solving polynomial systems with Puiseux series

Jan Verschelde
(joint work with Danko Adrovic)

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
http://www.math.uic.edu/~jan
jan@math.uic.edu

the 2013 Michigan Computational Algebraic Geometry meeting
3-4 May 2013, Western Michigan University, Kalamazoo
Outline

1. Introduction
   - solving sparse polynomial systems
   - space curves and initial forms

2. Solving Binomial Systems
   - unimodular coordinate transformations
   - computation of the degree and affine sets

3. Polyhedral Methods for Algebraic Sets
   - computing pretropisms with the Cayley embedding
   - Puiseux series for algebraic sets
   - application to the cyclic $n$-roots problem
polynomial systems

Consider $f(x) = 0$, a system of equations defined by

- $N$ polynomials $f = (f_0, f_1, \ldots, f_{N-1})$,
- in $n$ variables $x = (x_0, x_1, \ldots, x_{n-1})$.

A polynomial in $n$ variables consists of a vector of nonzero complex coefficients with corresponding exponents in $A$:

$$f_k(x) = \sum_{a \in A_k} c_a x^a, \quad c_a \in \mathbb{C} \setminus \{0\}, \quad x^a = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

Input data:

1. $A = (A_0, A_1, \ldots, A_{N-1})$ are sets of exponents, the supports. For $a \in \mathbb{Z}^n$, we consider Laurent polynomials, $f_k \in \mathbb{C}[x^{\pm 1}] \Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
2. $c_A = (c_{A_0}, c_{A_1}, \ldots, c_{A_{N-1}})$ are vectors of complex coefficients. Although $A$ is exact, the coefficients may be approximate.
the cyclic 4-roots system

\[ f(x) = \begin{cases} 
  x_0 + x_1 + x_2 + x_3 = 0 \\
  x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 = 0 \\
  x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_3 x_0 x_1 = 0 \\
  x_0 x_1 x_2 x_3 - 1 = 0
\end{cases} \]

Cyclic 4-roots \( x = (x_0, x_1, x_2, x_3) \) correspond to complex circulant Hadamard matrices:

\[
H = \begin{bmatrix}
  x_0 & x_1 & x_2 & x_3 \\
  x_3 & x_0 & x_1 & x_2 \\
  x_2 & x_3 & x_0 & x_1 \\
  x_1 & x_2 & x_3 & x_0
\end{bmatrix}, \quad |x_k| = 1, k = 1, 2, 3, 4
\]

\[ H^* H = 4I_4. \]

- Haagerup: for prime \( p \), there are \( \binom{2p - 2}{p - 1} \) isolated roots.
- Backelin: for \( n = \ell m^2 \), there are infinitely many cyclic \( n \)-roots.
Systems like cyclic $n$-roots are

- Sparse: relative to the degrees of the polynomials, few monomials appear with nonzero coefficients $\Rightarrow$ fewer roots than the Bézout bounds.

- Symmetric: solutions are invariant under permutations, $n = 4$:
  
  \[
  (x_0, x_1, x_2, x_3) \rightarrow (x_1, x_2, x_3, x_0) \text{ and } (x_0, x_1, x_2, x_3) \rightarrow (x_3, x_2, x_1, x_0)
  \]

  generate the permutation group.

  In addition: $(x_0, x_1, x_2, x_3) \rightarrow (x_0^{-1}, x_1^{-1}, x_2^{-1}, x_3^{-1})$.

- Not pure dimensional, for prime $n$, all solutions are isolated, but for $n = \ell m^2$, we have positive dimensional solution sets.

Our solution is to apply a hybrid symbolic-numeric approach.
Puiseux series

The Newton polygon of \( f(x_0, x_1) \) is the convex hull, spanned by the exponents \((a_0, a_1)\) of monomials \(x_0^{a_0}x_1^{a_1}\) that occur in \( f \) with \( c_{(a_0, a_1)} \neq 0\).

**Theorem (the theorem of Puiseux)**

Let \( f(x_0, x_1) \in \mathbb{C}(x_0)[x_1] \): \( f \) is a polynomial in the variable \( x_1 \) and its coefficients are fractional power series in \( x_0 \).

The polynomial \( f \) has as many series solutions as the degree of \( f \).

Every series solution has the following form:

\[
\begin{align*}
    x_0 &= t^u \\
    x_1 &= ct^v(1 + O(t)), \quad c \in \mathbb{C}^*
\end{align*}
\]

where \((u, v)\) is an inner normal to an edge of the lower hull of the Newton polygon of \( f \).

