Numerical Homotopies for Decomposing Solution Sets of Polynomial Systems

Jan Verschelde

Department of Math, Stat & CS

University of Illinois at Chicago

Chicago, IL 60607-7045, USA

`jan@math.uic.edu`   `www.math.uic.edu/~jan`

Joint with Andrew J. Sommese (U of Notre Dame)

`sommese@nd.edu`   `www.nd.edu/~sommese`

and Charles W. Wampler (General Motors R & D)

`charles.w.wampler@gm.com`

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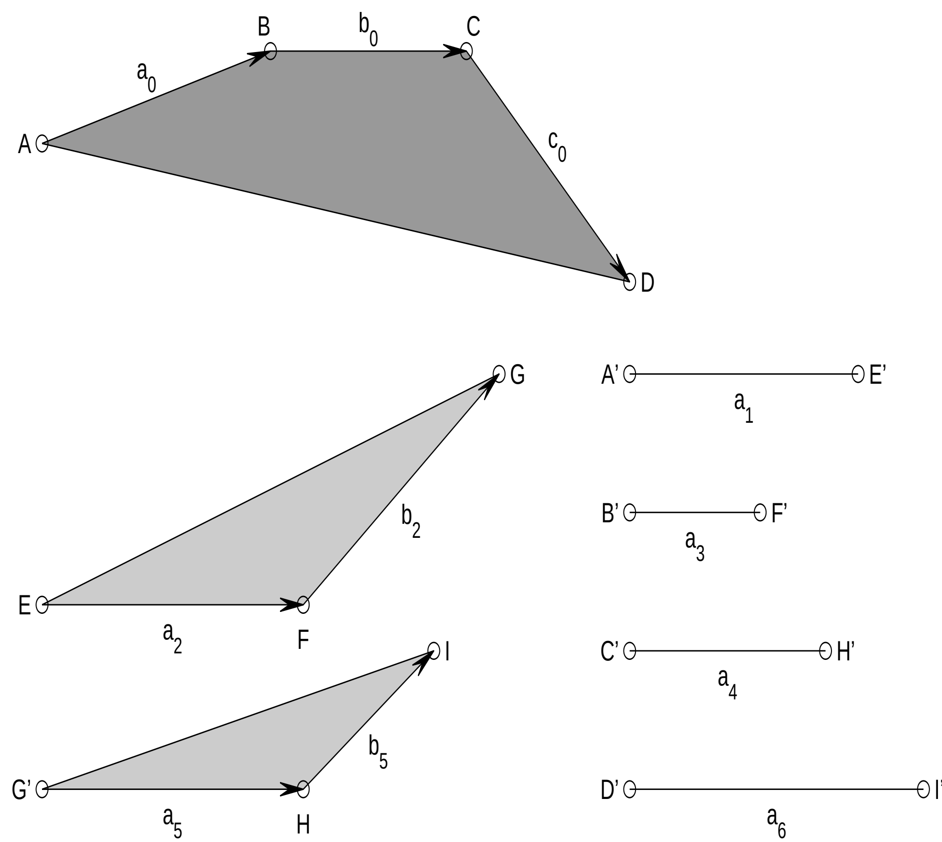
Outline of Talk

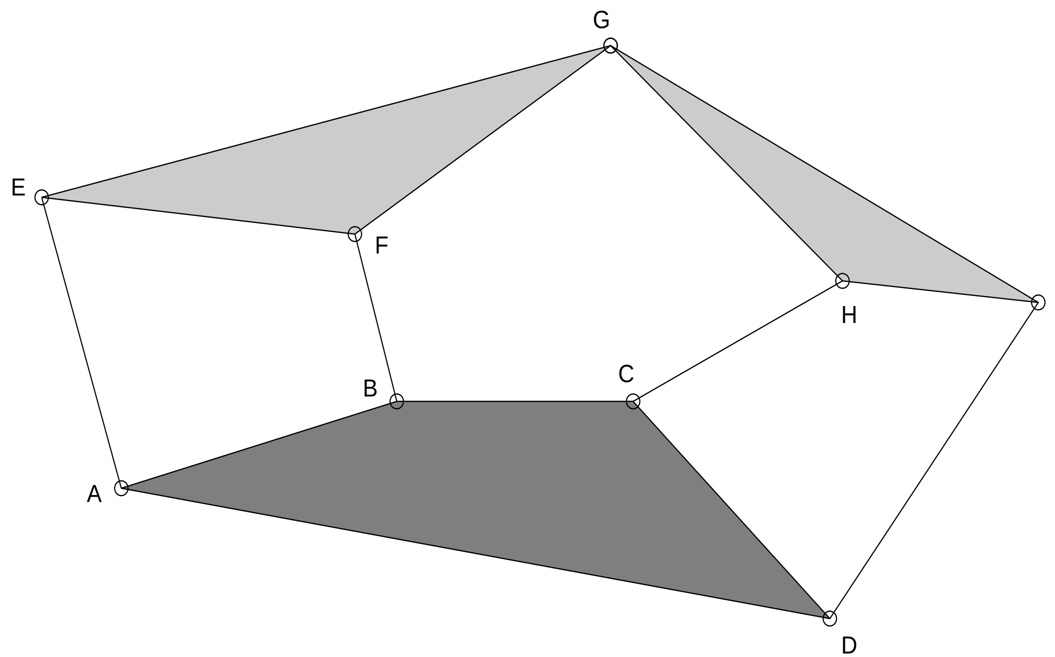
1. Some Motivating Examples
2. Numerical Algebraic Geometry
   - homotopy continuation methods
   - numerical irreducible decomposition
3. Incrementally Solving Polynomial Systems
   - diagonal homotopies to intersect components
   - intrinsic and extrinsic representations
4. Results on the Examples

Example 1. A Seven-Bar Structure



Problem: Find all possible assemblies of these pieces.

One possible assembly



- Generally, 18 solutions. (This example, 8 real, 10 complex.)
- Intersection of two four-bar coupler curves.

Question:

What if the four-bars have the same coupler curve (Roberts cognates)?

