The Numerical Solution of Polynomial Systems Arising in Engineering and Science

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Why bother?

practical and theoretical motivations:

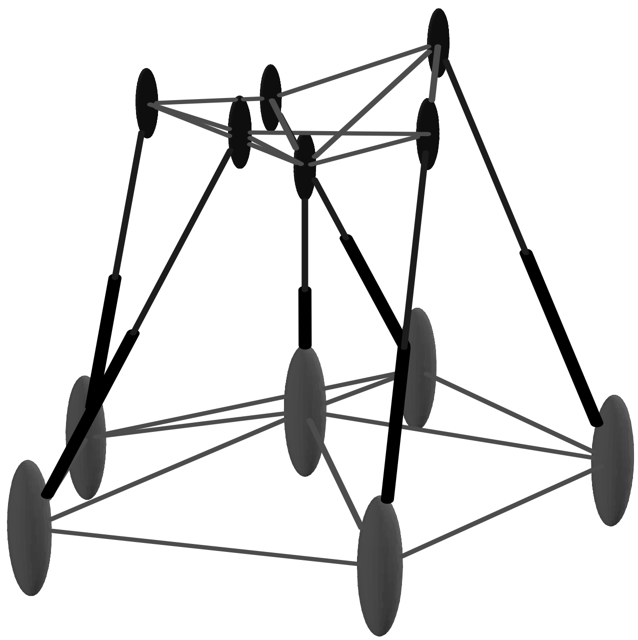
1. polynomial systems arise in practical problems
   $\rightarrow$ geometric constraint solving, real algebraic geometry, computation of Nash equilibria, signal processing, mechanical design, vision, global optimization, and computational physics

2. polynomial systems are studied in pure mathematics
   $\rightarrow$ algebraic geometry offers us tools

Solving polynomial systems involves

- engineering, in particular: mechanical design
- algebraic geometry: we are interested in constructive results about solutions of polynomial systems
- computational mathematics: numerical analysis, symbolic computation, discrete & computational geometry
- computer science: development of mathematical software and parallel computation to solve large problems

Assembly of Stewart-Gough Platforms



end plate, the platform

is connected by legs to

a stationary base

Forward Displacement Problem:

Given: position of base and leg lengths.

Wanted: position of end plate.

The Equations for the Platform Problem

workspace $\mathbb{R}^3 \times \mathrm{SO}(3)$: position and orientation

$$\mathrm{SO}(3) = \{ \ A \in \mathbb{C}^{3\times 3} \mid A^H A = I, \ \det(A) = 1 \ \}$$

more efficient to use Study (or soma) coordinates:

$[e : g] = [e_0 : e_1 : e_2 : e_3 : g_0 : g_1 : g_2 : g_3] \in \mathbb{P}^7$ quaternions on the Study quadric: $f_0(e, g) = e_0 g_0 + e_1 g_2 + e_2 g_2 + e_3 g_3 = 0$, excluding those e for which $ee' = 0$, $e' = (e_0, -e_1, -e_2, -e_3)$

given leg lengths L_i, find $[e : g]$ leads to

$$f_i(e, g) = gg' + (bb_i' + a_i a_i' - L_i^2) ee' + (g b_i' e' + e b_i g') - (g e' a_i' + a_i e g') - (e b_i e' a_i' + a_i e b_i' e') = 0, \quad i = 1, 2, \ldots 6$$

$\Rightarrow$ solve $f = (f_0, f_1, \ldots, f_6)$, 7 quadrics in $[e : g] \in \mathbb{P}^7$

expecting $2^7 = 128$ solutions...

Literature on Stewart-Gough platforms

M. Raghavan: **The Stewart platform of general geometry has 40 configurations.** *ASME J. Mech. Design* 115:277–282, 1993.

B. Mourrain: **The 40 generic positions of a parallel robot.** In *Proceedings of the International Symposium on Symbolic and Algebraic Computation*, ed. by M. Bronstein, pages 173–182, ACM 1993.

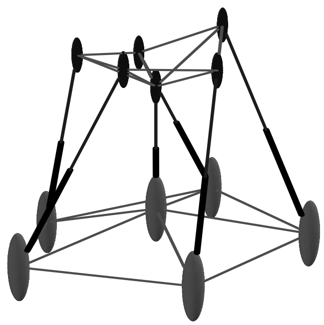
F. Ronga and T. Vust: **Stewart platforms without computer?** In *Real Analytic and Algebraic Geometry, Proceedings of the International Conference, (Trento, 1992)*, pages 196–212, Walter de Gruyter 1995.