The series are computed with the polyhedral Newton-Puiseux method.
limits of space curves

Assume \( f(x) = 0 \) has a solution curve \( C \), which intersects \( x_0 = 0 \) at a regular point.

For \( v = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}^n \), consider \( x = z t^v (1 + O(t)) \):

- \( x_0 = z_0 t^{v_0} \), for \( t \) close to zero, \( z_0 \neq 0 \) and
- for \( k = 1, 2, \ldots, n - 1 \): \( x_k = z_k t^{v_k} (1 + O(t)), \ z_k \neq 0 \).

Substitute \( x_0 = z_0 t^{v_0}, x_k = z_k t^{v_k} (1 + O(t)) \) in \( f_\ell(x) = \sum_{a \in A_\ell} c_\ell x^a \):

\[
f_\ell(x) = z t^v (1 + O(t)) = \sum_{a \in A_\ell} c_\ell z_0^{a_0} t^{a_0 v_0} \prod_{k=1}^{n-1} z_k t^{a_k v_k} (1 + O(t)) = \sum_{a \in A_\ell} c_\ell z^a t^{a_0 v_0 + a_1 v_1 + \cdots + a_{n-1} v_{n-1}} (1 + O(t)).
\]

Because \( z \in (\mathbb{C}^\ast)^n \), there must be at least two terms in \( f_\ell \) left as \( t \to 0 \).
initial forms and tropisms

Denote the inner product of vectors \( u \) and \( v \) as \( \langle u, v \rangle \).

**Definition**

Let \( v \in \mathbb{Z}^n \setminus \{0\} \) be a direction vector. Consider \( f(x) = \sum_{a \in A} c_a x^a \).

The *initial form of \( f \) in the direction \( v \)* is

\[
\text{in}_v(f) = \sum_{\langle a, v \rangle = m, a \in A} c_a x^a, \quad \text{where } m = \min\{ \langle a, v \rangle | a \in A \}.
\]

**Definition**

Let the system \( f(x) = 0 \) define a curve. A *tropism* consists of the leading powers \( (v_0, v_1, \ldots, v_{n-1}) \) of a Puiseux series of the curve.

The leading coefficients of the Puiseux series satisfy \( \text{in}_v(f)(x) = 0 \).
curves of cyclic 4-roots

\[ f(x) = \begin{cases} 
  x_0 + x_1 + x_2 + x_3 = 0 \\
  x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\
  x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\
  x_0x_1x_2x_3 - 1 = 0 
\end{cases} \]

One tropism \( \mathbf{v} = (+1, -1, +1, -1) \) with \( \text{in}_\mathbf{v}(f)(\mathbf{z}) = 0 \):

\[ \text{in}_\mathbf{v}(f)(x) = \begin{cases} 
  x_1 + x_3 = 0 \\
  x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\
  x_1x_2x_3 + x_3x_0x_1 = 0 \\
  x_0x_1x_2x_3 - 1 = 0. 
\end{cases} \]

We look for solutions of the form

\[(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1}).\]
solving the initial form system

Substitute \((x_0 = t^{-1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{-1}, x_3 = z_3 t^{-1})\):

\[\text{inv}_v(f)(x_0 = t^{-1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{-1}, x_3 = z_3 t^{-1}) = \begin{cases} 
(1 + z_2) t^{-1} = 0 \\
 z_1 + z_1 z_2 + z_2 z_3 + z_3 = 0 \\
 (z_1 z_2 + z_3 z_1) t^{-1} = 0 \\
 z_1 z_2 z_3 - 1 = 0.
\end{cases} \]

We find two solutions: \((z_1 = -1, z_2 = -1, z_3 = +1)\) and \((z_1 = +1, z_2 = -1, z_3 = -1)\).

Two space curves \((t, -t^{-1}, -t, t^{-1})\) and \((t, t^{-1}, -t, -t^{-1})\) satisfy the entire cyclic 4-roots system.
overview of our polyhedral method

- finding pretropisms and solving initial forms
  
  Initial forms with at least two monomials in every equation define the intersection points of the solution set with the coordinate hyperplanes.

- unimodular coordinate transformations
  
  Via the Smith normal form we obtain nice representations for solutions at infinity. Solutions of binomial systems are monomial maps.