- Structure has mobility $= 0$.
- The common four-bar coupler curve (degree 6) is a solution.
- *Is the four-bar curve the only solution?*
- This is an overconstrained mechanism.
  - *How do we treat it numerically?*

Example 2. Spatial Six-Positions

Planar Body Guidance (Burmester 1874)

- 5 positions determine 6 circle-point/center-point pairs
- 4 positions give cubic circle-point & center-point curves

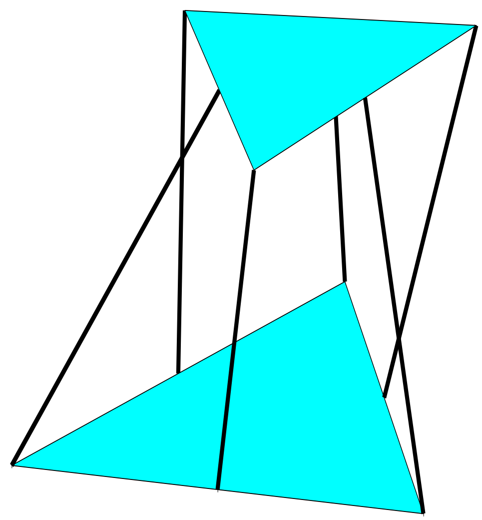
Spatial Body Guidance (Shoenflies 1886)

- 7 positions determine 20 sphere-point/center-point pairs
- 6 positions give 10^{th}-degree sphere-point & center-point curves

Question: *Can we confirm this result using continuation?*

Example 3. Stewart-Gough Platforms

Special Griffis-Duffy type



- Base and endplate are equilateral triangles.
- Legs connect vertices to midpoints.

Results of Husty and Karger

Self-motions of Griffis-Duffy type parallel manipulators. In *Proc. 2000 IEEE Int. Conf. Robotics and Automation* (CDROM), 2000.

The special Griffis-Duffy platforms *move*:

- Case 1: Plates not equal, legs not equal.
  - Curve is degree 20 in Euler parameters.
  - Curve is degree 40 in position.
- Case 2: Plates congruent, legs all equal.
  - Factors are degrees $(4+4)+6+2=16$ in Euler parameters.
  - Factors are degrees $(8+8)+12+4=32$ in position.

Question: *Can we confirm these results numerically?*

2. Numerical Homotopy Continuation Methods

If we wish to solve $f(\mathbf{x}) = \mathbf{0}$, then we construct a system $g(\mathbf{x}) = \mathbf{0}$ whose solutions are known. Consider the *homotopy*

$$H(\mathbf{x}, t) := (1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}.$$

By *continuation*, we trace the paths starting at the known solutions of $g(\mathbf{x}) = \mathbf{0}$ to the desired solutions of $f(\mathbf{x}) = \mathbf{0}$, for t from 0 to 1.

homotopy continuation methods are *symbolic-numeric*:

homotopy methods treat polynomials as algebraic objects,
continuation methods use polynomials as functions.

Solution sets to polynomial systems

Polynomial in One Variable	System of Polynomials
one equation, one variable	n equations, N variables
solutions are points	points, lines, surfaces, ...
double roots	sets with multiplicity
Factorization: $\prod_i (x - a_i)^{\mu_i}$	**Irreducible Decomposition**

Numerical Representation	
set of points	set of witness point sets

An Illustrative Example

$$f(x,y,z) = \begin{cases} (y-x^2)(x^2+y^2+z^2-1)(x-0.5) = 0 \\ (z-x^3)(x^2+y^2+z^2-1)(y-0.5) = 0 \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-0.5) = 0 \end{cases}$$

Irreducible decomposition of $Z = f^{-1}(\mathbf{0})$ is

$Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$

with

1. Z_{21} is the sphere $x^2+y^2+z^2-1=0$,
2. Z_{11} is the line $(x=0.5, z=0.5^3)$,
3. Z_{12} is the line $(x=\sqrt{0.5}, y=0.5)$,
4. Z_{13} is the line $(x=-\sqrt{0.5}, y=0.5)$,
5. Z_{14} is the twisted cubic $(y-x^2=0, z-x^3=0)$,
6. Z_{01} is the point $(x=0.5, y=0.5, z=0.5)$.

Witness Point Sets

A witness point is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component

  equals the $\boxed{\textbf{dimension}}$ of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes

  equals the $\boxed{\textbf{degree}}$ of the solution component.

A witness point set for a k-dimensional solution component consists of k random hyperplanes and a set of isolated solutions of the system cut with those hyperplanes.

Membership Test

Does the point $\mathbf{z}$ *belong to a component?*

Given: a point in space $\mathbf{z} \in \mathbb{C}^N$; a system $f(\mathbf{x}) = \mathbf{0}$;

and a witness point set W, $W = (Z, L)$:

for all $\mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0}$ and $L(\mathbf{w}) = \mathbf{0}$.

1. Let $L_{\mathbf{z}}$ be a set of hyperplanes through $\mathbf{z}$, and define

$$H(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ L_{\mathbf{z}}(\mathbf{x})t + L(\mathbf{x})(1-t) = \mathbf{0} \end{cases}$$