M.L. Husty: **An algorithm for solving the direct kinematics of general Stewart-Gough Platforms.** *Mech. Mach. Theory,* 31(4):365–380, 1996.

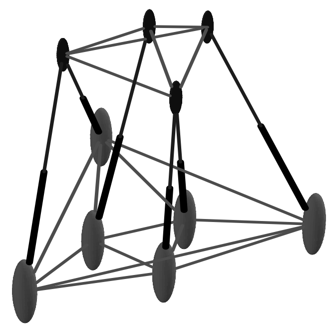
C.W. Wampler: **Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates.** *Mech. Mach. Theory* 31(3): 331–337, 1996.

P. Dietmaier: **The Stewart-Gough platform of general geometry can have 40 real postures.** In *Advances in Robot Kinematics: Analysis and Control*, ed. by J. Lenarcic and M.L. Husty, pages 1–10. Kluwer 1998.

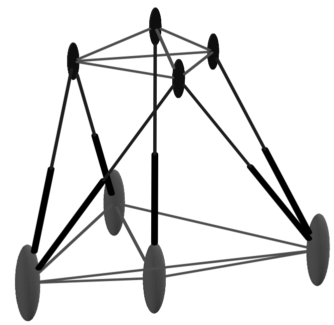
A family of Stewart-Gough platforms



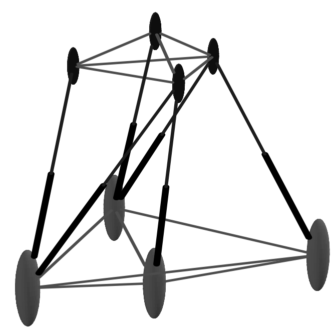
6-6, 40 solutions



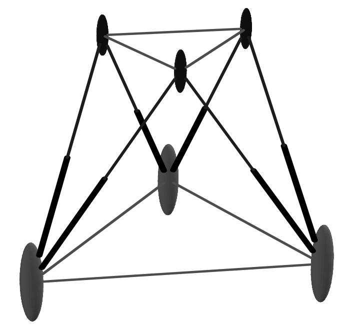
4-6, 32 solutions



4-4a, 16 solutions



4-4b, 24 solutions



3-3, 16 solutions

thanks to Charles Wampler

Numerical Homotopy Continuation Methods

If we wish to solve $f(\mathbf{x}) = \mathbf{0}$, then we construct a system $g(\mathbf{x}) = \mathbf{0}$ whose solutions are known. Consider the *homotopy*

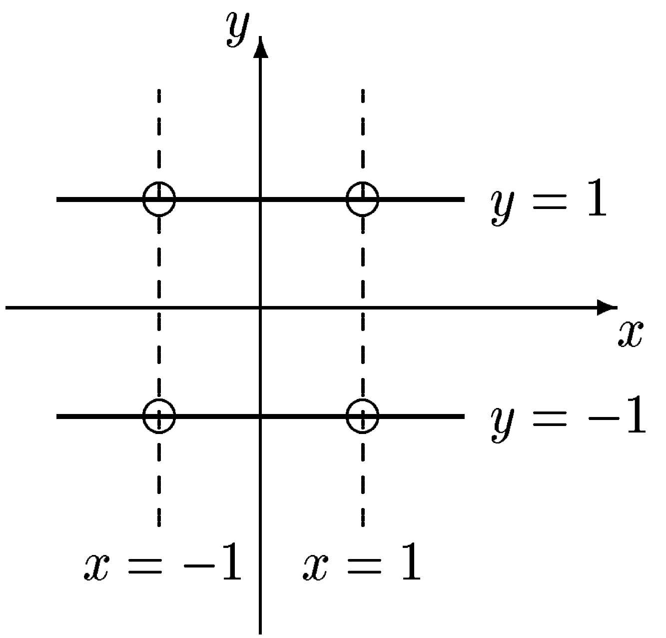
$$H(\mathbf{x}, t) := (1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}.$$

