Numerical Homotopies for Decomposing Solution Sets of Polynomial Systems

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Outline of Talk

1. Some Motivating Examples

2. Numerical Algebraic Geometry
   - homotopy continuation methods
   - numerical irreducible decomposition

3. Incrementally Solving Polynomial Systems
   - diagonal homotopies to intersect components
   - intrinsic and extrinsic representations

4. Results on the Examples
Example 1. A Seven-Bar Structure

Problem: Find all possible assemblies of these pieces.
• Generally, 18 solutions. (This example, 8 real, 10 complex.)
• Intersection of two four-bar coupler curves.
Question:

What if the four-bars have the same coupler curve (Roberts cognates)?

- Structure has mobility $= 0$.
- The common four-bar coupler curve (degree 6) is a solution.
- *Is the four-bar curve the only solution?*
- This is an overconstrained mechanism.
  - *How do we treat it numerically?*
Example 2. Spatial Six-Positions

Planar Body Guidance (Burmester 1874)
- 5 positions determine 6 circle-point/center-point pairs
- 4 positions give cubic circle-point & center-point curves

Spatial Body Guidance (Shoenflies 1886)
- 7 positions determine 20 sphere-point/center-point pairs
- 6 positions give 10\textsuperscript{th}-degree sphere-point & center-point curves

Question: Can we confirm this result using continuation?
Example 3. Stewart-Gough Platforms

Special Griffis-Duffy type

- Base and endplate are equilateral triangles.
- Legs connect vertices to midpoints.
Results of Husty and Karger


The special Griffis-Duffy platforms *move*:

- **Case 1**: Plates not equal, legs not equal.
  - Curve is degree 20 in Euler parameters.
  - Curve is degree 40 in position.

- **Case 2**: Plates congruent, legs all equal.
  - Factors are degrees \((4 + 4) + 6 + 2 = 16\) in Euler parameters.
  - Factors are degrees \((8 + 8) + 12 + 4 = 32\) in position.

**Question**: *Can we confirm these results numerically?*
2. Numerical Homotopy Continuation Methods

If we wish to solve \( f(x) = 0 \), then we construct a system \( g(x) = 0 \) whose solutions are known. Consider the homotopy

\[
H(x, t) := (1 - t)g(x) + tf(x) = 0.
\]

By continuation, we trace the paths starting at the known solutions of \( g(x) = 0 \) to the desired solutions of \( f(x) = 0 \), for \( t \) from 0 to 1.

**homotopy continuation** methods are *symbolic-numeric*:
- homotopy methods treat polynomials as algebraic objects,
- continuation methods use polynomials as functions.
### Solution sets to polynomial systems

<table>
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<tr>
<th>Polynomial in One Variable</th>
<th>System of Polynomials</th>
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<tr>
<td>one equation, one variable</td>
<td>$n$ equations, $N$ variables</td>
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<tr>
<td>solutions are points</td>
<td>points, lines, surfaces, …</td>
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<tr>
<td>double roots</td>
<td>sets with multiplicity</td>
</tr>
<tr>
<td>Factorization: $\prod_{i}(x - a_i)^{\mu_i}$</td>
<td>Irreducible Decomposition</td>
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#### Numerical Representation

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<tbody>
<tr>
<td>set of points</td>
<td>set of witness point sets</td>
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</table>
An Illustrative Example

\[ f(x, y, z) = \begin{cases} 
(y - x^2)(x^2 + y^2 + z^2 - 1)(x - 0.5) = 0 \\
(z - x^3)(x^2 + y^2 + z^2 - 1)(y - 0.5) = 0 \\
(y - x^2)(z - x^3)(x^2 + y^2 + z^2 - 1)(z - 0.5) = 0 
\end{cases} \]

Irreducible decomposition of \( Z = f^{-1}(0) \) is

\[ Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\} \]

with

1. \( Z_{21} \) is the sphere \( x^2 + y^2 + z^2 - 1 = 0 \),
2. \( Z_{11} \) is the line \( (x = 0.5, z = 0.5^3) \),
3. \( Z_{12} \) is the line \( (x = \sqrt{0.5}, y = 0.5) \),
4. \( Z_{13} \) is the line \( (x = -\sqrt{0.5}, y = 0.5) \),
5. \( Z_{14} \) is the twisted cubic \( (y - x^2 = 0, z - x^3 = 0) \),
6. \( Z_{01} \) is the point \( (x = 0.5, y = 0.5, z = 0.5) \).
Witness Point Sets

A witness point is a solution of a polynomial system which lies on a set of generic hyperplanes.

