of multiple double precision arithmetic, provided by QDlib [Y. Hida, X. S Li, and D. H. Bailey, 2001] and by CAMPARY [M. Joldes, J.-M. Muller, V. Popescu, and W. Tucker, 2014].

Taylor series of roots of a polynomial homotopy

Consider the homotopy

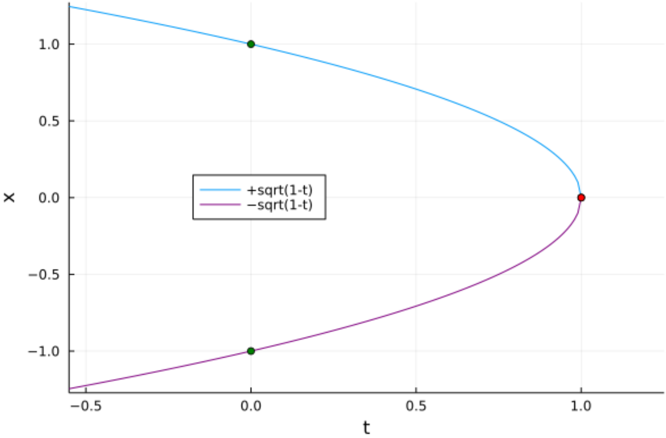
$$h(x,t) = x^2 - 1 + t = 0,$$

where x is the variable and t the parameter.

- At $t = 0$, the solutions are $x = \pm 1$.
- At $t = 1$, we have the double root $x = 0$.

In this test problem, starting at $t = 0$,
we compute 1 as the nearest singularity.

paths defined by $h(x, t) = x^2 - 1 + t = 0$



slow convergence

The homotopy $h(x,t) = x^2 - 1 + t = 0$ is equivalent to $x^2 = 1 - t$.

Develop the solution $x(t) = \sqrt{1-t}$ in a Taylor series about $t = 0$.

The ratio of the coefficients $\displaystyle\frac{c_n}{c_{n+1}}$ is $\displaystyle f(n) = \frac{2(n+1)}{2n-1}$.

Problem:

```
f(  2) = 2.00000000000000
f(  4) = 1.42857142857143
f(  8) = 1.20000000000000
f( 16) = 1.09677419354839
f( 32) = 1.04761904761905
f( 64) = 1.02362204724409
f(128) = 1.01176470588235
f(256) = 1.00587084148728
f(512) = 1.00293255131965
```

Very slow convergence!

the regularity of the errors

$\texttt{E} = |f(n) - 1|$ and $\texttt{R}$ is the ratio between two consecutive errors.

```
f(  2) = 2.00000000000000  E = 1.00e+00
f(  4) = 1.42857142857143  E = 4.29e-01   R = 2.33e+00
f(  8) = 1.20000000000000  E = 2.00e-01   R = 2.14e+00
f( 16) = 1.09677419354839  E = 9.68e-02   R = 2.07e+00
f( 32) = 1.04761904761905  E = 4.76e-02   R = 2.03e+00
f( 64) = 1.02362204724409  E = 2.36e-02   R = 2.02e+00
f(128) = 1.01176470588235  E = 1.18e-02   R = 2.01e+00
f(256) = 1.00587084148728  E = 5.87e-03   R = 2.00e+00
f(512) = 1.00293255131965  E = 2.93e-03   R = 2.00e+00
```

The error is proportional to $\frac{1}{n}$ and is halved each time we double n.

To gain one bit of accuracy the number of coefficients must be doubled.

applying extrapolation

Input: $f(2)$, $f(4)$, $f(8)$, …, $f(2^N)$.

Output: $R_{i,j}$, the triangular table of extrapolated values.