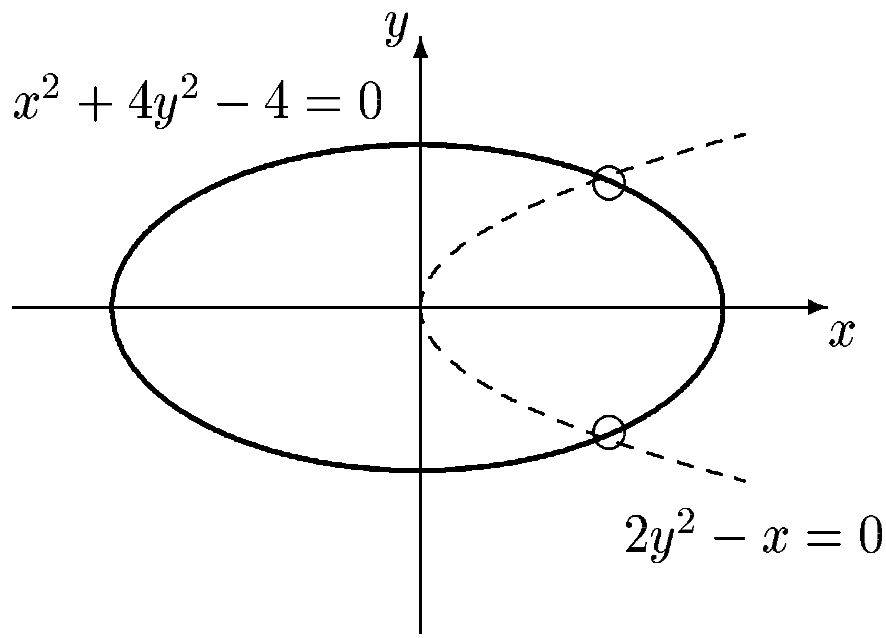
By *continuation*, we trace the paths starting at the known solutions of $g(\mathbf{x}) = \mathbf{0}$ to the desired solutions of $f(\mathbf{x}) = \mathbf{0}$, for t from 0 to 1.

homotopy continuation methods are *symbolic-numeric*:
  homotopy methods treat polynomials as algebraic objects,
  continuation methods use polynomials as functions.

geometric interpretation: move from general to special,
  solve special, and move solutions from special to general.

Product Deformations





$$\gamma \left(\left\{ \begin{array}{r} x^2-1=0 \\ y^2-1=0 \end{array} \right. \right)(1-t) \quad + \quad \left(\left\{ \begin{array}{r} x^2+4y^2-4=0 \\ 2y^2-x=0 \end{array} \right. \right) t, \quad \gamma \in \mathbb{C}$$

The theorem of Bézout

$f = (f_1, f_2, \ldots, f_n)$

$d_i = \deg(f_i)$

total degree D :

$$D = \prod_{i=1}^{n} d_i$$

$$g(\mathbf{x}) = \begin{cases} \alpha_1 x_1^{d_1} - \beta_1 = 0 \\ \alpha_2 x_2^{d_2} - \beta_2 = 0 \\ \vdots \\ \alpha_n x_n^{d_n} - \beta_n = 0 \end{cases}$$

start system $\alpha_i, \beta_i \in \mathbb{C}$ random

> Theorem: $f(\mathbf{x}) = \mathbf{0}$ *has at most D isolated solutions in $\mathbb{C}^n$, counted with multiplicities.*

Sketch of Proof: $V = \{\ (f, \mathbf{x}) \in \mathbb{P}(\mathcal{H}_D) \times \mathbb{P}(\mathbb{C}^n) \mid f(\mathbf{x}) = \mathbf{0}\ \}$

$\Sigma' = \{(f, \mathbf{x}) \in V \mid \det(D_{\mathbf{x}} f(\mathbf{x})) = 0\}$, $\Sigma = \pi_1(\Sigma'), \pi_1 : V \to \mathbb{P}(\mathcal{H}_D)$

Elimination theory: Σ is variety $\Rightarrow \mathbb{P}(\mathcal{H}_D) - \Sigma$ is connected.

Thus $h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$ avoids $\Sigma, \forall t \in [0, 1)$.

Implicitly defined curves

Consider a homotopy $h_k(x(t), y(t), t) = 0$, $k = 1, 2$.

By $\frac{\partial}{\partial t}$ on homotopy: $\frac{\partial h_k}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h_k}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial h_k}{\partial t}\frac{\partial t}{\partial t} = 0$, $k = 1, 2$.

Set $\Delta x := \frac{\partial x}{\partial t}$, $\Delta y := \frac{\partial y}{\partial t}$, and $\frac{\partial t}{\partial t} = 1$.

$$
\begin{array}{ll}
\text{Increment} & t := t + \Delta t \\
\text{Solve} & \left[\begin{array}{cc} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{array}\right] \left[\begin{array}{c} \Delta x \\ \Delta y \end{array}\right] = - \left[\begin{array}{c} \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial t} \end{array}\right] (Newton) \\
\text{Update} & \left\{\begin{array}{l} x := x + \Delta x \\ y := y + \Delta y \end{array}\right.
\end{array}
$$