- The number of generic hyperplanes used to isolate a point from a solution component equals the dimension of the solution component.

- The number of witness points on one component cut out by the same set of generic hyperplanes equals the degree of the solution component.

A witness point set for a $k$-dimensional solution component consists of $k$ random hyperplanes and a set of isolated solutions of the system cut with those hyperplanes.
Membership Test

Does the point $z$ belong to a component?

Given: a point in space $z \in \mathbb{C}^N$; a system $f(x) = 0$;
and a witness point set $W$, $W = (Z, L)$:

for all $w \in Z : f(w) = 0$ and $L(w) = 0$.

1. Let $L_z$ be a set of hyperplanes through $z$, and define

$$H(x, t) = \begin{cases} f(x) = 0 \\ L_z(x)t + L(x)(1 - t) = 0 \end{cases}$$

2. Trace all paths starting at $w \in Z$, for $t$ from 0 to 1.

3. The test $(z, 1) \in H^{-1}(0)$? answers the question above.
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<th>example in 3-space</th>
<th>Numerical Analysis</th>
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<td>variety</td>
<td>collection of points, algebraic curves, and</td>
<td>polynomial system</td>
</tr>
<tr>
<td></td>
<td>algebraic surfaces</td>
<td>+ union of witness point sets, see below</td>
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<tr>
<td></td>
<td></td>
<td>for the definition of a witness point</td>
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<tr>
<td>irreducible variety</td>
<td>a single point, or</td>
<td>polynomial system</td>
</tr>
<tr>
<td></td>
<td>a single curve, or</td>
<td>+ witness point set</td>
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<tr>
<td></td>
<td>a single surface</td>
<td>+ probability-one membership test</td>
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<tr>
<td>generic point</td>
<td>random point on</td>
<td>point in witness point set; a witness point</td>
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<td>on an irreducible</td>
<td>an algebraic</td>
<td>is a solution of polynomial system on the</td>
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<tr>
<td>variety</td>
<td>curve or surface</td>
<td>variety and on a random slice whose</td>
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<td></td>
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<td>codimension is the dimension of the variety</td>
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<tr>
<td>pure dimensional</td>
<td>one or more points, or</td>
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</tr>
<tr>
<td>variety</td>
<td>one or more curves, or</td>
<td>+ set of witness point sets of same dimension</td>
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<tr>
<td></td>
<td>one or more surfaces</td>
<td>+ probability-one membership tests</td>
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<tr>
<td>irreducible decomposition of a</td>
<td>several pieces</td>
<td>polynomial system</td>
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<tr>
<td>variety</td>
<td>of different dimensions</td>
<td>+ array of sets of witness point sets and</td>
</tr>
<tr>
<td></td>
<td></td>
<td>probability-one membership tests</td>
</tr>
</tbody>
</table>

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A Numerical Irreducible Decomposition of the Illustrative Example

**Witness Generate**
**Path following**

- Homotopy + Start Solutions
  - 139 paths
    - 99 at infinity
      - 2 solutions
    - 38 nonsolutions

**Witness Classify**
**Filter Points**

- Filter = ∅
  - 2 to classify

**Sample & Interpolate**

- 2 on Sphere
  - $W_2$
  - $W_1$
    - 8 on Sphere
      - 1 on Line 1
        - $J_1$
        - $W_1$
    - 1 on Line 2
    - 1 on Line 3
    - 3 on Cubic
      - $W_2$

- $W_0$
  - 13 on Sphere
    - 2 on Line 1
    - 2 on Line 2
    - 1 on Line 3
    - 0 on Cubic
    - 1 to classify
  - $J_0$
  - $W_0$
    - 1 Isolated
      - $W_{01}$
History of Numerical Irreducible Decomposition


Numerical Factorization of Multivariate Polynomials


Monodromy to Decompose Solution Components

Given: a system $f(x) = 0$; and $W = (Z, L)$:

for all $w \in Z : f(w) = 0$ and $L(w) = 0$.

