Robust Polyhedral Homotopy Continuation
(preliminary report)

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Robust Polyhedral Continuation

1. Problem Statement
   - robust path tracking
   - polyhedral homotopies
2. Linear Systems of Series with Real Powers
   - Newton's method for power series
   - the tropical Cramer rule
3. Chebyshev Least Squares Approximations
   - applying the ratio theorem of Fabry
   - least squares approximations

polynomial homotopy continuation

A polynomial homotopy is a system of polynomials in one parameter t, the solution trajectories are then also analytic functions in t, therefore apply *analytic continuation* to approximate the solutions.

Nearby singularities are problematic for convergence.

An algorithm is *robust* if it does not fail for small perturbations of degenerate inputs.

A robust path tracker applies apriori step size control:

- taking into account the curvature of the paths,
- detecting the nearby singularities.

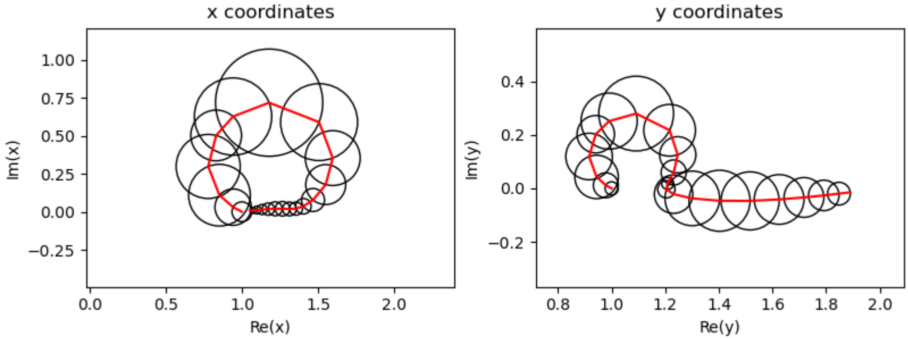
S. Telen, M. Van Barel, J. Verschelde: **A robust numerical path tracking algorithm for polynomial homotopy continuation.** *SIAM Journal on Scientific Computing*, 2020.

logarithmic convergence to a singular solution

The homotopy, for a random complex constant γ,

$$\gamma(1-t)\left(\begin{array}{rcl} x^2-1 & = & 0 \\ y^2-1 & = & 0 \end{array}\right) + t\left(\begin{array}{rcl} x^2+y-3 & = & 0 \\ x+0.125y^2-1.5 & = & 0 \end{array}\right)$$

defines three paths leading to a triple root $(1,2)$, at $t=1$.



the theorem of Fabry

Theorem (the ratio theorem, Fabry 1896)

If for the series $x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n + c_{n+1} t^{n+1} + \cdots$,
we have $\lim_{n \to \infty} c_n / c_{n+1} = z$, *then*

- z *is a singular point of the series, and*
- *it lies on the boundary of the circle of convergence of the series.*

Then the radius of this circle equals $|z|$.

The ratio c_n / c_{n+1} is the pole of Padé approximants of degrees $[n/1]$ (n is the degree of the numerator, with linear denominator).

Newton polytopes, mixed volumes, and homotopies

- The Newton polytope of a polynomial is the convex hull of the exponents of those monomials appearing with nonzero coefficient.
- A regular subdivision Δ of the polytopes defines homotopies, starting at solutions of systems supported on the faces of Δ.

Theorem (Bernshteǐn's theorems, 1975)

Let P be the Newton polytopes of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
An initial form system of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has faces of P as Newton polytopes.

1. *The mixed volume $V(P) \geq \#$isolated solutions of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.*
2. *If $V(P) > \#$isolated solutions of $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ in ${\mathbb{C}^*}^n$, then $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ has initial form systems with solutions in ${\mathbb{C}^*}^n$.*

- $V(P)$ is a generically sharp upper bound. For systems with fewer solutions, faces of Newton polytopes certify diverging paths.

numerical stability of polyhedral continuation

That high powers of the continuation parameter t cause numerical difficulties was addressed in the following papers:

- Placing points in the mixed-cell configurations.

  J. Verschelde, K. Gatermann, and R. Cools: **Mixed-volume computation by dynamic lifting applied to polynomial system solving.** *Discrete Comput. Geom.*, 16(1):69–112, 1996.

- Recomputing the lifting values, given a mixed-cell configuration.

  T. Gao, T.Y. Li, J. Verschelde, and M. Wu: **Balancing the lifting values to improve the numerical stability of polyhedral homotopy continuation methods.** *Applied Math. Comput.* 114:233–247, 2000.

- Change of $t \in [0, 1]$ as $s = \log(t)$, during path tracking.

  S. Kim and M. Kojima: **Numerical stability of path tracing in polyhedral homotopy continuation methods.** *Computing*, 73:329–348, 2004.

problem statement

We want to apply apriori step size control to track solution paths defined by polyhedral homotopies.