Predictor-Corrector Methods

loop

1. predict $\begin{cases} t_{k+1} := t_k + \Delta t \\ \mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + \Delta\mathbf{x} \end{cases}$

2. correct with Newton

3. if convergence

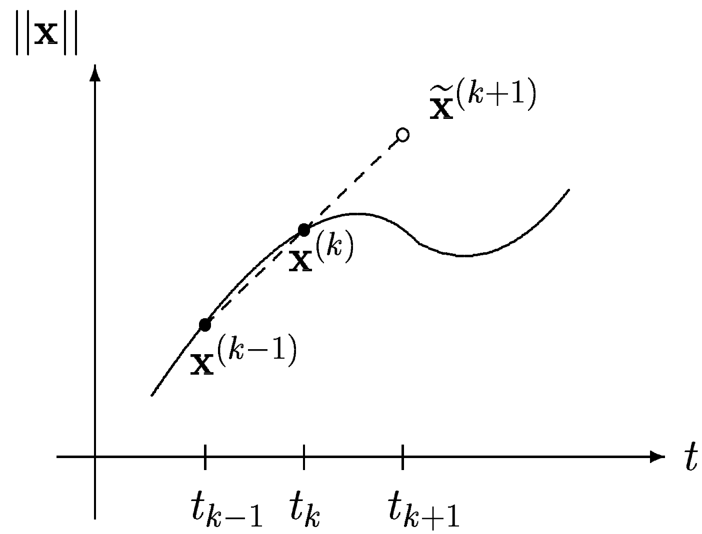
   then enlarge Δt

   continue with $k+1$

   else reduce Δt

   back up and restart at k

until $t = 1$.



$$\widetilde{\mathbf{x}}^{(k+1)} := \mathbf{x}^{(k)} + \lambda(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)})$$

Complexity Issues

The Problem: a hierarchy of complexity classes

$$P \quad : \quad \text{evaluation of a system at a point}$$

$$NP \quad : \quad \text{find one root of a system}$$

$$\#P \quad : \quad \text{find \textbf{all} roots of a system (\textit{intractable!})}$$

Complexity of Homotopies: for bounds on #Newton steps in a linear homotopy, see

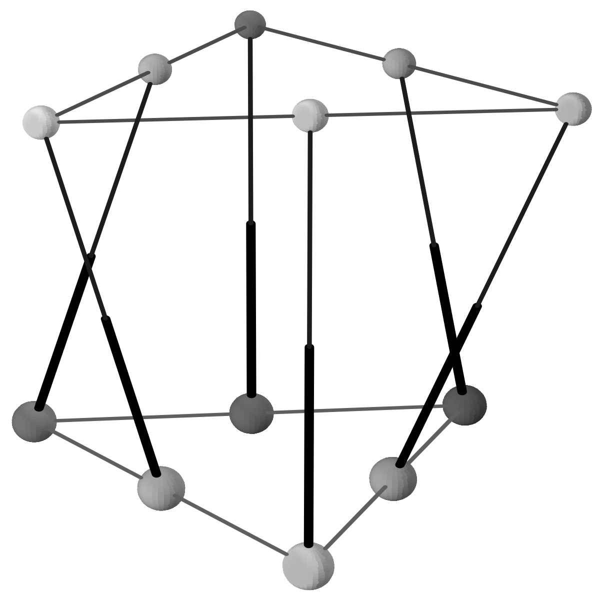
L. Blum, F. Cucker, M. Shub, and S. Smale: **Complexity and Real Computation**. Springer 1998.

M. Shub and S. Smale: **Complexity of Bezout's theorem V: Polynomial Time**. *Theoretical Computer Science* 133(1):141–164, 1994.

On average, we can find an approximate zero in polynomial time.

Architecturally Singular Platforms Move

M. Griffis and J. Duffy: **Method and apparatus for controlling geometrically simple parallel mechanisms with distinctive connections.** US Patent 5,179,525, 1993.



end plate, the platform

is connected by legs to

a stationary base

- Base and endplate are equilateral triangles.
- Legs connect vertices to midpoints.

Solution sets to polynomial systems

Polynomial in One Variable	**System of Polynomials**
one equation, one variable	n equations, N variables
solutions are points	points, lines, surfaces, ...
multiple roots	sets with multiplicity
Factorization: $\prod_i (x - a_i)^{\mu_i}$	**Irreducible Decomposition**

Numerical Representation	
set of points	set of witness sets

Witness Sets

A witness point is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component

  equals the $\boxed{\textbf{dimension}}$ of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes

  equals the $\boxed{\textbf{degree}}$ of the solution component.