Wanted: partition of $Z$ so that all points in a subset of $Z$
lie on the same irreducible factor.

Example: does $f(x, y) = xy - 1 = 0$ factor?

Consider $H(x, y, \theta) = \begin{cases} 
xy - 1 = 0 & \text{for } \theta \in [0, 2\pi]. \\
 x + y = 4e^{i\theta} & \text{for } \theta \in [0, 2\pi]. 
\end{cases}$

For $\theta = 0$, we start with two real solutions. At $\theta = \pi$, the real
solutions have turned complex. Back at $\theta = 2\pi$, we have again two
real solutions, but their order is permuted $\Rightarrow$ irreducible.
1. For two sets of hyperplanes $K$ and $L$, and a random $\gamma \in \mathbb{C}$

$$H(x, t, K, L, \gamma) = \begin{cases} f(x) = 0 \\ \gamma K(x)(1 - t) + L(x)t = 0 \end{cases}$$

We start paths at $t = 0$ and end at $t = 1$.

2. For $\alpha \in \mathbb{C}$, trace the paths defined by $H(x, t, K, L, \alpha) = 0$.
For $\beta \in \mathbb{C}$, trace the paths defined by $H(x, t, L, K, \beta) = 0$.

Compare start points of first path tracking with end points of second path tracking. Points which are permuted belong to the same irreducible factor.

3. Repeat the loop with other values of $\alpha$ and $\beta$. 
Linear Traces

Consider  
\[ f(x, y(x)) = (y - y_1(x))(y - y_2(x))(y - y_3(x)) \]
\[ = y^3 - t_1(x)y^2 + t_2(x)y - t_3(x) \]

We are interested in the linear trace:  
\[ t_1(x) = c_1x + c_0. \]

Sample the cubic at  \( x = x_0 \) and  \( x = x_1 \). The samples are  
\[ \{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\} \quad \text{and} \quad \{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}. \]

Solve  
\[ \begin{cases} 
  y_{00} + y_{01} + y_{02} = c_1x_0 + c_0 \\
  y_{10} + y_{11} + y_{12} = c_1x_1 + c_0
\end{cases} \]

\[ \quad \text{to find} \quad c_0, c_1. \]

With  \( t_1 \) we can predict the sum of the  \( y \)'s for a fixed choice of  \( x \). For example, samples at  \( x = x_2 \) are  
\[ \{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}. \]

Then,  
\[ t_1(x_2) = c_1x_2 + c_0 = y_{20} + y_{21} + y_{22}. \]
Validation of Breakup with Linear Trace

Do we have enough witness points on a factor?

- We may not have enough monodromy loops to connect all witness points on the same irreducible component.
- For a $k$-dimensional solution component, it suffices to consider a curve on the component cut out by $k - 1$ random hyperplanes. The factorization of the curve tells the decomposition of the solution component.
- We have enough witness points on the curve if the value at the linear trace can predict the sum of one coordinate of all points in the set.
Numerical Irreducible Decomposition

In computing a numerical irreducible decomposition of a given polynomial system, we typically run through the following steps:

1. **Embed** (phc -c)  
   - add #random hyperplanes = top dimension,  
   - add slack variables to make the system square

2. **Solve** (phc -b)  
   - solve the system constructed above

3. **WitnessGenerate** (phc -c)  
   - apply a sequence of homotopies to compute witness point sets on all solution components

4. **WitnessClassify** (phc -f)  
   - filter junk from witness point sets  
   - factor components into irreducible components

Especially step 2 is a computational bottleneck.