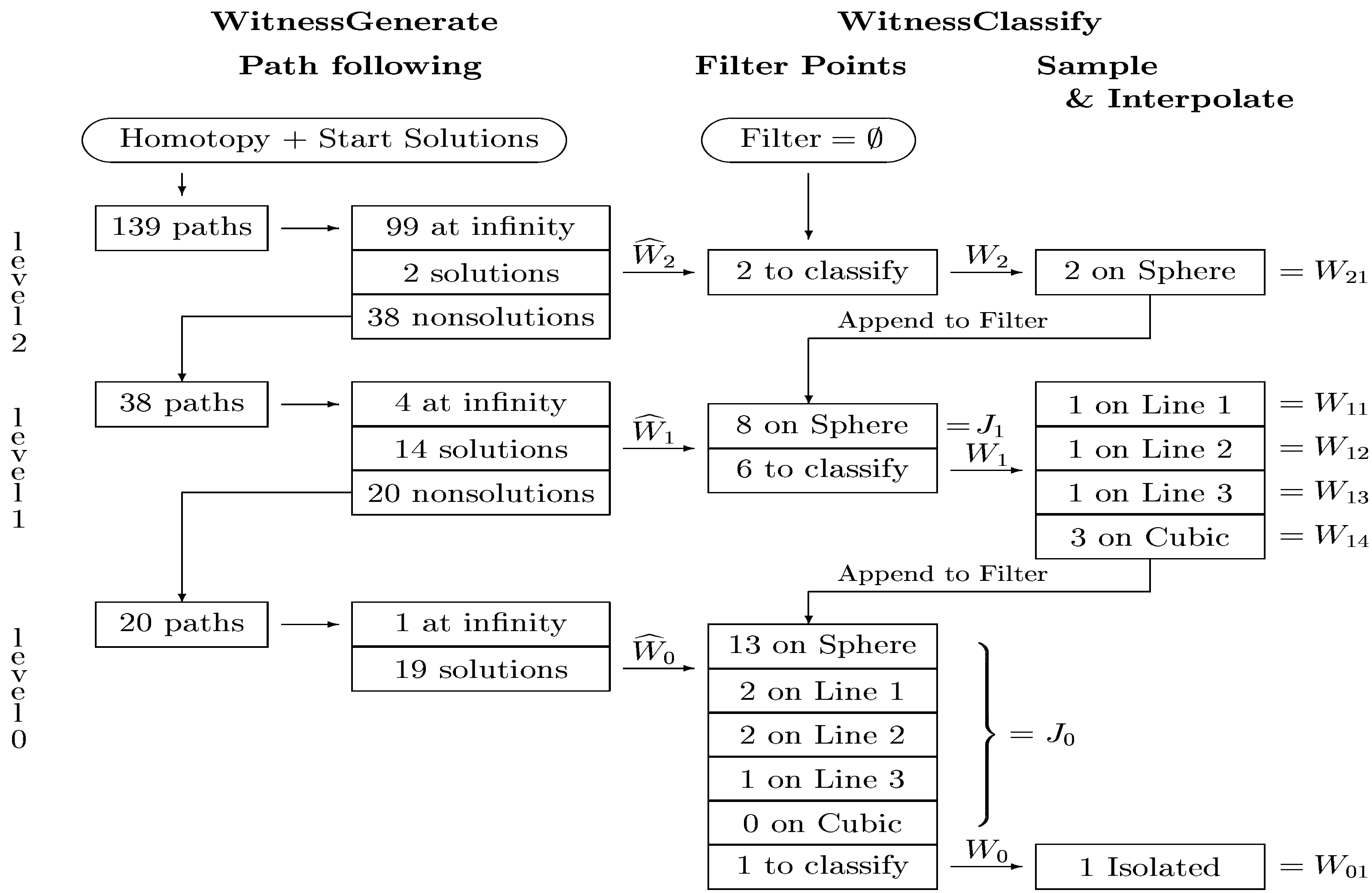
2. Trace all paths starting at $\mathbf{w} \in Z$, for t from 0 to 1.

3. The test $(\mathbf{z}, 1) \in H^{-1}(\mathbf{0})$? answers the question above.

Numerical Algebraic Geometry Dictionary

Algebraic Geometry	example in 3-space	Numerical Analysis
variety	collection of points, algebraic curves, and algebraic surfaces	polynomial system + union of witness point sets, see below for the definition of a witness point
irreducible variety	a single point, or a single curve, or a single surface	polynomial system + witness point set + probability-one membership test
generic point on an irreducible variety	random point on an algebraic curve or surface	point in witness point set; a witness point is a solution of polynomial system on the variety and on a random slice whose codimension is the dimension of the variety
pure dimensional variety	one or more points, or one or more curves, or one or more surfaces	polynomial system + set of witness point sets of same dimension + probability-one membership tests
irreducible decomposition of a variety	several pieces of different dimensions	polynomial system + array of sets of witness point sets and probability-one membership tests

A Numerical Irreducible Decomposition of the Illustrative Example



History of Numerical Irreducible Decomposition

A.J. Sommese and C.W. Wampler: **Numerical algebraic geometry.** In *The Mathematics of Numerical Analysis*, ed. by J. Renegar, M. Shub, and S. Smale, pages 749–763, AMS, 1996.

A.J. Sommese and J. Verschelde: **Numerical homotopies to compute generic points on positive dimensional algebraic sets.** *J. of Complexity* 16(3):572–602, 2000.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Numerical decomposition of the solution sets of polynomial systems into irreducible components.** *SIAM J. Numer. Anal.* 38(6):2022–2046, 2001.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Using monodromy to decompose solution sets of polynomial systems into irreducible components.** In *Application of Algebraic Geometry to Coding Theory, Physics and Computation*, ed. by C. Ciliberto, F. Hirzebruch, R. Miranda, and M. Teicher, pages 297–315, Kluwer, 2001.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Symmetric functions applied to decomposing solution sets of polynomial systems.** *SIAM J. Numer. Anal.*, to appear.

Numerical Factorization of Multivariate Polynomials

E. Kaltofen: **Challenges of symbolic computation: my favorite open problems.** *J. Symbolic Computation* 29(6): 891–919, 2000.

Y. Huang, W. Wu, H.J. Stetter, and L. Zhi: **Pseudofactors of multivariate polynomials.** In *Proceedings of ISSAC 2000*, ed. by C. Traverso, pages 161–168, ACM 2000.

R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas and S.M. Watt: **Towards factoring bivariate approximate polynomials.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 85–92, ACM 2001.

A. Galligo and D. Rupprecht: **Semi-numerical determination of irreducible branches of a reduced space curve.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 137–142, ACM 2001.

A. Galligo and D. Rupprecht: **Irreducible decomposition of curves.** *J. Symbolic Computation* 33(5):661–677, 2002.

T. Sasaki: **Approximate multivariate polynomial factorization based on zero-sum relations.** In *Proceedings of ISSAC 2001*, ed. by B. Mourrain, pages 284–291, ACM 2001.