A witness set for a k-dimensional solution component consists of k random hyperplanes and a set of isolated solutions of the system cut with those hyperplanes.

Membership Test

Does the point $\mathbf{z}$ *belong to a component?*

Given: a point in space $\mathbf{z} \in \mathbb{C}^N$; a system $f(\mathbf{x}) = \mathbf{0}$;

and a witness set W, $W = (Z, L)$:

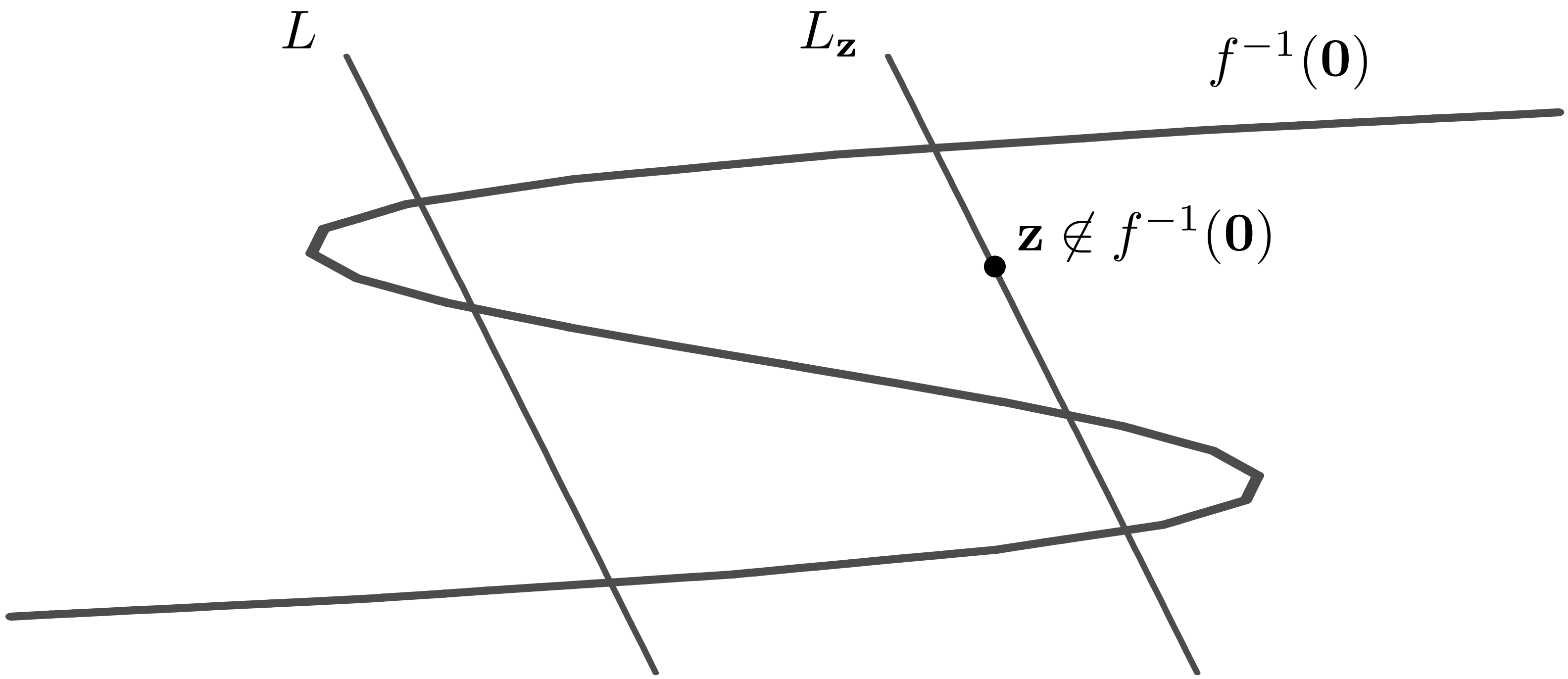
for all $\mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0}$ and $L(\mathbf{w}) = \mathbf{0}$.

1. Let $L_{\mathbf{z}}$ be a set of hyperplanes through $\mathbf{z}$, and define

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{z}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

2. Trace all paths starting at $\mathbf{w} \in Z$, for t from 0 to 1.
3. The test $(\mathbf{z}, 1) \in h^{-1}(\mathbf{0})$? answers the question above.

