Polyhedral Methods to Solve Polynomial Systems II: computing mixed dimensional solution sets

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based on joint work with Danko Adrovic[1], Nathan Bliss[1], Anders Jensen[2], and Jeff Sommars[1]

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Outline

1. Introduction
   - an illustrative example
   - looking for solution sets
   - asymptotics of solution sets
2. Gauss-Newton for Power Series
   - linearization in the regular case
   - three scenarios in the singular case
   - the circles of Apollonius
3. Polyhedral Cascades to Process the Prevariety
   - reformulating polyhedral homotopies

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an illustrative example

$$f(x_0, x_1, x_2) = \begin{cases} (x_1 - x_0^2)(x_0^2 + x_1^2 + x_2^2 - 1)(x_0 - 0.5) = 0 \\ (x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_1 - 0.5) = 0 \\ (x_1 - x_0^2)(x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_2 - 0.5) = 0 \end{cases}$$

$$f^{-1}(\mathbf{0}) = Z = Z_2 \cup Z_1 \cup Z_0 = \{Z_{21}\} \cup \{Z_{11} \cup Z_{12} \cup Z_{13} \cup Z_{14}\} \cup \{Z_{01}\}$$

1. Z_{21} is the sphere $x_0^2 + x_1^2 + x_2^2 - 1 = 0$,
2. Z_{11} is the line $(x_0 = 0.5, x_2 = 0.5^3)$,
3. Z_{12} is the line $(x_0 = \sqrt{0.5}, x_1 = 0.5)$,
4. Z_{13} is the line $(x_0 = -\sqrt{0.5}, x_1 = 0.5)$,
5. Z_{14} is the twisted cubic $(x_1 - x_0^2 = 0, x_2 - x_0^3 = 0)$,
6. Z_{01} is the point $(x_0 = 0.5, x_1 = 0.5, x_2 = 0.5)$.

The Illustrative Example

numerically computing positive dimensional solution sets

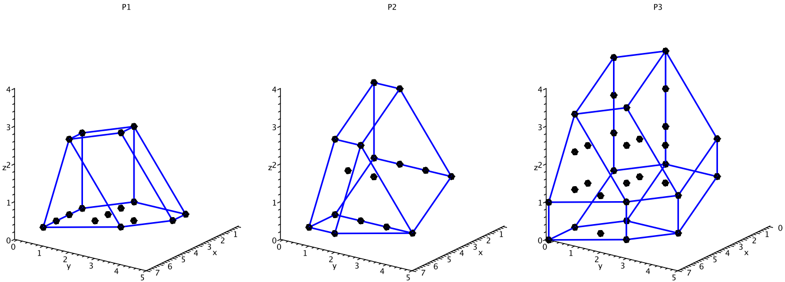
Used in two papers on numerical algebraic geometry:

- first cascade of homotopies: 197 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Numerical decomposition of the solution sets of polynomial systems into irreducible components*. SIAM J. Numer. Anal. 38(6):2022–2046, 2001.

- equation-by-equation solver: 13 paths
  A.J. Sommese, J. Verschelde, and C.W. Wampler: *Solving polynomial systems equation by equation*. In Algorithms in Algebraic Geometry, Volume 146 of The IMA Volumes in Mathematics and Its Applications, pages 133–152, Springer-Verlag, 2008.

The mixed volume of the Newton polytopes of this system is 124.
By theorem A of Bernshteĭn, the mixed volume is an upper bound on the number of isolated solutions.

three Newton polytopes



$$f(x_0, x_1, x_2) = \left\{ \begin{array}{r} (x_1 - x_0^2)(x_0^2 + x_1^2 + x_2^2 - 1)(x_0 - 0.5) = 0 \\ (x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_1 - 0.5) = 0 \\ (x_1 - x_0^2)(x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_2 - 0.5) = 0 \end{array} \right.$$

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looking for solution curves

The twisted cubic is $(x_0 = t, x_1 = t^2, x_2 = t^3)$.