Two problems:

1. Compute series developments of the solution paths of homotopies with real powers of the continuation parameter.
2. The ratio theorem of Fabry applies only to Taylor series.

"implement a numerically robust path tracker for tropical homotopies" is stated as one of the main challenges by:
P. A. Helminck, O. Henriksson, and Y. Ren. **A tropical method for solving parametrized polynomial systems.**
`arXiv:2409.13288v1 [math.AG] 20 Sep 2024.`

series with real powers

As the powers of the continuation parameter in polyhedral homotopies are real numbers, the solution paths have series developments where the powers are real. Our series are converging.

Definition (series with real powers)

A *series with real powers* $c(t)$ is defined as

$$c(t) = c_0 + c_1 t^{\gamma_1} + c_2 t^{\gamma_2} + \cdots, \quad 0 < \gamma_1 < \gamma_2 < \ldots,$$

with $c_k \in \mathbb{C}$, $\gamma_k \in \mathbb{R}^+$.

For higher order terms: $c(t) = c_0 + c_1 t^{\gamma_1} + O(t^{\gamma_1 + \epsilon})$, for some $\epsilon > 0$.

A generalized Newton-Puiseux algorithm was defined by T. Markwig: **A Field of Generalized Puiseux Series for Tropical Geometry.** *Rend. Sem. Math. Univ. Politec. Torino*, 68(1): 79–92, 2010.

Coefficients of series with real powers relate to *fractional derivatives*.

linear systems of series with real powers

The series with real powers we consider arise in linear systems with random complex coefficients.
The systems are determined by as many equations as variables.

Definition (series with real powers)

$A(t)\mathbf{x}(t) = \mathbf{b}(t)$ is *linear system of series with real powers*, with

1. $A(t) = A_0 + A_t$, a matrix of series where $A_0 = A(0)$, and all elements in A_t are $a\,t^\alpha + O(t^{\alpha+\epsilon})$, $a \in \mathbb{C}$, $\alpha, \epsilon \in \mathbb{R}^+$.
2. $\mathbf{b}(t) = \mathbf{b}_0 + \mathbf{b}_t$, a vector of series where $\mathbf{b}_0 = \mathbf{b}(0)$, and all elements in $\mathbf{b}_t$ are $b\,t^\beta + O(t^{\beta+\epsilon})$, $b \in \mathbb{C}$, $\beta, \epsilon \in \mathbb{R}^+$.

As the coefficients are random, the leading coefficients $\mathbf{x}_0 = \mathbf{x}(0)$ of the solution $\mathbf{x}(t)$ satisfy the regular linear system $A_0\mathbf{x}_0 = \mathbf{b}_0$.

Newton's method

The regularity of A_0 guarantees a unique solution.

Newton's method can be applied to compute the terms in the power series developments of the solution paths.

In each step, we solve a linear system of series of real powers to compute the updates to the developments.

about the tropical Cramer rule

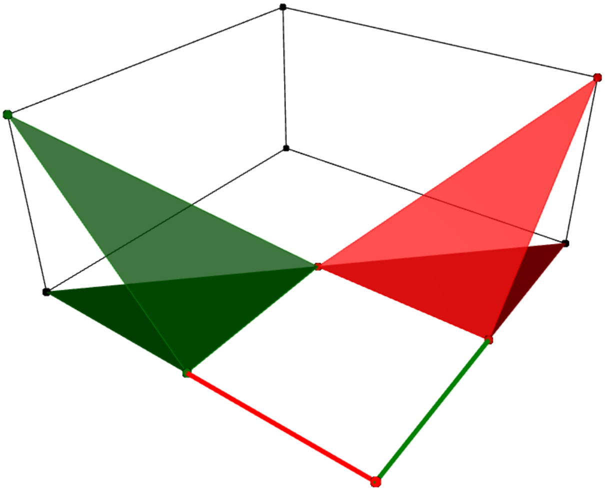
literature in chronological order

Starting with tropical geometry ...

- J. Richter-Gebert, B. Sturmfels, and T. Teobald: **First Steps in Tropical Geometry.** *Contemp. Math.* 377: 289–317, 2005.
- M. Akian, S. Gaubert, and A. Guterman: **Tropical Cramer Determinants Revisited.** *Contemp. Math.* 616:1–45, 2014.
- B. Assarf, E. Gawrilow, K. Herr, M. Joswig, B. Lorenz, A. Paffenholz, and T. Rehn: **Computing Convex Hulls and Counting Integer Points with `polymake`.** *Math. Program. Comput.* 9(1): 1–38, 2017.
- M. Joswig: **Essentials of Tropical Combinatorics.** A.M.S., 2021.