Membership Test – an example



$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{z}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

Numerical Algebraic Geometry Dictionary

Algebraic Geometry	example in 3-space	Numerical Analysis
variety	collection of points, algebraic curves, and algebraic surfaces	polynomial system + union of witness sets, see below for the definition of a witness set
irreducible variety	a single point, or a single curve, or a single surface	polynomial system + witness set + probability-one membership test
generic point on an irreducible variety	random point on an algebraic curve or surface	point in witness set; a witness point is a solution of polynomial system on the variety and on a random slice whose codimension is the dimension of the variety
pure dimensional variety	one or more points, or one or more curves, or one or more surfaces	polynomial system + set of witness sets of same dimension + probability-one membership tests
irreducible decomposition of a variety	several pieces of different dimensions	polynomial system + array of sets of witness sets and probability-one membership tests

Factoring Solution Components

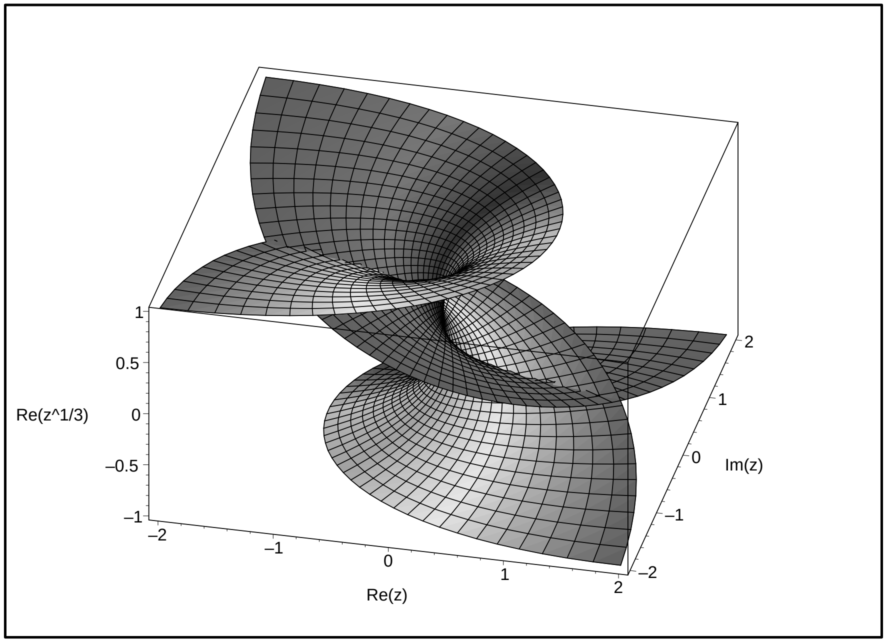
Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system with a positive dimensional solution component, represented by witness set.

coefficients of f known approximately, work with limited precision

Wanted: decompose the component into irreducible factors, for each factor, give its degree and multiplicity.

Symbolic-Numeric issue: essential numerical information (such as degree and multiplicity of each factor), is obtained much faster than the full symbolic representation.

The Riemann Surface of $z^3 - w = 0$:



R.M. Corless and D.J. Jeffrey: **Graphing elementary Riemann surfaces.** *SIGSAM Bulletin* 32(1):11–17, 1998.

Monodromy to Decompose Solution Components

Given: a system $f(\mathbf{x}) = \mathbf{0}$; and $W = (Z, L)$:

$$\text{for all } \mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0} \text{ and } L(\mathbf{w}) = \mathbf{0}.$$

Wanted: partition of Z so that all points in a subset of Z lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

$$\text{Consider } H(x, y, \theta) = \begin{cases} xy - 1 = 0 \\ x + y = 4e^{i\theta} \end{cases} \quad \text{for } \theta \in [0, 2\pi].$$

For $\theta = 0$, we start with two real solutions. When $\theta > 0$, the solutions turn complex, real again at $\theta = \pi$, then complex until at $\theta = 2\pi$. Back at $\theta = 2\pi$, we have again two real solutions, but their order is permuted $\Rightarrow$ irreducible.

Connecting Witness Points

1. For two sets of hyperplanes K and L, and a random $\gamma \in \mathbb{C}$

$$H(\mathbf{x}, t, K, L, \gamma) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ \gamma K(\mathbf{x})(1-t) + L(\mathbf{x})t = \mathbf{0} \end{cases}$$

   We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, K, L, \alpha) = \mathbf{0}$.

   For $\beta \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, L, K, \beta) = \mathbf{0}$.

   Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other hyperplanes.