We recently discovered and implemented a new algorithm.
3. Solving Systems Incrementally

- Extrinsic and Intrinsic Deformations
  - **extrinsic**: defined by explicit equations
  - **intrinsic**: following the actual geometry

- Diagonal Homotopies
  - to intersect pure dimensional solution sets

- Intersecting with Hypersurfaces
  - adding the polynomial equations one after the other we arrive at an incremental polynomial system solver.
Extrinsic Homotopy Deformations

\[ f(x) = 0 \] has \( k \)-dimensional solution components. We cut with \( k \) hyperplanes to find isolated solutions = \textit{witness points sets}:

\[
a_{i0} + \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, 2, \ldots, k, \quad a_{ij} \in \mathbb{C} \text{ random}
\]

Sample

\[
\begin{align*}
\left\{ \begin{array}{l}
f(x) + \gamma z = 0 \\
\sum_{j=1}^{n} a_{ij}(t) x_j = 0
\end{array} \right. \\
z = \text{slack}
\end{align*}
\]

\[
\#\text{witness points} = \sum \deg(C)
\]

\[
C \subseteq f^{-1}(0)
\]

\[
\dim(C) = k
\]
Embedding with Slack Variables

The cyclic 4-roots system defines 2 quadrics in $\mathbb{C}^4$:

\[
\begin{align*}
& x_1 + x_2 + x_3 + x_4 + \gamma_1 z = 0 \\
& x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_1 + \gamma_2 z = 0 \\
& x_1 x_2 x_3 + x_2 x_3 x_4 + x_3 x_4 x_1 + x_4 x_1 x_2 + \gamma_3 z = 0 \\
& x_1 x_2 x_3 x_4 - 1 + \gamma_4 z = 0 \\
& a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + z = 0
\end{align*}
\]

Original system: 4 equations in $x_1$, $x_2$, $x_3$, and $x_4$.

Cut with random hyperplane to find isolated points.

Slack variable $z$ with random $\gamma_i$, $i = 1, 2, 3, 4$: square system.

Solve embedded system to find $4 = 2 + 2$ witness points as isolated solutions with $z = 0$. 
Intrinsic Homotopy Deformations

\[ f(\mathbf{x}) = 0 \] has \( k \)-dimensional solution components. We cut with a random affine \((n - k)\)-plane to find witness points:

\[
\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i \in \mathbb{C}^n
\]

The vectors \( \mathbf{b} \) and \( \mathbf{v}_i \) are chosen at random.

Sample

\[
f \left( \mathbf{x}(\lambda, t) = \mathbf{b}(t) + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i(t) \right) = 0
\]

Points on the moving \((n - k)\)-plane are determined by \( n - k \) independent variables \( \lambda_i, i = 1, 2, \ldots, n - k \).
Independent variables = co-dimension

\( f(\mathbf{x}) = 0 \) is a system with \( \mathbf{x} \in \mathbb{C}^n \), \( \mathbf{x} \) lies on an affine \((n-k)\)-plane:

\[
\mathbf{x}(\lambda) = \mathbf{b} + \sum_{i=1}^{n-k} \lambda_i \mathbf{v}_i \in \mathbb{C}^n
\]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-k}) \) contains all independent variables.

Correct with Newton on \( f(\mathbf{x}(\lambda)) = 0 \), a system in \( \lambda \).

Solve

\[
\left[ \frac{\partial f}{\partial \lambda} \right] \lambda = -f(\mathbf{x}(\lambda)) \quad \text{with} \quad \frac{\partial f_i}{\partial \lambda_j} = \sum_{l=1}^{n-k} \frac{\partial f_i}{\partial x_l} \frac{\partial x_l}{\partial \lambda_j}.
\]

Overdetermined case moved from global to local level!

no slack variables needed...
Intersecting Hypersurfaces Extrinsicially

\[
\begin{align*}
\begin{cases}
  f_1(x) = 0 & x \in \mathbb{C}^n \\
  L_1(x) = 0 & n-1 \text{ hyperplanes}
\end{cases}
\quad \begin{cases}
  f_2(y) = 0 & y \in \mathbb{C}^n \\
  L_2(y) = 0 & n-1 \text{ hyperplanes}
\end{cases}
\end{align*}
\]

**diagonal homotopy**

\[
\begin{pmatrix}
  f_1(x) = 0 \\
  f_2(y) = 0 \\
  L_1(x) = 0 \\
  L_2(y) = 0
\end{pmatrix}
\]

\[ t + \begin{pmatrix}
  f_1(x) = 0 \\
  f_2(y) = 0 \\
  x - y = 0 \\
  M(y) = 0
\end{pmatrix} (1 - t) = 0
\]