We look for solutions of the form

$$\begin{cases} x_0 = t^{v_0}, & v_0 > 0, \\ x_1 = c_1 t^{v_1}, & c_1 \in \mathbb{C} \setminus \{0\}, \\ x_2 = c_2 t^{v_2}, & c_2 \in \mathbb{C} \setminus \{0\}. \end{cases}$$

Substitute $x_0 = t, x_1 = c_1 t^2, x_2 = c_2 t^3$ into f

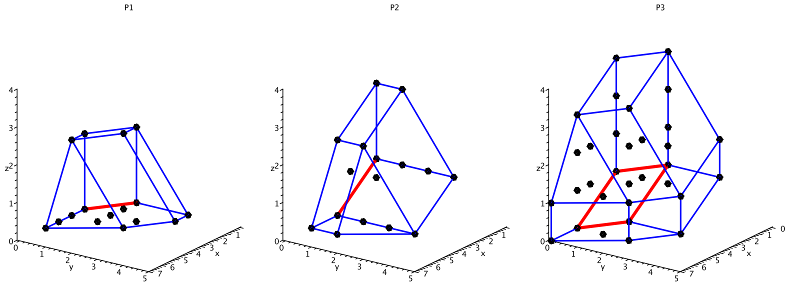
$$f(x_0 = t, x_1 = c_1 t^2, x_2 = c_2 t^3) = \begin{cases} (0.5c_1 - 0.5)t^2 + O(t^3) = 0 \\ (0.5c_2 - 0.5)t^3 + O(t^5) = 0 \\ 0.5(c_1 - 1.0)(c_2 - 1.0)t^5 + O(t^7) \end{cases}$$

$\rightarrow$ conditions on c_1 and c_2.

How to find $(v_0, v_1, v_2) = (1, 2, 3)$?

faces of Newton polytopes

Looking at the Newton polytopes in the direction $\mathbf{v} = (1, 2, 3)$:



Selecting those monomials supported on the faces

$$\mathrm{in}_{\mathbf{v}} f(x_0, x_1, x_2) = \begin{cases} 0.5x_1 - 0.5x_0^2 = 0 \\ 0.5x_2 - 0.5x_0^3 = 0 \\ -0.5x_1x_0^3 - 0.5x_2x_0^2 + 0.5x_2x_1 + 0.5x_0^5 = 0 \end{cases}$$

degenerating the sphere

$$f(x_0, x_1, x_2) = \begin{cases} (x_1 - x_0^2)(x_0^2 + x_1^2 + x_2^2 - 1)(x_0 - 0.5) = 0 \\ (x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_1 - 0.5) = 0 \\ (x_1 - x_0^2)(x_2 - x_0^3)(x_0^2 + x_1^2 + x_2^2 - 1)(x_2 - 0.5) = 0 \end{cases}$$

As $x_0 = t \rightarrow 0$:

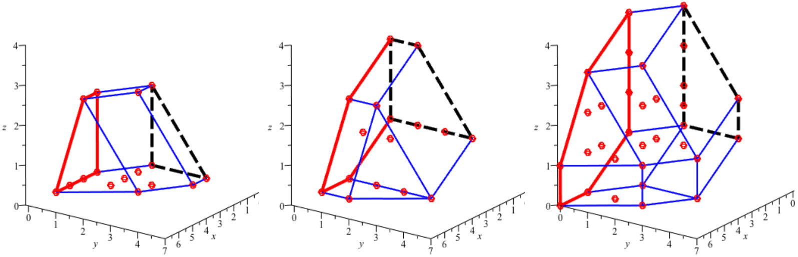
$$\text{in}_{(1,0,0)} f(x_0, x_1, x_2) \begin{cases} x_1(x_1^2 + x_2^2 - 1)(-0.5) = 0 \\ x_2(x_1^2 + x_2^2 - 1)(x_1 - 0.5) = 0 \\ x_1 x_2(x_1^2 + x_2^2 - 1)(x_2 - 0.5) = 0 \end{cases}$$

As $x_1 = s \rightarrow 0$:

$$\text{in}_{(0,1,0)} f(x_0, x_1, x_2) \begin{cases} -x_0^2(x_0^2 + x_2^2 - 1)(x_0 - 0.5) = 0 \\ (x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(-0.5) = 0 \\ -x_0^2(x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(x_2 - 0.5) = 0 \end{cases}$$

more faces of Newton polytopes

Looking at the Newton polytopes along $\mathbf{v} = \mathbf{(1,0,0)}$ and $\mathbf{v} = (0,1,0)$:



$$\mathrm{in}_{(1,0,0)} f(x_0, x_1, x_2) = \left\{ \begin{array}{r} x_1(x_1^2 + x_2^2 - 1)(-0.5) \\ x_2(x_1^2 + x_2^2 - 1)(x_1 - 0.5) \\ x_1 x_2(x_1^2 + x_2^2 - 1)(x_2 - 0.5) \end{array} \right.$$

$$\mathrm{in}_{(0,1,0)} f(x_0, x_1, x_2) = \left\{ \begin{array}{r} -x_0^2(x_0^2 + x_2^2 - 1)(x_0 - 0.5) \\ (x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(-0.5) \\ -x_0^2(x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(x_2 - 0.5) \end{array} \right.$$

faces of faces

The sphere degenerates to circles at the coordinate planes.