R.M. Corless, A. Galligo, I.S. Kotsireas, and S.M. Watt: **A geometric-numeric algorithm for absolute factorization of multivariate polynomials.** In *Proceedings of ISSAC 2002*, ed. by T. Mora, pages 37–45, ACM 2002.

Monodromy to Decompose Solution Components

Given: a system $f(\mathbf{x}) = \mathbf{0}$; and $W = (Z, L)$:

for all $\mathbf{w} \in Z : f(\mathbf{w}) = \mathbf{0}$ and $L(\mathbf{w}) = \mathbf{0}$.

Wanted: partition of Z so that all points in a subset of Z lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

Consider $H(x, y, \theta) = \begin{cases} xy - 1 = 0 \\ x + y = 4e^{i\theta} \end{cases}$ for $\theta \in [0, 2\pi]$.

For $\theta = 0$, we start with two real solutions. At $\theta = \pi$, the real solutions have turned complex. Back at $\theta = 2\pi$, we have again two real solutions, but their order is permuted $\Rightarrow$ irreducible.

Connecting Witness Points

1. For two sets of hyperplanes K and L, and a random $\gamma \in \mathbb{C}$

$$H(\mathbf{x}, t, K, L, \gamma) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ \gamma K(\mathbf{x})(1-t) + L(\mathbf{x})t = \mathbf{0} \end{cases}$$

   We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, K, L, \alpha) = \mathbf{0}$.

   For $\beta \in \mathbb{C}$, trace the paths defined by $H(\mathbf{x}, t, L, K, \beta) = \mathbf{0}$.

   Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other values of α and β.

Linear Traces

$$\text{Consider} \quad f(x,y(x)) \;=\; (y-y_1(x))(y-y_2(x))(y-y_3(x))$$
$$= \; y^3 - t_1(x)y^2 + t_2(x)y - t_3(x)$$

We are interested in the linear trace: $t_1(x) = c_1 x + c_0$.

Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \left\{ \begin{array}{l} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{array} \right. \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x. For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$. Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.

Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- For a k-dimensional solution component, it suffices to consider a curve on the component cut out by $k-1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.

Numerical Irreducible Decomposition

In computing a numerical irreducible decomposition of a given polynomial system, we typically run through the following steps:

1.	**Embed** (phc -c)	add #random hyperplanes = top dimension,
		add slack variables to make the system square
2.	**Solve** (phc -b)	solve the system constructed above
3.	**WitnessGenerate**	apply a sequence of homotopies to compute
	(phc -c)	witness point sets on all solution components
4.	**WitnessClassify**	filter junk from witness point sets
	(phc -f)	factor components into irreducible components

Especially step 2 is a computational bottleneck.
We recently discovered and implemented a new algorithm.

3. Solving Systems Incrementally

- Extrinsic and Intrinsic Deformations

  extrinsic : defined by explicit equations

  intrinsic : following the actual geometry

- Diagonal Homotopies

  $\rightarrow$ to intersect pure dimensional solution sets

- Intersecting with Hypersurfaces

  adding the polynomial equations one after the other we arrive at an incremental polynomial system solver.

Extrinsic Homotopy Deformations

$f(\mathbf{x}) = \mathbf{0}$ has k-dimensional solution components. We cut with k hyperplanes to find isolated solutions = *witness points sets* :

$$a_{i0} + \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, 2, \ldots, k, \quad a_{ij} \in \mathbb{C} \text{ random}$$

$$\text{Sample} \quad \left\{ \begin{array}{cc} f(\mathbf{x}) + \gamma \mathbf{z} = 0 & \mathbf{z} = \textit{slack} \\ a_{i0}(t) + \displaystyle\sum_{j=1}^{n} a_{ij}(t) x_j = 0 & \textit{moving} \end{array} \right.$$

$$\#\text{witness points} \ = \sum_{\substack{C \subseteq f^{-1}(0) \\ \dim(C) = k}} \deg(C)$$

Embedding with Slack Variables

The cyclic 4-roots system defines 2 quadrics in $\mathbb{C}^4$:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{lllllcl} x_1 & + x_2 & + x_3 & + x_4 & + & \gamma_1 z & = & 0 \\ x_1 x_2 & + x_2 x_3 & + x_3 x_4 & + x_4 x_1 & + & \gamma_2 z & = & 0 \\ x_1 x_2 x_3 & + x_2 x_3 x_4 & + x_3 x_4 x_1 & + x_4 x_1 x_2 & + & \gamma_3 z & = & 0 \\ x_1 x_2 x_3 x_4 & - \quad 1 & & & + & \gamma_4 z & = & 0 \end{array} \right. \\ \qquad a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + z = 0 \end{array} \right.$$

Original system : 4 equations in x_1, x_2, x_3, and x_4.

Cut with random hyperplane to find isolated points.

Slack variable z with random γ_i, $i = 1,2,3,4$: square system.

Solve embedded system to find $4 = 2{+}2$ witness points as isolated solutions with $z = 0$.

Intrinsic Homotopy Deformations

$f(\mathbf{x}) = \mathbf{0}$ has k-dimensional solution components. We cut with a random affine $(n-k)$-plane to find witness points :

$$\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i \in \mathbb{C}^n$$

The vectors $\mathbf{b}$ and $\mathbf{v}_i$ are choosen at random.

$$\text{Sample} \quad f\left(\mathbf{x}(\lambda, t) = \mathbf{b}(t) + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i(t)\right) = \mathbf{0}$$

Points on the moving $(n-k)$-plane are determined by $n-k$ independent variables λ_i, $i = 1, 2, \ldots, n-k$.

#independent variables = co-dimension

$f(\mathbf{x}) = \mathbf{0}$ is a system with $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}$ lies on an affine $(n-k)$-plane:

$$\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i \in \mathbb{C}^n$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-k})$ contains all independent variables.