- computing terms of Puiseux series
  
  Although solutions to any initial forms may be monomial maps, in general we need a second term in the Puiseux series expansion to distinguish between
  
  - a positive dimensional solution set, and
  - an isolated solution at infinity.
some references


binomial systems

Definition

A binomial system has exactly two monomials with nonzero coefficient in every equation.

The binomial equation $c_a x^a - c_b x^b = 0$, $a, b \in \mathbb{Z}^n$, $c_a, c_b \in \mathbb{C} \setminus \{0\}$, has normal representation $x^{a-b} = c_b / c_a$.

A binomial system of $N$ equations in $n$ variables is then defined by an exponent matrix $A \in \mathbb{Z}^{N \times n}$ and a coefficient vector $c \in (\mathbb{C}^*)^N$: $x^A = c$.

Motivations to study binomial systems:

1. A unimodular coordinate transformation provides a monomial parametrization for the solution set.
2. The leading coefficients of a Puiseux series satisfy a system of binomial equations.
3. Finding all solutions with zero coordinates can happen via a generalized permanent calculation.
some references


an example

Consider as an example for $x^A = c$ the system

$$\begin{align*}
    x_0^2x_1x_2^4x_3^3 - 1 &= 0 \\
    x_0x_1x_2x_3 - 1 &= 0 \\
\end{align*}$$

$$A = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^T \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

As basis of the null space of $A$ we can for example take $u = (-3, 2, 1, 0)$ and $v = (-2, 1, 0, 1)$.

The vectors $u$ and $v$ are tropisms for a two dimensional algebraic set.

Placing $u$ and $v$ in the first two rows of a matrix $M$, extended so $\det(M) = 1$, we obtain a coordinate transformation, $x = y^M$:

$$M = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{array}{l}
    x_0 = y_0^{-3}y_1^{-2}y_2 \\
    x_1 = y_0^2y_1y_3 \\
    x_2 = y_0 \\
    x_3 = y_1.
\end{array}$$
monomial transformations

By construction, as $Au = 0$ and $Av = 0$:

$$MA = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = B.$$

The corresponding monomial transformation $x = y^M$ performed on $x^A = c$ yields $y^{MA} = y^B = c$, eliminating the first two variables:

$$\begin{cases} y_2^2 y_3 - 1 = 0 \\ y_2 y_3 - 1 = 0 \end{cases}.$$

Solving this reduced system gives values $z_2$ and $z_3$ for $y_2$ and $y_3$. Leaving $y_0$ and $y_1$ as parameters $t_0$ and $t_1$ we find as solution

$$(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1).$$
unimodular coordinate transformations

**Definition**

A *unimodular coordinate transformation* $x = y^M$ is determined by an invertible matrix $M \in \mathbb{Z}^{n \times n}$: $\det(M) = \pm 1$.

For a $d$ dimensional solution set of a binomial system:

1. The null space of $A$ gives $d$ tropisms, stored in the rows of a $d$-by-$n$-matrix $B$.
2. Compute the Smith normal form $S$ of $B$: $UBV = S$.
3. There are three cases:
   1. $U = I \Rightarrow M = V^{-1}$
   2. If $U \neq I$ and $S$ has ones on its diagonal, then extend $U^{-1}$ with an identity matrix to form $M$.
   3. Compute the Hermite normal form $H$ of $B$

and let $D$ be the diagonal elements of $H$, then $M = \begin{bmatrix} D^{-1}B \\ 0 & I \end{bmatrix}$.
To compute the degree of \((x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1)\)
we use two random linear equations:

\[
\begin{align*}
    c_{10} x_0 + c_{11} x_1 + c_{12} x_2 + c_{13} x_3 + c_{14} &= 0 \\
    c_{20} x_0 + c_{21} x_1 + c_{22} x_2 + c_{23} x_3 + c_{24} &= 0
\end{align*}
\]

after substitution:

\[
\begin{align*}
    c'_{10} t_0^{-3} t_1^{-2} + c'_{11} t_0^2 t_1 + c_{12} t_0 + c_{13} t_1 + c_{14} &= 0 \\
    c'_{20} t_0^{-3} t_1^{-2} + c'_{21} t_0^2 t_1 + c_{22} t_0 + c_{23} t_1 + c_{24} &= 0
\end{align*}
\]

Theorem (Koushnirenko’s Theorem)

*If all n polynomials in \(\mathbf{f}\) share the same Newton polytope \(P\), then the number of isolated solutions of \(\mathbf{f}(\mathbf{x}) = 0\) in \((\mathbb{C}^*)^n \leq \) the volume of \(P\).*

As the area of the Newton polygon equals 8, the surface has degree 8.
affine solution sets

An incidence matrix $M$ of a bipartite graph:

$$f(x) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \end{cases} \quad M[x^a, x_k] = \begin{cases} 1 & \text{if } a_k > 0 \\ 0 & \text{if } a_k = 0. \end{cases}$$

Meaning of $M[x^a, x_k] = 1$: $x_k = 0 \Rightarrow x^a = 0$.