Linear Traces – an example

$$\text{Consider}\quad \begin{aligned} f(x,y(x)) &= (y-y_1(x))(y-y_2(x))(y-y_3(x)) \\ &= y^3 - t_1(x)y^2 + t_2(x)y - t_3(x) \end{aligned}$$

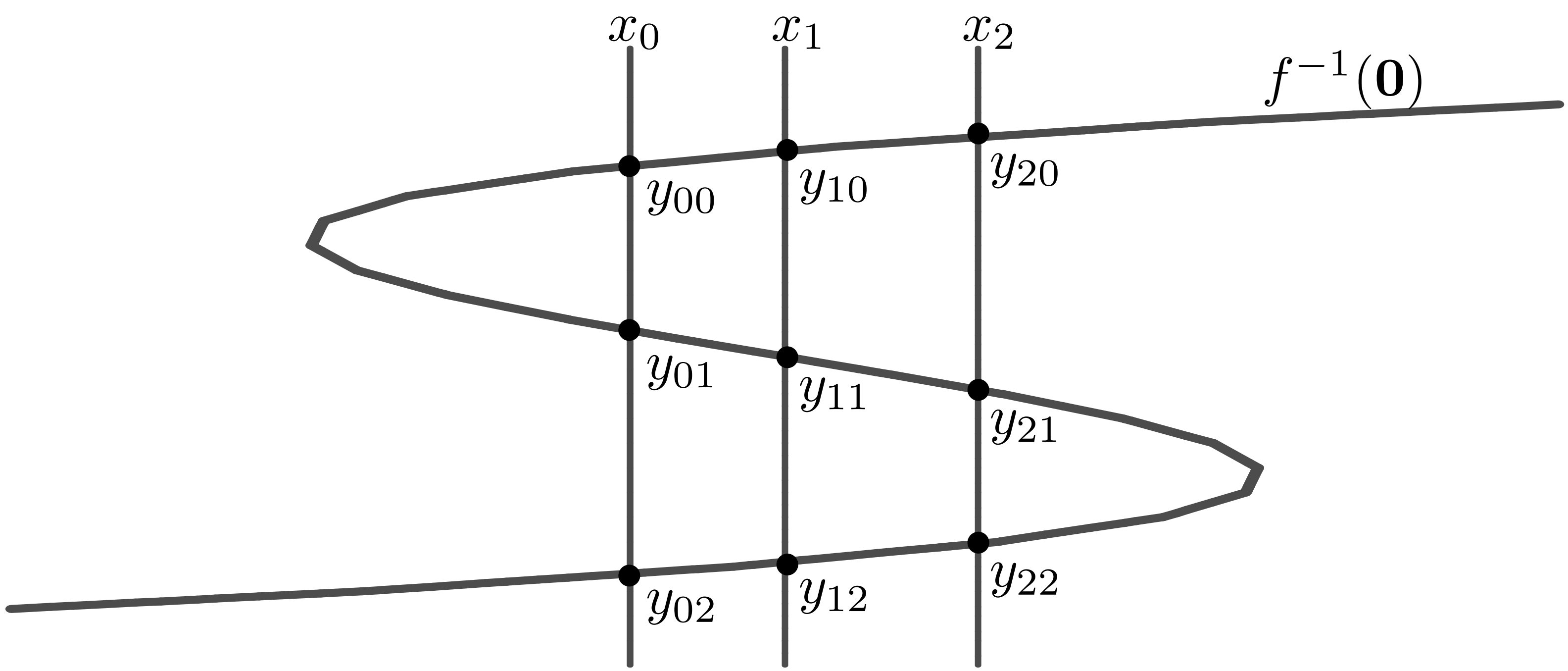
We are interested in the linear trace: $t_1(x) = c_1 x + c_0$.

Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve}\quad \left\{ \begin{array}{l} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{array} \right. \quad \text{to find}\quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.

Linear Traces – example continued



Use $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$ to find the linear trace $t_1(x) = c_0 + c_1 x$.

At $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$: $c_0 + c_1 x_2 = y_{20} + y_{21} + y_{22}$?

Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- For a k-dimensional solution component, it suffices to consider a curve on the component cut out by $k-1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.

Notice: Instead of monodromy, we may enumerate all possible factors and use linear traces to certify. While the complexity of this enumeration is exponential, it works well for low degrees.

Results of Husty and Karger

Self-motions of Griffis-Duffy type parallel manipulators. In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

The special Griffis-Duffy platforms *move*:

- Case 1: Plates not equal, legs not equal.
  - Curve is degree 20 in Euler parameters.
  - Curve is degree 40 in position.
- Case 2: Plates congruent, legs all equal.
  - Factors are degrees $(4+4)+6+2 = 16$ in Euler parameters.
  - Factors are degrees $(8+8)+12+4 = 32$ in position.