$$\mathrm{in}_{(1,0,0)} f(x_0, x_1, x_2) = \left\{ \begin{array}{r} x_1(x_1^2 + x_2^2 - 1)(-0.5) \\ x_2(x_1^2 + x_2^2 - 1)(x_1 - 0.5) \\ x_1 x_2(x_1^2 + x_2^2 - 1)(x_2 - 0.5) \end{array} \right.$$

$$\mathrm{in}_{(0,1,0)} f(x_0, x_1, x_2) = \left\{ \begin{array}{r} -x_0^2(x_0^2 + x_2^2 - 1)(x_0 - 0.5) \\ (x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(-0.5) \\ -x_0^2(x_2 - x_0^3)(x_0^2 + x_2^2 - 1)(x_2 - 0.5) \end{array} \right.$$

Degenerating even more:

$$\mathrm{in}_{(0,1,0)}\mathrm{in}_{(1,0,0)} f(x_0, x_1, x_2) = \left\{ \begin{array}{r} x_1(x_2^2 - 1)(-0.5) \\ x_2(x_2^2 - 1)(-0.5) \\ x_1 x_2(x_2^2 - 1)(x_2 - 0.5) \end{array} \right.$$

The factor $x_2^2 - 1$ is shared with $\mathrm{in}_{(1,0,0)}\mathrm{in}_{(0,1,0)} f(x_0, x_1, x_2)$.

representing a solution surface

The sphere is two dimensional, x_1 and x_2 are free:

$$\left\{ \begin{array}{l} x_0 = t_0 \\ x_1 = t_1 \\ x_2 = 1 + c_0 t_0^2 + c_1 t_1^2. \end{array} \right.$$

For $t_0 = 0$ and $t_1 = 0$, $x_2 = 1$ is a solution of $x^3 - 1 = 0$.

Substituting $(x_0 = t_0, x_1 = t_1, x_2 = 1 + c_0 t_0^2 + c_1 t_1^2)$ into the original system gives linear conditions on the coefficients of the second term: $c_0 = -0.5$ and $c_1 = -0.5$.

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asymptotics of solution sets

Getting generic points on a two dimensional surface:

$$\left\{ \begin{array}{c} f(\mathbf{x}) = 0 \\ c_{10} + c_{11}x_0 + c_{12}x_1 + c_{13}x_2 = 0 \\ c_{20} + c_{21}x_0 + c_{22}x_1 + c_{23}x_2 = 0 \end{array} \right. \rightarrow \left\{ \begin{array}{c} f(\mathbf{x}) = 0 \\ c_{10} + c_{11}x_0 = 0 \\ c_{20} + c_{22}x_1 = 0 \end{array} \right.$$

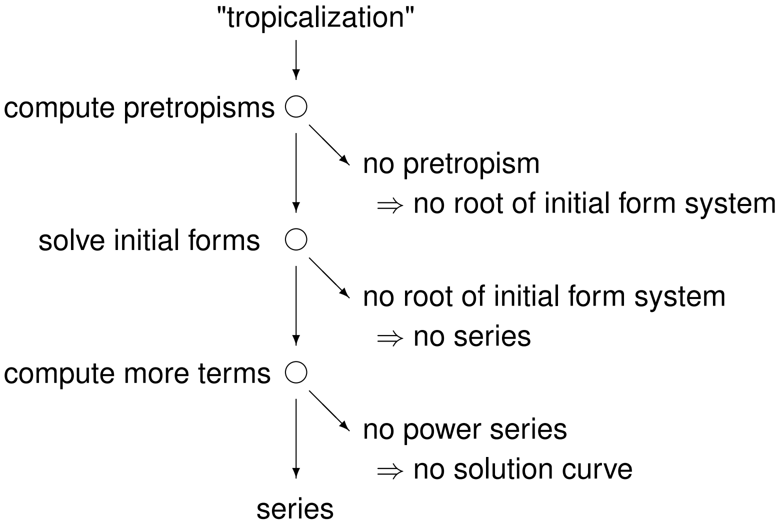
Specializing the two planes more:

$$\left\{ \begin{array}{c} f(\mathbf{x}) = 0 \\ x_0 = t_0 \\ x_1 = t_1 \end{array} \right.$$

As $t_0 \rightarrow 0$ and $t_1 \rightarrow 0$,
the leading powers of the power series solution define a tropism.