Correct with Newton on $f(\mathbf{x}(\lambda)) = \mathbf{0}$, a system in λ.

$$\text{Solve} \quad \left[\frac{\partial f}{\partial \lambda} \right] \lambda = -f(\mathbf{x}(\lambda)) \quad \text{with} \quad \frac{\partial f_i}{\partial \lambda_j} = \sum_{l=1}^{n-k} \frac{\partial f_i}{\partial x_l} \frac{\partial x_l}{\partial \lambda_j}.$$

Overdetermined case moved from global to local level!

no slack variables needed...

Intersecting Hypersurfaces Extrinsicially

$$\begin{cases} f_1(\mathbf{x}) = 0 & \mathbf{x} \in \mathbb{C}^n \\ L_1(\mathbf{x}) = \mathbf{0} & {\scriptstyle n-1 \text{ hyperplanes}} \end{cases} \qquad \begin{cases} f_2(\mathbf{y}) = 0 & \mathbf{y} \in \mathbb{C}^n \\ L_2(\mathbf{y}) = \mathbf{0} & {\scriptstyle n-1 \text{ hyperplanes}} \end{cases}$$

diagonal homotopy *extrinsic version*

$$\left(\begin{cases} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{y}) = 0 \\ L_1(\mathbf{x}) = \mathbf{0} \\ L_2(\mathbf{y}) = \mathbf{0} \end{cases} \right) t + \left(\begin{cases} f_1(\mathbf{x}) = 0 \\ f_2(\mathbf{y}) = 0 \\ \mathbf{x} - \mathbf{y} = \mathbf{0} \\ M(\mathbf{y}) = \mathbf{0} \end{cases} \right) (1-t) = \mathbf{0}$$

At $t = 1$ **:** $\deg(f_1) \times \deg(f_2)$ solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n \times n}$.

At $t = 0$ **:** witness points $(\mathbf{x} = \mathbf{y} \in \mathbb{C}^n)$ on $f_1^{-1}(0) \cap f_2^{-1}(0)$ cut out by $n-2$ hyperplanes M.

Intersecting Hypersurfaces Intrinsically

Consider a general affine line $\mathbf{x}(\lambda) = \mathbf{b} + \lambda \mathbf{v} \in \mathbb{C}^n$.

$$\underset{\deg(f_1) \text{ values for } \lambda}{f_1(\mathbf{x}(\lambda) = \mathbf{b} + \lambda \mathbf{v})} \quad \bigcap \quad \underset{\deg(f_2) \text{ values for } \mu}{f_2(\mathbf{y}(\mu) = \mathbf{b} + \mu \mathbf{v})}$$

diagonal homotopy

$$\left(\begin{array}{c} f_1 \\ f_2 \end{array} \right) \left(\left[\begin{array}{c} \mathbf{x}(t) \\ \mathbf{y}(t) \end{array} \right] \right) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

intrinsic version

$$\left[\begin{array}{c} \mathbf{x}(t) \\ \mathbf{y}(t) \end{array} \right] = \left[\begin{array}{c} \mathbf{b} \\ \mathbf{b} \end{array} \right] + \lambda \left(\left[\begin{array}{c} \mathbf{v} \\ \mathbf{0} \end{array} \right] t + \left[\begin{array}{c} \mathbf{u}_1 \\ \mathbf{u}_1 \end{array} \right] (1-t) \right) + \mu \left(\left[\begin{array}{c} \mathbf{0} \\ \mathbf{v} \end{array} \right] t + \left[\begin{array}{c} \mathbf{u}_2 \\ \mathbf{u}_2 \end{array} \right] (1-t) \right)$$

At $t = 1$ **:** $\deg(f_1) \times \deg(f_2)$ solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n \times n}$.

At $t = 0$ **:** witness points on $\mathbf{x} = \mathbf{b} + \lambda \mathbf{u}_1 + \mu \mathbf{u}_2$, a general 2-plane defined by a random point $\mathbf{b}$ and 2 random vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

Intersecting with Hypersurfaces

Let $f(\mathbf{x}) = \mathbf{0}$ have k-dimensional solution components described by witness points on a general $(n-k)$-dimensional affine plane, i.e.:

$$f\left(\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i\right) = \mathbf{0}.$$

Let $g(\mathbf{x}) = 0$ be a hypersurface with witness points on a general affine line, i.e.:

$$g(\mathbf{x}(\mu) = \mathbf{b} + \mu\mathbf{w}) = 0.$$

Assuming $g(\mathbf{x}) = 0$ properly cuts one degree of freedom from $f^{-1}(\mathbf{0})$, we want to find witness points on all $(k-1)$-dimensional components of $f^{-1}(\mathbf{0}) \cap g^{-1}(0)$.

Intrinsic Hypersurface Intersection

The **diagonal homotopy** for (f, g) on $(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{n \times n}$ starts at

$$\begin{bmatrix} \mathbf{x}(1) \\ \mathbf{y}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} + \sum_{i=1}^{n-k} \lambda_i \begin{bmatrix} \mathbf{v}_i \\ \mathbf{0} \end{bmatrix} + \mu \begin{bmatrix} \mathbf{0} \\ \mathbf{w} \end{bmatrix}$$

and ends at

$$\begin{bmatrix} \mathbf{x}(0) \\ \mathbf{y}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \end{bmatrix} + \sum_{i=1}^{n-k} \lambda_i \begin{bmatrix} \mathbf{v}_i \\ \mathbf{v}_i \end{bmatrix} + \mu \begin{bmatrix} \mathbf{w} \\ \mathbf{w} \end{bmatrix}.$$

The diagonal homotopy

$$\begin{pmatrix} f \\ g \end{pmatrix} \left(\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}(1) \\ \mathbf{y}(1) \end{bmatrix} t + \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{y}(0) \end{bmatrix} (1-t) \right) = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}$$

has $n - k + 1$ independent variables $(\lambda_1, \lambda_2, \ldots, \lambda_{n-k}, \mu)$.