**At** \( t = 1 \): \( \deg(f_1) \times \deg(f_2) \) solutions \( (x, y) \in \mathbb{C}^{n \times n} \).

**At** \( t = 0 \): witness points \( (x = y \in \mathbb{C}^n) \) on \( f_1^{-1}(0) \cap f_2^{-1}(0) \) cut out by \( n - 2 \) hyperplanes \( M \).
Consider a general affine line $x(\lambda) = b + \lambda v \in \mathbb{C}^n$.

$$f_1(x(\lambda) = b + \lambda v) \quad \cap \quad f_2(y(\mu) = b + \mu v)$$

deg($f_1$) values for $\lambda$ \quad deg($f_2$) values for $\mu$

\text{diagonal} \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ intrinsic version}

\text{homotopy} \quad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix} + \lambda \begin{bmatrix} v \\ 0 \end{bmatrix} t + \begin{bmatrix} u_1 \\ u_1 \end{bmatrix} (1-t) + \mu \begin{bmatrix} 0 \\ v \end{bmatrix} t + \begin{bmatrix} u_2 \\ u_2 \end{bmatrix} (1-t)

\text{At} \quad t = 1 : \quad \text{deg($f_1$) \times deg($f_2$) solutions} (x, y) \in \mathbb{C}^{n \times n}$.

\text{At} \quad t = 0 : \quad \text{witness points on} \quad x = b + \lambda u_1 + \mu u_2, \text{ a general 2-plane defined by a random point} \quad b \text{ and 2 random vectors} \quad u_1 \text{ and} \quad u_2.$
Intersecting with Hypersurfaces

Let $f(x) = 0$ have $k$-dimensional solution components described by witness points on a general $(n - k)$-dimensional affine plane, i.e.:

$$f\left(x(\lambda) = b + \sum_{i=1}^{n-k} \lambda_i v_i\right) = 0.$$

Let $g(x) = 0$ be a hypersurface with witness points on a general affine line, i.e.:

$$g(x(\mu) = b + \mu w) = 0.$$

Assuming $g(x) = 0$ properly cuts one degree of freedom from $f^{-1}(0)$, we want to find witness points on all $(k - 1)$-dimensional components of $f^{-1}(0) \cap g^{-1}(0)$.
The diagonal homotopy for \((f, g)\) on \((x, y) \in \mathbb{C}^{n \times n}\) starts at
\[
\begin{bmatrix}
x(1) \\
y(1)
\end{bmatrix} =
\begin{bmatrix}
b \\
b
\end{bmatrix} + \sum_{i=1}^{n-k} \lambda_i \begin{bmatrix}
v_i \\
0
\end{bmatrix} + \mu \begin{bmatrix}
0 \\
w
\end{bmatrix}
\]
and ends at
\[
\begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix} =
\begin{bmatrix}
b \\
b
\end{bmatrix} + \sum_{i=1}^{n-k} \lambda_i \begin{bmatrix}
v_i \\
v_i
\end{bmatrix} + \mu \begin{bmatrix}
w \\
w
\end{bmatrix}.
\]

The diagonal homotopy
\[
\begin{pmatrix}
f \\
g
\end{pmatrix}
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
x(1) \\
y(1)
\end{bmatrix} t + \begin{bmatrix}
x(0) \\
y(0)
\end{bmatrix} (1 - t) =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
has \(n - k + 1\) independent variables \((\lambda_1, \lambda_2, \ldots, \lambda_{n-k}, \mu)\).
Computing Nonsingular Solutions Incrementally

Suppose \((f_1, f_2, \ldots, f_k)\) defines the system \(f(x) = 0, \ x \in \mathbb{C}^n\),
whose solution set is pure dimensional of multiplicity one for all \(k = 1, 2, \ldots, N \leq n\), i.e.: we find only nonsingular roots if we slice the solution set of \(f(x) = 0\) with a generic linear space of dimension \(n - k\).

Main loop in the solver:

for \(k = 2, 3, \ldots, N - 1\) do

use a diagonal homotopy to intersect \((f_1, f_2, \ldots, f_k)^{-1}(0)\) with \(f_{k+1}(x) = 0\),
to find witness points on all \((n - k - 1)\)-dimensional solution components.
Outcomes of Hypersurface Intersections

Let $V$ be an $(n - k)$-dimensional irreducible component of $(f_1, ..., f_k)^{-1}(0)$ and $g^{-1}(0)$ be an irreducible hypersurface.