... leading into *combinatorial scientific computing.*

one mixed cell



computing tropisms

the leading powers of series

Consider $A(t)\mathbf{x}(t) = \mathbf{b}(t)$ as $(A_0 + A_t)(\mathbf{x}_0 + \mathbf{x}_t) = \mathbf{b}_0 + \mathbf{b}_t$, where A_0, $\mathbf{x}_0$, and $\mathbf{b}_0$ are the constant coefficients.

Solve $A_0\mathbf{x}_0 = \mathbf{b}_0$ and $(A_0 + A_t)(\mathbf{x}_0 + \mathbf{x}_t) = \mathbf{b}_0 + \mathbf{b}_t$ simplifies into

$$A_0\mathbf{x}_t + A_t\mathbf{x}_t = \mathbf{b}_t - A_t\mathbf{x}_0,$$

where we look for $\mathbf{x}_t$ which has components of the form

$$c_k t^{\gamma_k}, \quad c_k \in \mathbb{C} \setminus \{0\}, \quad \gamma_k > 0.$$

The leading exponents γ of $\mathbf{x}_t$ are *tropisms*.

the tropical Cramer rule

Cramer's rule expresses each component of the solution of a linear system as the ratio of determinants.

Definition (tropical determinant)

The *tropical determinant* is the tropicalization of the determinant. For an $n \times n$ matrix X, we have

$$\begin{aligned} \mathrm{tdet}(X) &= \bigoplus_{\sigma \in S_n} x_{1,\sigma(1)} \odot x_{2,\sigma(2)} \odot \cdots \odot x_{n,\sigma(n)} \\ &= \min_{\sigma \in S_n} x_{1,\sigma(1)} + x_{2,\sigma(2)} + \cdots + x_{n,\sigma(n)} \end{aligned}$$

Viewed as an instance of the weighted bipartite maching problem, it is solved by the Hungarian method in $O(n^3)$ time.

steps in the algorithm

Given a linear system of series with real powers $A(t)\mathbf{x}(t) = \mathbf{b}(t)$, the steps to compute the leading terms in $\mathbf{x}(t)$ are as follows:

0. Solve $A_0\mathbf{x}_0 = \mathbf{b}_0$ for the constant coefficients $\mathbf{x}_0$.
1. Set up $A_0\mathbf{x}_t + A_t\mathbf{x}_t = \mathbf{b}_t - A_t\mathbf{x}_0$.
2. Apply the tropical Cramer rule
   to compute the next smallest exponent of $\mathbf{x}_t$.
3. Compute the coefficient corresponding to the smallest exponent.
4. Substitute and go to step 2.

The computation of the first k exponents of an n-by-n linear system of series will real powers can happen in running time $O(k\,n^3)$, in fixed precision.

the ratio theorem of Fabry for orthogonal series

Problem: The theorem of Fabry applies to Taylor series.

V. I. Buslaev: **On the Fabry Ratio Theorem for Orthogonal Series.** *Proceedings of the Steklov Institute of Mathematics*, 253: 8–21, 2006.

We suggest the following solution:

1. given a series with real powers, compute a least squares approximation using orthogonal polynomials, and
2. expand the orthogonal polynomials in the power monomial basis so the theorem of Fabry applies again.

least squares of series with real powers

The Chebyshev polynomials T_n form an orthogonal basis with respect to the inner product

$$\langle p, q \rangle = \frac{2}{\pi} \int_{-1}^{1} \frac{p(x)q(x)}{\sqrt{1 - x^2}} dx.$$

For any function $f(x)$, we compute an approximation as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k T_k(x),$$

where the coefficients are computed via the inner products:

$$a_i = \langle f, T_i \rangle, \quad i = 0, 1, \ldots, n.$$

By the orthogonality of the Chebyshev basis,
the approximation minimizes the square of the errors.

shifted Chebyshev polynomials

Functions such as $\sqrt{x}$ have a singularity at $x = 0$.

The shifted Chebyshev polynomials are $T_n^*(x) = T_n(2x - 1)$, for $x \in [0, 1]$, forming an orthogonal basis with respect to the weight

$$w(x) = \frac{1}{\sqrt{x - x^2}}.$$

Steps in estimating the nearest singularity:

1. Run shifted Gauss-Chebyshev quadrature for the coefficients of the least square approximations of the series with real powers.
2. Expand the approximations in the power basis.
3. Apply the Fabry ratio theorem.
4. Extrapolate with the rho algorithm.

conclusions and future directions

After the introduction of apriori step size control in a robust path tracker, two applications followed:

1. extrapolation algorithms towards isolated singularities at the end of the solutions paths, and
2. real power series algorithms to improve the numerical stability of polyhedral homotopies.

In the (near) future, this will lead to a robust blackbox solver to solve systems of polynomial quations.

The cost overhead will be compensated by parallel computing.