Computing a Series Expansion

a staggered approach to find a certificate for a regular solution curve



three separate stages

1. compute candidate tropisms:
   $\rightarrow$ a pretropism is perpendicular to a facet that is a sum of edges of the Newton polytopes

2. find the leading coefficients of the power series:

   1. change coordinates so one variable cancels
   2. apply a solver to a much sparser system

3. compute more terms of the power series:
   $\rightarrow$ run Newton's method applying linearization

Solving Polynomial Systems by Polyhedral Methods II

linearization

Working with truncated power series, computing modulo $O(t^d)$, is doing arithmetic over the field of formal Laurent series $C((t))$.

Linearization: consider $\mathbb{C}^n((t))$ instead of $\mathbb{C}((t))^n$. Instead of a vector of power series, we consider a power series with vectors as coefficients.

Solve $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} \in \mathbb{C}^{n \times n}((t))$, $\mathbf{b}, \mathbf{x} \in \mathbb{C}^n((t))$.

$$\begin{aligned} \mathbf{A} &= A_0 t^a + A_1 t^{a+1} + \cdots , \\ \mathbf{b} &= \mathbf{b}_0 t^b + \mathbf{b}_1 t^{b+1} + \cdots \\ \mathbf{x} &= \mathbf{x}_0 t^{b-a} + \mathbf{x}_1 t^{b-a+1} + \cdots \end{aligned}$$

where $A_i \in \mathbb{C}^{n \times n}$ and $\mathbf{b}_i, \mathbf{x}_i \in \mathbb{C}^n$.

block linear algebra

Computing the first d terms of the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\begin{array}{l} \left(A_0 t^a + A_1 t^{a+1} + A_2 t^{a+2} + \cdots + A_d t^{a+d}\right) \\ \cdot \ \left(\mathbf{x}_0 t^{b-a} + \mathbf{x}_1 t^{b-a+1} + \mathbf{x}_2 t^{b-a+2} + \cdots + \mathbf{x}_d t^{b-a+d}\right) \\ = \mathbf{b}_0 t^b + \mathbf{b}_1 t^{b+1} + \mathbf{b}_2 t^{b+2} + \cdots + \mathbf{b}_d t^{b+d}. \end{array}$$

Written in matrix format:

$$\begin{bmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_d & A_{d-1} & A_{d-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_d \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_d \end{bmatrix}.$$

If A_0 is regular, then solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ is straightforward.

biunimodular vectors and cyclic n-roots

$$
\begin{cases}
x_0 + x_1 + \cdots + x_{n-1} = 0 \\
i = 2, 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\
x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0.
\end{cases}
$$

The system arises in the study of biunimodular vectors.
A vector $\mathbf{u} \in \mathbb{C}^n$ of a unitary matrix A is biunimodular if for $k = 1, 2, \ldots, n$: $|u_k| = 1$ and $|v_k| = 1$ for $\mathbf{v} = A\mathbf{u}$.

- J. Backelin: *Square multiples n give infinitely many cyclic n-roots.* Technical Report, 1989.
- H. Führ and Z. Rzeszotnik. On biunimodular vectors for unitary matrices. *Linear Algebra and its Applications* 484:86–129, 2015.

series developments for cyclic 8-roots

Cyclic 8-roots has solution curves not reported by Backelin.

With Danko Adrovic (ISSAC 2012, CASC 2013): a tropism is $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$, the leading exponents of the series.

The corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ is

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} x_0 = z_0 \\ x_1 = z_1 z_0^{-1} \\ x_2 = z_2 \\ x_3 = z_3 z_0 \\ x_4 = z_4 \\ x_5 = z_5 \\ x_6 = z_6 z_0^{-1} \\ x_7 = z_7. \end{array}$$

Solving $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ gives the leading term of the series.

version 2.4.21 of PHCpack and 0.5.0 of phcpy

The source code (GNU GPL License) is availale at `github`.