The matrix linking monomials to variables is

$$M[x^a, x_k] = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$ 

Observe: overlapping columns $x_{12}$ with $x_{22}$ gives all ones.
enumerating all candidate affine solution sets

Apply row expansion on the matrix

$$M[x^a, x_k] = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\
x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\
x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\
x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\
x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}.$$  

- Selecting 1 means setting the corresponding variable to zero.
- Monomials must be considered in pairs: if one monomial in an equation vanishes, then so must the other one.
- For all affine sets, we must skip pairs of rows, preventing from certain variables to be set to zero.
- To decide whether one candidate set $C_1$ belongs to another set $C_2$, we construct the defining equations $I(C_1)$ and $I(C_2)$ and apply $C_1 \subseteq C_2 \iff I(C_1) \supseteq I(C_2)$. 

Jan Verschelde (UIC) solving with Puiseux series
the Cayley embedding – an example

\[
\begin{align*}
    p &= (x_0 - x_1^2)(x_0 + 1) = x_0^2 + x_0 - x_1^2 x_0 - x_1^2 = 0 \\
    q &= (x_0 - x_1^2)(x_1 + 1) = x_0 x_1 + x_0 - x_1^3 - x_1^2 = 0
\end{align*}
\]

The Cayley polytope is the convex hull of

\[
\{(2, 0, 0), (1, 0, 0), (1, 2, 0), (0, 2, 0)\} \cup \{(1, 1, 1), (1, 0, 1), (0, 3, 1), (0, 2, 1)\}.
\]
facet normals and initial forms

The Cayley polytope
has facets spanned by
one edge of the
Newton polygon of \( p \)
and
one edge of the
Newton polygon of \( q \).

Consider \( v = (2, 1, 0) \).

\[
\begin{align*}
\text{in}_{(2,1)}(p) &= \text{in}_{(2,1)} \left( x_0^2 + x_0 - x_1^2 x_0 - x_1^2 \right) = x_0 - x_1^2 \\
\text{in}_{(2,1)}(q) &= \text{in}_{(2,1)} \left( x_0 x_1 + x_0 - x_1^3 - x_1^2 \right) = x_0 - x_1^2
\end{align*}
\]
computing all pretropisms

**Definition**
A nonzero vector \( \mathbf{v} \) is a *pretropism* for the system \( \mathbf{f}(\mathbf{x}) = \mathbf{0} \) if \( \# \text{in}_\mathbf{v}(f_k) \geq 2 \) for all \( k = 0, 1, \ldots, N - 1 \).

Application of the Cayley embedding to \( (A_0, A_1, \ldots, A_{N-1}) \):

\[
E = \{ (a, 0) \mid a \in A_0 \} \cup \bigcup_{k=1}^{N-1} \{ (a, e_k) \mid a \in A_k \} \subset \mathbb{Z}^{n+N-1},
\]

where \( 0, e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_{N-1} = (0, 0, \ldots, 1) \) span the standard unit simplex in \( \mathbb{R}^{N-1} \).

The set of all facet normals to the convex hull of \( E \) contains all normals to facets spanned by at least two points of each support.

We used *cddlib* to compute all pretropisms of the cyclic \( n \)-roots system, up to \( n = 12 \) (148.5 hours on a 3.07GHz CPU with 4GB RAM).
cones of pretropisms

**Definition**

A *cone of pretropism* is a polyhedral cone spanned by pretropisms.

If we are looking for an algebraic set of dimension $d$ and

- if there are no cones of vectors perpendicular to edges of the Newton polytopes of $f(x) = 0$ of dimension $d$, then the system $f(x) = 0$ has no solution set of dimension $d$ that intersects the first $d$ coordinate planes properly; otherwise

- if a $d$-dimensional cone of vectors perpendicular to edges of the Newton polytopes exists, then that cone defines a part of the tropical prevariety.