Three cases for $V \cap g^{-1}(0)$:

1. $V \subseteq g^{-1}(0)$
   
   All witness points of $V$ satisfy $g(x) = 0$.

2. $\dim(V \cap g^{-1}(0)) = k - 1$
   
   The diagonal homotopy gives witness points on all $(k - 1)$-dimensional components of the intersection.

3. $V \cap g^{-1}(0) = \emptyset$
   
   All paths in the diagonal homotopy diverge.
New WitnessGenerate for the Illustrative Example

\begin{align*}
\text{level 2} & \quad \deg(f_1) = 5 \\
& \quad \deg(f_2) = 6 \quad \rightarrow \\
& \quad 2 \text{ common satisfy } f_3 \\
& \quad (5 - 2, 6 - 2) \text{ to continue}
\end{align*}

\begin{align*}
\text{level 1} & \quad (5 - 2) \times (6 - 2) = 12 \\
& \quad \text{solution paths} \quad \rightarrow \\
& \quad 5 \text{ at infinity} \\
& \quad 6 \text{ satisfy } f_3 \text{ to classify} \\
& \quad 1 \text{ to continue}
\end{align*}

\begin{align*}
\text{level 0} & \quad \deg(f_3) = 8 \quad \rightarrow \\
& \quad 8 - 4 \text{ satisfy } f_1 \\
& \quad - 3 \text{ satisfy } f_2 = 1 \text{ left} \\
& \quad (8 - 7) \times 1 = 1 \\
& \quad \text{solution paths} \quad \rightarrow \\
& \quad 1 \text{ to classify}
\end{align*}
4. Test Polynomial Systems

Example 1 a 7-bar mechanism in the plane
Example 2 a spatial Burmester problem
Example 3 the Griffis-Duffy platform


Ex 1. A Seven-Bar Structure: Solution

Roberts cognate 7-bar moves on a degree-6 curve (coupler curve) AND ...
AND ... has six isolated solutions

- two at each double point of coupler curve
- here, only 1 of 3 double points is real
Sphere-point/center-point curves are irreducible, degree 10. An illustration of Numerical Elimination.
Witness Points for the Spatial Burmester Problem

- The input polynomial system consists of five quadrics in six unknowns \((x, y)\).
- The new incremental solver computes 20 witness points in 7s 181ms on Pentium III 1Ghz Windows 2000 PC.
- Projection onto \(x\) or \(y\) reduces the degree from 20 to 10.
### Ex 3. Griffis-Duffy Platforms: Solution

#### Solution components by degree

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<th>Husty &amp; Karger</th>
<th>SVW</th>
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<tr>
<td>Euler/Position</td>
<td>Study/Position</td>
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#### General Case

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<tbody>
<tr>
<td>20</td>
<td>40</td>
<td>28</td>
<td>40</td>
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#### Legs equal, Plates equal

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<td>16</td>
<td>32</td>
<td>28</td>
<td>40</td>
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</tbody>
</table>
Case A: One irreducible component of degree 28 (general case).

Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

<table>
<thead>
<tr>
<th>user cpu on 800Mhz</th>
<th>Case A</th>
<th>Case B</th>
</tr>
</thead>
<tbody>
<tr>
<td>witness points</td>
<td>1m 12s 480ms</td>
<td></td>
</tr>
<tr>
<td>monodromy breakup</td>
<td>33s 430ms</td>
<td>27s 630ms</td>
</tr>
<tr>
<td>Newton interpolation</td>
<td>1h 19m 13s 110ms</td>
<td>2m 34s 50ms</td>
</tr>
</tbody>
</table>

32 decimal places used to interpolate polynomial of degree 28

Linear traces replace Newton interpolation:

⇒ time to factor independent of geometry!
Griffis-Duffy Platforms: an Animation
Conclusions

- Feasible in practice to decompose the solution set of a polynomial system by standard machine arithmetic. Multi-precision arithmetic is needed for singular components...

- The incremental solving method with diagonal homotopies promises to unify solvers for isolated and solvers for components of solutions.

  exploitation of structure in progress...