Computing Nonsingular Solutions Incrementally

Suppose $(f_1, f_2, \ldots, f_k)$ defines the system $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \mathbb{C}^n$, whose solution set is pure dimensional of multiplicity one for all $k = 1, 2, \ldots, N \leq n$, i.e.: we find only nonsingular roots if we slice the solution set of $f(\mathbf{x}) = \mathbf{0}$ with a generic linear space of dimension $n - k$.

Main loop in the solver :

for $k = 2, 3, \ldots, N - 1$ do
  use a diagonal homotopy to intersect
    $(f_1, f_2, \ldots, f_k)^{-1}(\mathbf{0})$ with $f_{k+1}(\mathbf{x}) = 0$,
  to find witness points on all $(n - k - 1)$-dimensional solution components.

Outcomes of Hypersurface Intersections

Let V be an $(n-k)$-dimensional irreducible component of $(f_1, ..., f_k)^{-1}(\mathbf{0})$ and $g^{-1}(0)$ be an irreducible hypersurface.

Three cases for $V \cap g^{-1}(0)$:

1. $V \subseteq g^{-1}(0)$

   All witness points of V satisfy $g(\mathbf{x}) = 0$.

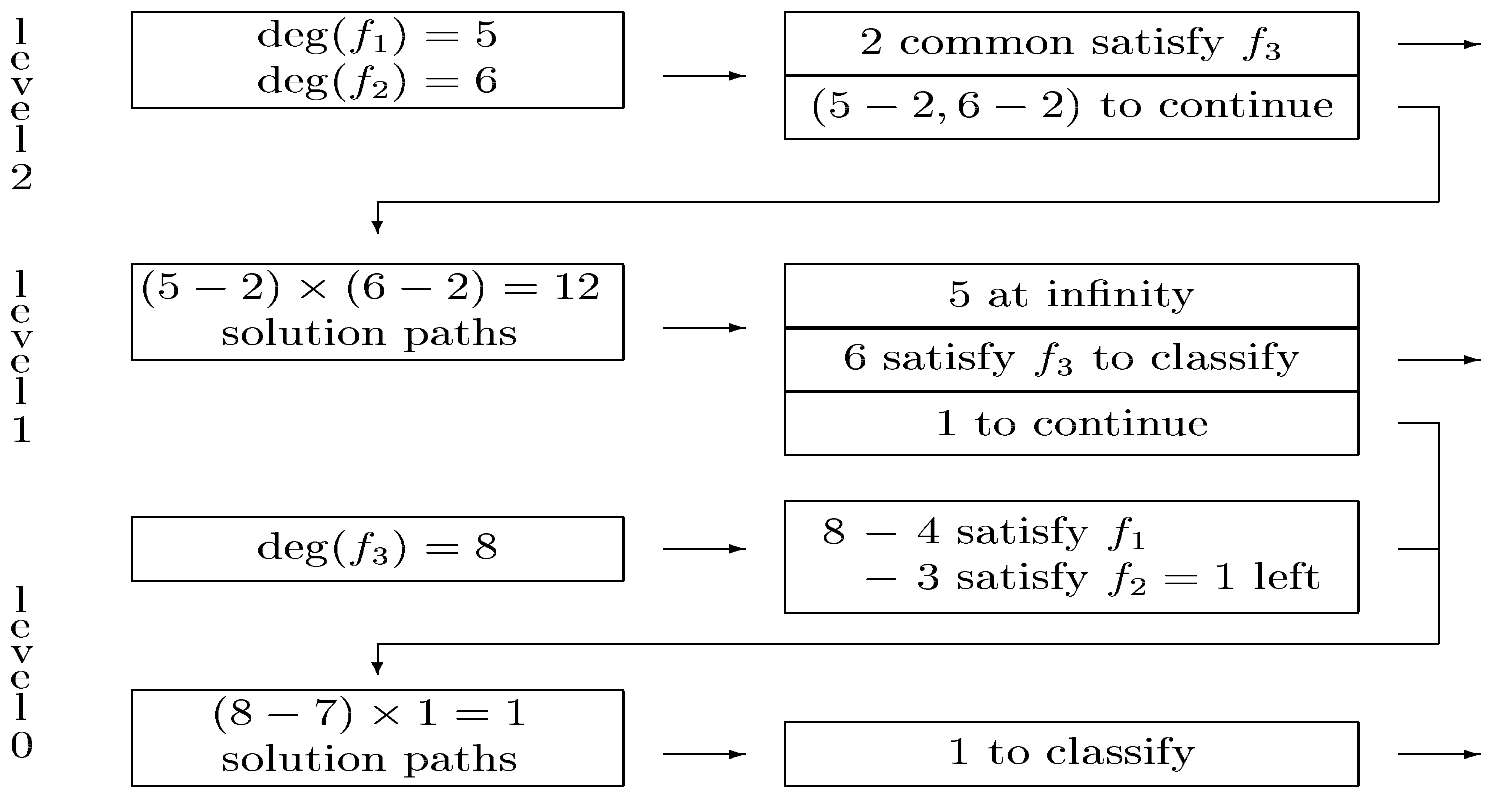
2. $\dim(V \cap g^{-1}(0)) = k - 1$

   The diagonal homotopy gives witness points on all $(k-1)$-dimensional components of the intersection.

3. $V \cap g^{-1}(0) = \emptyset$

   All paths in the diagonal homotopy diverge.