Question: *Can we confirm these results numerically?*

Components of Griffis-Duffy Platforms

Solution components by degree

Husty & Karger		SVW	
Euler	Position	Study	Position
General Case			
20	40	28	40
Legs equal, Plates equal			
		6	8
4	8	6	8
4	8	6	8
6	12	6	12
2	4	4	4
16	32	28	40

Griffis-Duffy Platforms: Factorization

Case A: One irreducible component of degree 28 (general case).

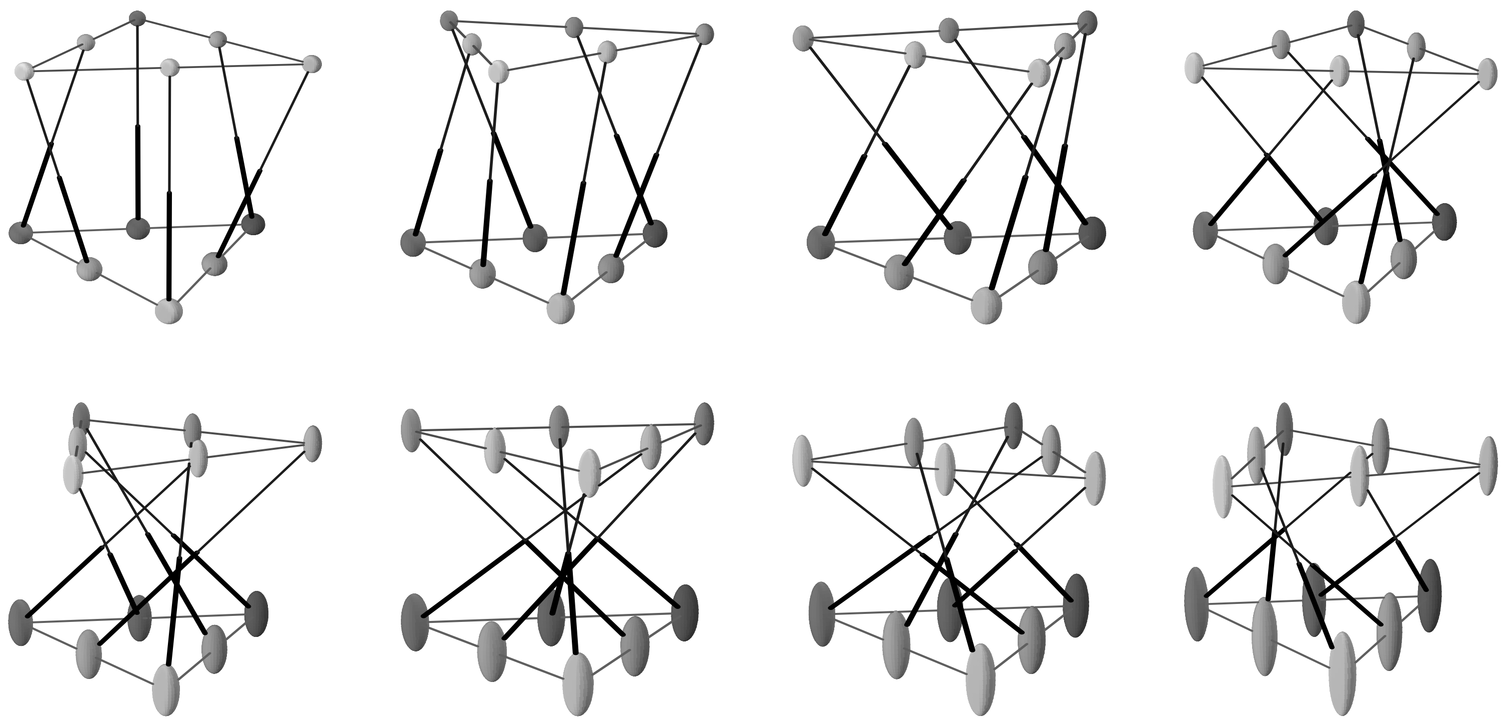
Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

user cpu on 800Mhz	Case A	Case B
witness points	1m 12s 480ms	
monodromy breakup	33s 430ms	27s 630ms
Newton interpolation	1h 19m 13s 110ms	2m 34s 50ms
32 decimal places used to interpolate polynomial of degree 28		
linear trace	4s 750ms	4s 320ms

Linear traces replace Newton interpolation:

$\Rightarrow$ **time to factor independent of geometry!**

Griffis-Duffy Platforms: an Animation



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  Call #68796, 1 hour credit

- MCS 563 Analytic Symbolic Computation

  will likely run in 2004-2005