1. The first column: $R_{i,1} = f(2^i)$, for $i = 1, 2, 3, \ldots, N$.
2. The next columns in the table are computed via

$$R_{i,j} = \frac{2^i R_{i,j-1} - R_{1,j-1}}{2^i - 1},$$

for $j = i, i+1, \ldots, N$ and for $i = 2, 3, \ldots, N$.

the results of the extrapolation

```
f( 2) = 2.00000000000000
f( 4) = 1.42857142857143    R( 4, 4) = 0.85714285714286
f( 8) = 1.20000000000000    R( 8, 8) = 1.00952380952381
f(16) = 1.09677419354839    R(16,16) = 0.99969278033794
f(32) = 1.04761904761905    R(32,32) = 1.00000487650257
f(64) = 1.02362204724409    R(64,64) = 0.99999996160234
```

The errors in the extrapolation $|R_{i,j} - 1|$:

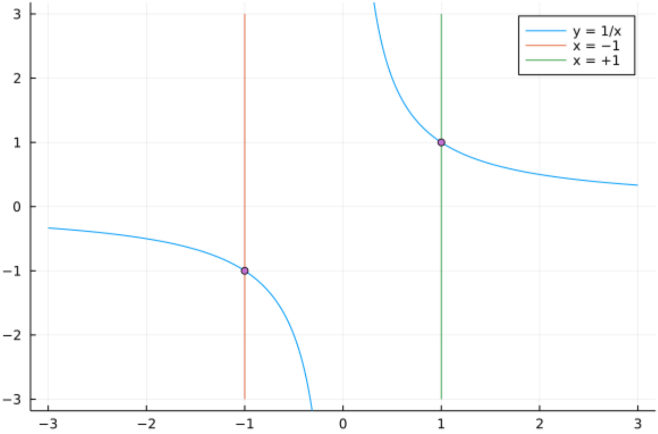
```
 2 : 1.0e+00
 4 : 4.3e-01 1.4e-01
 8 : 2.0e-01 6.7e-02 9.5e-03
16 : 9.7e-02 3.2e-02 4.6e-03 3.1e-04
32 : 4.8e-02 1.6e-02 2.3e-03 1.5e-04 4.9e-06
64 : 2.4e-02 7.9e-03 1.1e-03 7.5e-05 2.4e-06 3.8e-08
```

The extrapolated value is `0.99999996160234` with error `3.8e-08`.

a hyperbola twice cut

$$\begin{cases} x^2 = 1 - t \\ xy = 1 - t \end{cases}$$

At $t = 0$:



triangular structure and substitution

$$\left\{ \begin{array}{rcl} x_1^2 & = & 1 - t \\ x_1 x_2 & = & 1 - t \end{array} \right.$$

The Taylor series for $x_1(t)$

$$-1 + a_1 t + a_2 t^2 + \cdots \quad \text{and} \quad +1 + b_1 t + b_2 t^2 + \cdots$$

are invertible because $x(0) \neq 0$.

Substitute the series into the next equation:

$$x_2(t) = \frac{1 - t}{x_1(t)}$$

to reduce the many variables case to the one variable case.

unimodular coordinate transformations

Reduce $x_1^a x_2^b$ to y_1^d by the greatest common divisor:

$$d = \gcd(a, b) = ka + \ell b.$$

Then a, b, k, ℓ define the unimodular coordinate transformation U:

$$U \left[\begin{array}{c} a \\ b \end{array} \right] = \left[\begin{array}{cc} k & \ell \\ -\frac{b}{d} & \frac{a}{d} \end{array} \right] \left[\begin{array}{c} a \\ b \end{array} \right] = \left[\begin{array}{c} d \\ 0 \end{array} \right].$$

The matrix-vector multiplication is a coordinate transformation:

$$x_1^a x_2^b = \left(y_1^k y_2^{-b/d} \right)^a \left(y_1^\ell y_2^{a/d} \right)^b = y_1^d.$$

software

PHCpack is software for Polynomial Homotopy Continuation, to solve systems of polynomial equations.

Support for a priori step size control via `phc -u`, added to version 2.4.72, released 1 September 2019.