New WitnessGenerate for the Illustrative Example



4. Test Polynomial Systems

Example 1 a 7-bar mechanism in the plane

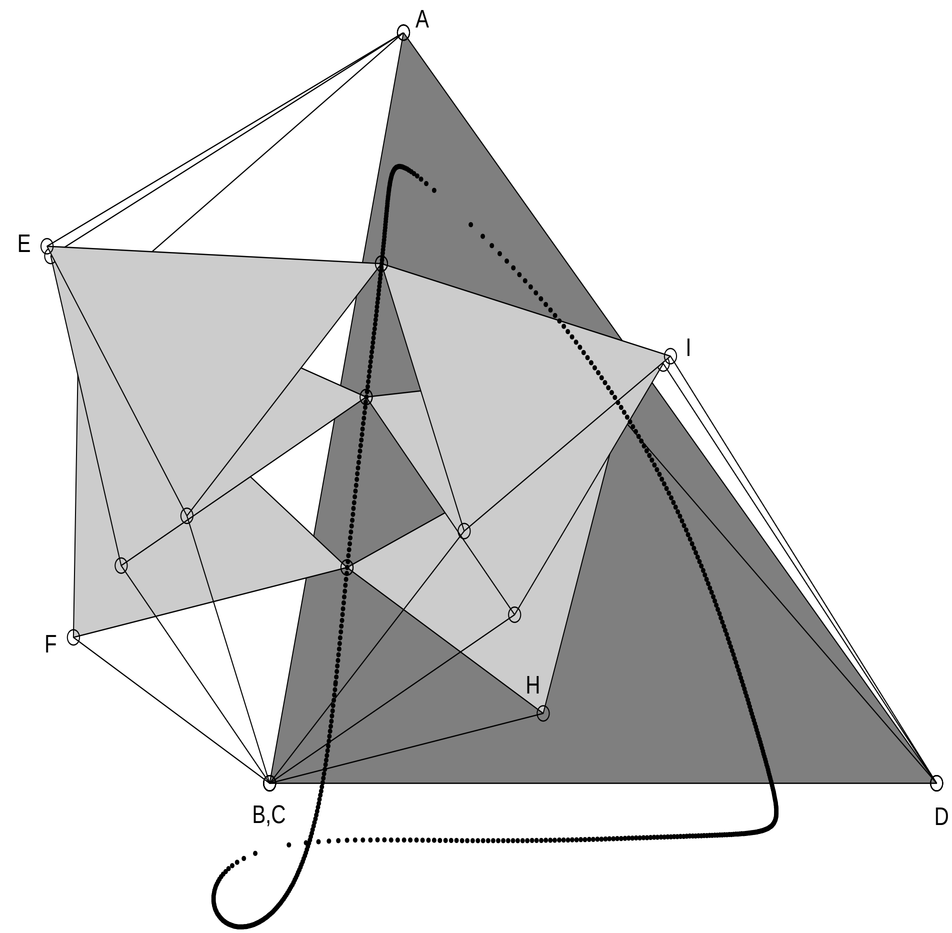
Example 2 a spatial Burmester problem

Example 3 the Griffis-Duffy platform

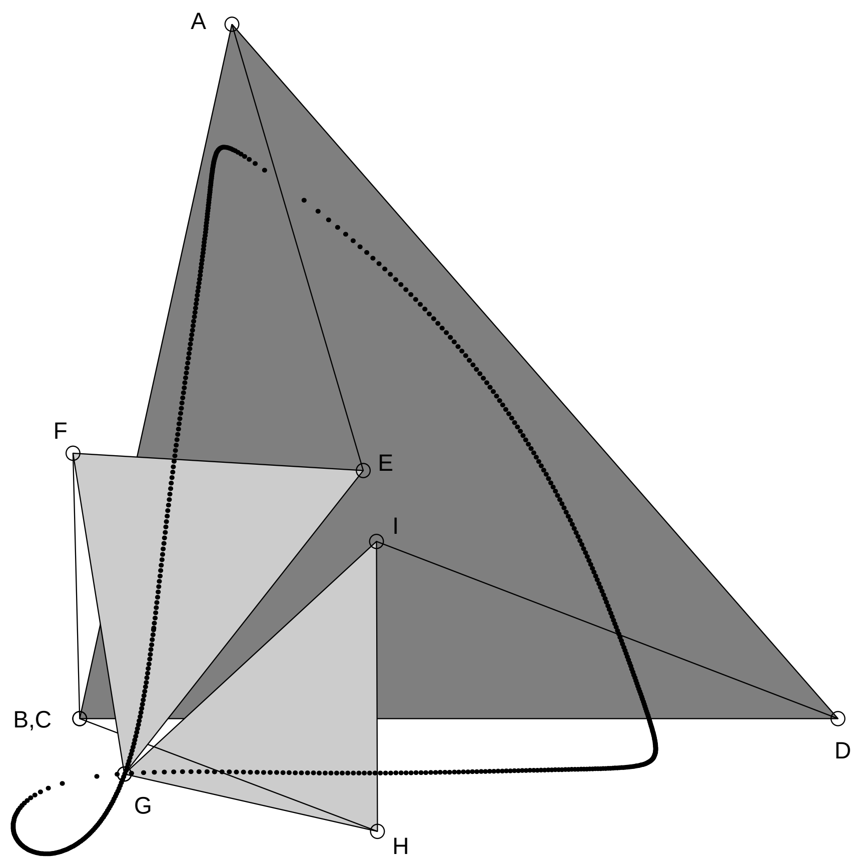
A.J. Sommese, J. Verschelde, and C.W. Wampler: **Advances in polynomial continuation for solving problems in kinematics.** In *Proc. ASME Design Engineering Technical Conf.* (CDROM), 2002.

A.J. Sommese, J. Verschelde, and C.W. Wampler: **Numerical irreducible decomposition using PHCpack.** In *Algebra, Geometry, and Software Systems* ed. by M. Joswig and N. Takayama. Springer-Verlag, to appear.