After 2 Newton steps with `phc -u`, the series for z_1:

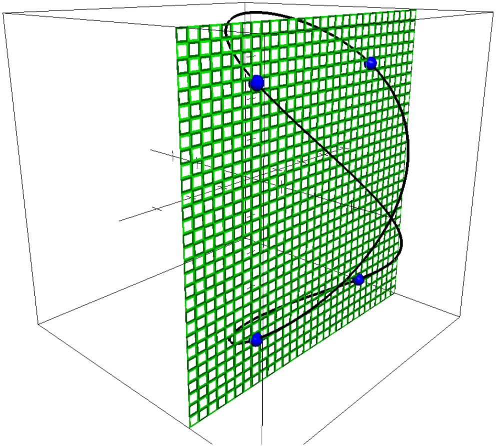
```
(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 2.37676980513323E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

After 3 Newton steps with `phc -u`, the series for z_1:

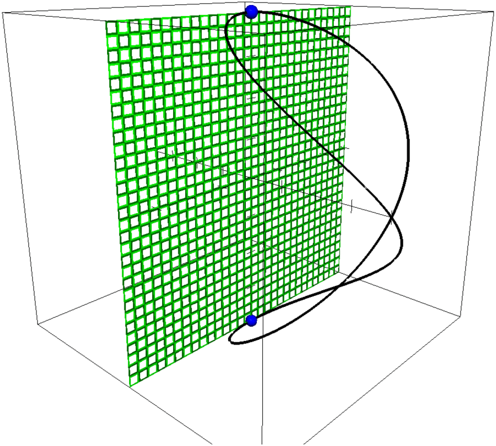
```
( 7.12500000000000E+00 + 7.12500000000000E+00*i)*z0^4
+(-1.52745512076048E-16 - 4.25000000000000E+00*i)*z0^3
+(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 1.45255178343636E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

Solving Polynomial Systems by Polyhedral Methods II

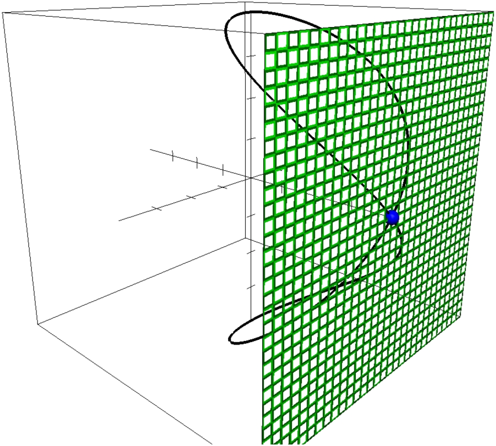
Viviani's curve – the regular case



Viviani's curve – two turning points



Viviani's curve – turning at a crossing point



three possible scenarios

We develop a power series for $x_0 = t$.

Geometric interpretation: we cut the curve with the plane perpendicular to the first coordinate axis.

We assume: the curve does not lie in the coordinate plane $x_0 = 0$.

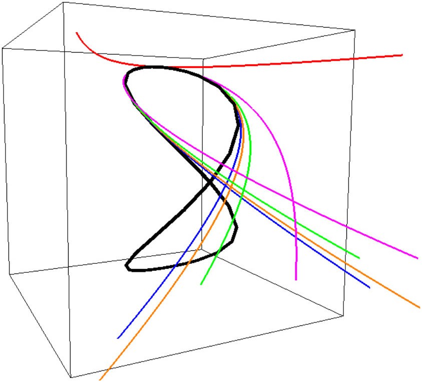
There are three different cases at an intersection point:

1. The plane cuts the curve transversally (regular).
2. The plane touches the curve at the point.
3. The plane intersects at a crossing point.

As in the crossing point of the Viviani curve,
the crossing point may occur at a turning point.

Viviani's curve at a turning point

Viviani's curve expanded around $(0, 0, 2)$:



Viviani's curve at a turning point

Consider:

$$\mathbf{f} = (x_0^2 + x_1^2 + x_2^2 - 4, (x_0 - 1)^2 + x_1^2 - 1), \quad \mathbf{p} = (0, 0, 2).$$

We apply the transformation $x_1 \to 2t^2$ and start from $\mathbf{z} = (2t, 2)$.

$$\begin{bmatrix} 4t & 4 \\ 4t & 0 \end{bmatrix} \Delta\mathbf{z} = - \begin{bmatrix} 4t^2 + 4t^4 \\ 4t^4 \end{bmatrix}.$$

The matrix is invertible over $\mathbb{C}((t))$.