For the cyclic 9-roots system, we found a two dimensional cone of pretropisms.
the tropical prevariety of cyclic $n$-roots

All facets normals of the Cayley polytope computed with cddlib on a 3.07GHz Linux computer with 4Gb RAM:

<table>
<thead>
<tr>
<th>$n$</th>
<th>#normals</th>
<th>#pretropisms</th>
<th>#generators</th>
<th>user cpu time</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>831</td>
<td>94</td>
<td>11</td>
<td>&lt; 1 sec</td>
</tr>
<tr>
<td>9</td>
<td>4,840</td>
<td>276</td>
<td>17</td>
<td>1 sec</td>
</tr>
<tr>
<td>12</td>
<td>907,923</td>
<td>38,229</td>
<td>290</td>
<td>148 hours 27 min</td>
</tr>
</tbody>
</table>

Tropical intersections with Gfan on a 2.26GHz MacBook:

<table>
<thead>
<tr>
<th>$n$</th>
<th>#rays</th>
<th>f-vector</th>
<th>user cpu time</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>94</td>
<td>1 94 108 48</td>
<td>15 sec</td>
</tr>
<tr>
<td>9</td>
<td>276</td>
<td>1 276 222 54</td>
<td>1 min 11 sec</td>
</tr>
<tr>
<td>12</td>
<td>5,582</td>
<td>1 5582 37786 66382 42540 8712</td>
<td>21 hours 1 min</td>
</tr>
</tbody>
</table>

Note that Gfan can exploit permutation symmetry.
Puiseux series for algebraic sets

Proposition

If \( f(x) = 0 \) is in Noether position and defines a \( d \)-dimensional solution set in \( \mathbb{C}^n \), intersecting the first \( d \) coordinate planes in regular isolated points, then there are \( d \) linearly independent tropisms \( v_0, v_1, \ldots, v_{d-1} \in \mathbb{Q}^n \) so that the initial form system

\[
\text{in}_{v_0}(\text{in}_{v_1}(\cdots \text{in}_{v_{d-1}}(f) \cdots))(x = y^M) = 0
\]

has a solution \( c \in (\mathbb{C} \setminus \{0\})^{n-d} \).

This solution and the tropisms are the leading coefficients and powers of a generalized Puiseux series expansion for the algebraic set:

\[
\begin{align*}
    x_0 &= t_0^{v_{0,0}} \\
    x_1 &= t_0^{v_{0,1}} t_1^{v_{1,1}} \\
    & \vdots \\
    x_{d-1} &= t_0^{v_{0,d-1}} t_1^{v_{1,d-1}} \cdots t_{d-1}^{v_{d-1,d-1}} \\
    x_d &= c_0 t_0^{v_{0,d}} t_1^{v_{1,d}} \cdots t_{d-1}^{v_{d-1,d}} + \cdots \\
    x_{d+1} &= c_1 t_0^{v_{0,d+1}} t_1^{v_{1,d+1}} \cdots t_{d-1}^{v_{d-1,d+1}} + \cdots \\
    & \vdots \\
    x_n &= c_{n-d-1} t_0^{v_{0,n-1}} t_1^{v_{1,n-1}} \cdots t_{d-1}^{v_{d-1,n-1}} + \cdots 
\end{align*}
\]
our polyhedral approach

For every $d$-dimensional cone $C$ of pretropisms:

1. We select $d$ linearly independent generators to form the $d$-by-$n$ matrix $A$ and the unimodular transformation $x = y^M$.

2. If $\text{in}_{v_0}(\text{in}_{v_1}(\cdots \text{in}_{v_{d-1}}(f) \cdots ))(x = y^M) = 0$ has no solution in $(\mathbb{C}^*)^{n-d}$, then return to step 1 with the next cone $C$, else continue.

3. If the leading term of the Puiseux series satisfies the entire system, then we report an explicit solution of the system and return to step 1 to process the next cone $C$. Otherwise, we take the current leading term to the next step.

4. If there is a second term in the Puiseux series, then we have computed an initial development for an algebraic set and report this development in the output.