Available under the GPL-3.0 license at
`https://github.com/janverschelde/PHCpack`

a random 4-dimensional monomial homotopy

Consider

$$\mathbf{x}^A = 1 - t, \quad A = \begin{bmatrix} 7 & 7 & 0 & 0 \\ 7 & 3 & 5 & 7 \\ 7 & 2 & 1 & 2 \\ 7 & 0 & 1 & 2 \end{bmatrix}, \quad \det(A) = -42.$$

The monomial homotopy is

$$h(\mathbf{x}, t) = \begin{cases} x_1^7 x_2^7 x_3^7 x_4^7 & = & 1 - t \\ x_1^7 x_2^3 x_3^2 & = & 1 - t \\ x_2^5 x_3^4 & = & 1 - t \\ x_2^7 x_3^2 x_4^2 & = & 1 - t. \end{cases}$$

At $t = 0$, (1,1,1,1) is one of the 42 solutions.

in double precision, extrapolating on $x_1(t)$

```
f( 2) = 1.96874999999999
f( 4) = 1.41891891892034    R( 4, 4) =  0.86908783784070
f( 8) = 1.19620253165939    R( 8, 8) =  1.00828557991769
f(16) = 1.09509202184358    R(16,16) =  0.99974583285467
f(32) = 1.04682884602505    R(32,32) =  1.00000726557726
f(64) = 1.02320337390705    R(64,64) =  0.99987907919821
```

The errors in the extrapolation $|R_{i,j} - 1|$:

```
 2 : 9.7e-01
 4 : 4.2e-01 1.3e-01
 8 : 2.0e-01 6.1e-02 8.3e-03
16 : 9.5e-02 3.0e-02 4.0e-03 2.5e-04
32 : 4.7e-02 1.5e-02 2.0e-03 1.2e-04 7.3e-06
64 : 2.3e-02 7.3e-03 9.4e-04 1.1e-04 5.7e-05 1.2e-04
```

The coefficients of the original series are not accurate enough.

on coefficients computed in double double precision

```
f( 2) = 1.96875000000000
f( 4) = 1.41891891891892    R( 4, 4) =  0.86908783783784
f( 8) = 1.19620253164557    R( 8, 8) =  1.00828557988368
f(16) = 1.09509202453988    R(16,16) =  0.99974584110786
f(32) = 1.04682779456193    R(32,32) =  1.00000383925819
f(64) = 1.02323838080960    R(64,64) =  0.99999997121995
```

The errors in the extrapolation $|R_{i,j} - 1|$:

```
 2 : 9.7e-01
 4 : 4.2e-01 1.3e-01
 8 : 2.0e-01 6.1e-02 8.3e-03
16 : 9.5e-02 3.0e-02 4.0e-03 2.5e-04
32 : 4.7e-02 1.5e-02 2.0e-03 1.3e-04 3.8e-06
64 : 2.3e-02 7.3e-03 9.8e-04 6.2e-05 1.9e-06 2.9e-08
```

Extrapolated on coefficients computed with 32 decimal places.

setting the convergence radius to one

Consider c_n the coefficient of t^n in the Taylor series.

What happens if n grows:

$$\left| \frac{c_n}{c_{n+1}} \right| \rightarrow \begin{cases} < 1 & : \text{ coefficients increase,} \\ = 1 & : \text{ coefficients are constant,} \\ > 1 & : \text{ coefficients decrease.} \end{cases}$$

Let $x(t)$ satisfy $h(x(t), t) = 0$, then

$$x \left(\left| \frac{c_n}{c_{n+1}} \right| t \right)$$

has convergence radius one.

Therefore, work with the homotopy $h(x, s) = 0$, where $s = \left| \frac{c_n}{c_{n+1}} \right| t$.

conclusions

Let c_n be the n-th coefficient of a Taylor series, according to Fabry:

$$\text{as} \quad n \rightarrow \infty, \quad \frac{c_n}{c_{n+1}} \rightarrow \text{ nearest singularity.}$$

Preliminary experimental results:

- The convergence of the series tends to be very slow.
- Extrapolation on series with 66 coefficients can already give 8 decimal places of accuracy.
- Monomial homotopies are good test cases.
- Multiple double precision may be needed.

Once the radius $R = |c_n/c_{n+1}|$ is accurately computed, work with $s = R\,t$ in the polynomial homotopy.