Ex 1. A Seven-Bar Structure: Solution



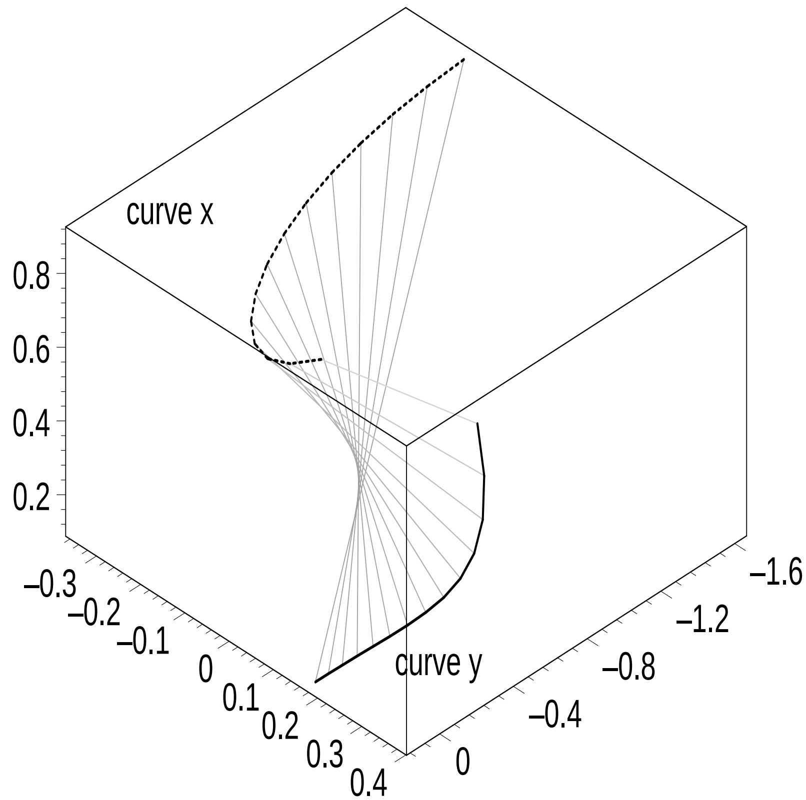
Roberts cognate 7-bar moves on a degree-6 curve (coupler curve)
AND . . .



AND . . . has six isolated solutions

- two at each double point of coupler curve
- here, only 1 of 3 double points is real

Ex 2. Six Spatial Positions: Solution



Sphere-point/center-point curves are irreducible, degree 10.

An illustration of Numerical Elimination.

Witness Points

for the Spatial Burmester Problem

- The input polynomial system consists of five quadrics in six unknowns $(\mathbf{x}, \mathbf{y})$.
- The new incremental solver computes 20 witness points in 7s 181ms on Pentium III 1Ghz Windows 2000 PC.
- Projection onto $\mathbf{x}$ or $\mathbf{y}$ reduces the degree from 20 to 10.

Ex 3. Griffis-Duffy Platforms: Solution

Solution components by degree

Husty & Karger		SVW	
Euler	Position	Study	Position
General Case			
20	40	28	40
Legs equal, Plates equal			
		6	8
4	8	6	8
4	8	6	8
6	12	6	12
2	4	4	4
16	32	28	40

Griffis-Duffy Platforms: Factorization

Case A: One irreducible component of degree 28 (general case).

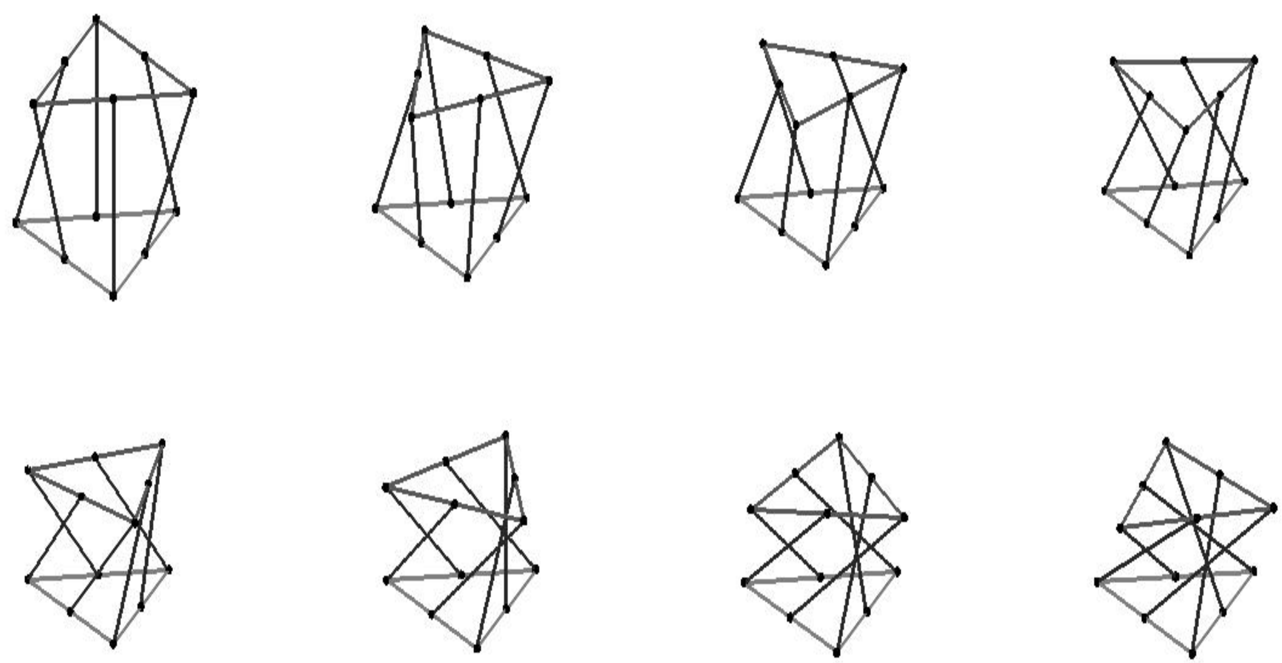
Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

user cpu on 800Mhz	Case A	Case B
witness points	1m 12s 480ms	
monodromy breakup	33s 430ms	27s 630ms
Newton interpolation	1h 19m 13s 110ms	2m 34s 50ms
32 decimal places used to interpolate polynomial of degree 28		
linear trace	4s 750ms	4s 320ms

Linear traces replace Newton interpolation:

$\Rightarrow$ **time to factor independent of geometry!**

Griffis-Duffy Platforms: an Animation



Conclusions

- Feasible in practice to decompose the solution set of a polynomial system by standard machine arithmetic.

  multi-precision arithmetic is needed for singular components...

- The incremental solving method with diagonal homotopies promises to unify solvers for isolated *and* solvers for components of solutions.

  exploitation of structure in progress...