Note: to ensure the solution of the initial form system is not isolated, it suffices to compute a series development for a curve.
our approach depicted in stages

inner normals

1. compute pretropisms
   - no tropism
     ⇒ no root at ∞

2. solve initial forms
   - no root at ∞
     ⇒ no series
   - singular roots
     ⇒ deflate factor

3. evaluate initial terms
   - initial term satisfies
     ⇒ a binomial factor

4. compute 2nd term
   - no series
     ⇒ no factor
relevant software

- **cddlib** by Komei Fukuda and Alain Prodon implements the double description method to efficiently enumerate all extreme rays of a general polyhedral cone.
- **Gfan** by Anders Jensen to compute Gröbner fans and tropical varieties uses **cddlib**.
- **The Singular library tropical.lib** by Anders Jensen, Hannah Markwig and Thomas Markwig for computations in tropical geometry.
- **Macaulay2 interfaces to Gfan**.
- **Sage interfaces to Gfan**.
- **PHCpack** *(published as Algorithm 795 ACM TOMS)* provides our numerical blackbox solver.
positive dimensional sets of cyclic $n$-roots

- $n = 8$: Tropisms, their cyclic permutations, and degrees:

\[
\begin{align*}
(1, -1, 1, -1, 1, -1, 1, -1) & \quad 8 \times 2 = 16 \\
(1, -1, 0, 1, 0, 0, -1, 0) \rightarrow (1, 0, 0, -1, 0, 1, -1, 0) & \quad 8 \times 2 + 8 \times 2 = 32 \\
(1, 0, -1, 0, 0, 1, 0, -1) \rightarrow (1, 0, -1, 1, 0, -1, 0, 0) & \quad 8 \times 2 + 8 \times 2 = 32 \\
(1, 0, -1, 1, 0, -1, 0, 0) \rightarrow (1, 0, -1, 0, 0, 1, 0, -1) & \quad 8 \times 2 + 8 \times 2 = 32 \\
(1, 0, 0, -1, 0, 1, -1, 0) \rightarrow (1, -1, 0, 1, 0, 0, -1, 0) & \quad 8 \times 2 + 8 \times 2 = 32 \\
\end{align*}
\]

TOTAL $= 144$

- $n = 9$: A 2-dimensional cone of tropisms spanned by 
  \[ v_0 = (1, 1, -2, 1, 1, -2, 1, 1, -2) \] and \[ v_1 = (0, 1, -1, 0, 1, -1, 0, 1, -1). \]
  Denoting by \( u = e^{i2\pi/3} \) the primitive third root of unity, \( u^3 - 1 = 0 \):

\[
\begin{align*}
x_0 &= t_0 \\
x_1 &= t_0 t_1 \\
x_2 &= u^2 t_0^{-2} t_1^{-1} \\
x_3 &= u t_0 \\
x_4 &= u t_0 t_1 \\
x_5 &= t_0^{-2} t_1^{-1} \\
x_6 &= u^2 t_0 \\
x_7 &= u^2 t_0 t_1 \\
x_8 &= u t_0^{-2} t_1^{-1}.
\end{align*}
\]

- $n = 12$: Computed 77 quadratic space curves.
results in the literature

Our results for $n = 9$ and $n = 12$ are in agreement with


a tropical version of Backelin’s Lemma

Lemma (Tropical Version of Backelin’s Lemma)

For \( n = m^2 \ell \), where \( \ell \in \mathbb{N} \setminus \{0\} \) and \( \ell \) is no multiple of \( k^2 \), for \( k \geq 2 \), there is an \((m-1)\)-dimensional set of cyclic \( n \)-roots, represented exactly as

\[
\begin{align*}
x_{km+0} &= u^k t_0 \\
x_{km+1} &= u^k t_0 t_1 \\
x_{km+2} &= u^k t_0 t_1 t_2 \\
& \vdots \\
x_{km+m-2} &= u^k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} &= \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{align*}
\]

for \( k = 0, 1, 2, \ldots, m-1 \), free parameters \( t_0, t_1, \ldots, t_{m-2} \), constants \( u = e^{\frac{i2\pi}{m\ell}} \), \( \gamma = e^{\frac{i\pi \beta}{m\ell}} \), with \( \beta = (\alpha \mod 2) \), and \( \alpha = m(m\ell - 1) \).
summary

Promising results on the cyclic $n$-roots problem give a proof of concept for a new polyhedral method to compute algebraic sets.

For the computation of pretropisms, we rely on

- `cddlib` on the Cayley embedding of the Newton polytopes, or
- `Gfan` for the tropical intersection.

To process the pretropisms, we

- use `Sage` to extract initial form systems and look for the second term in the Puiseux series;
- solve initial form systems with the blackbox solver of `PHCpack`. 