Its inverse begins with negative exponents of t:

$$\begin{bmatrix} 0 & 1/4 \\ 1/4\ t^{-1} & -1/4\ t^{-1} \end{bmatrix}.$$

linearization

The linearized block form is

$$\left[\begin{array}{cc|cc|cc} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{array}\right] \mathbf{x} = \left[\begin{array}{r} -4 \\ 0 \\ 0 \\ 0 \\ -4 \\ -4 \end{array}\right].$$

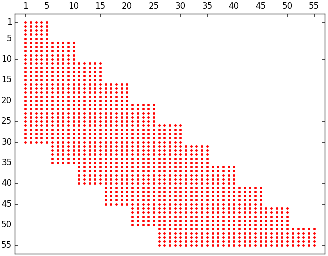
Solving gives the Newton update

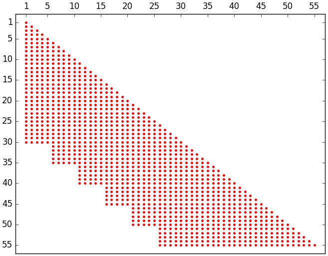
$$\Delta \mathbf{z} = \left[\begin{array}{c} -t^3 \\ -t^2 \end{array}\right].$$

Substituting $\mathbf{z} + \Delta\mathbf{z} = (2t - t^3, 2 - t^2)$ into the Viviani equations gives $(x_0^6 + x_0^4, x_0^6)$, the desired cancellation of terms.

Lower Triangular Echelon Form

The banded block structure of a generic matrix for $n = 5$ at the left, with its lower triangular echelon form at right:





Viviani's curve, continued

The block matrix reduction:

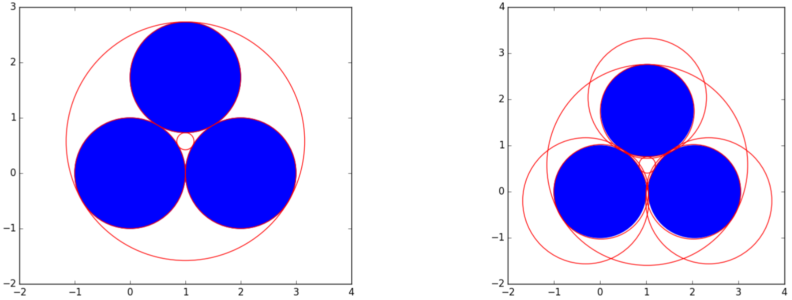
$$\begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 \end{bmatrix}.$$

Solving Polynomial Systems by Polyhedral Methods II

the circles of Apollonius

Given three circles, find all circles which touch all three given circles.

If the three given circles touch each other, then the solutions are the given circles (with multiplicity two) and other two regular circles.



a singular configuration of Apollonius circles

The system is $\mathbf{f}(t, x_0, x_1, r) =$

$$\left\{ \begin{array}{rcl} x_0^2 + 3x_1^2 - r^2 - 2r - 1 & = & 0 \\ x_0^2 + 3x_1^2 - r^2 - 4x_0 - 2r + 3 & = & 0 \\ 3t^2 + x_0^2 - 6tx_1 + 3x_1^2 - r^2 + 6t - 2x_0 - 6x_1 + 2r + 3 & = & 0. \end{array} \right.$$

We examine at the point $(t, x_0, x_1, r) = (0, 1, 1, 1) = \mathbf{p}$.

We obtain

$$\left\{ \begin{array}{rcl} x_0 & = & 1 \\ x_1 & = & 1 + 7.464t + 45.017t^2 + 290.992t^3 + \cdots \\ r & = & 1 + 11.196t + 77.971t^2 + 504.013t^3 + \cdots . \end{array} \right.$$

The growth of the coefficients explains why one circle grows large.

Solving Polynomial Systems by Polyhedral Methods II

a reformulation of polyhedral homotopies

We need to reformulate polyhedral homotopies.

1. Why? We need to explore cones of the prevariety.
   Polyhedral work only for square systems,
   systems that have the same number of equations as variables.
2. How? We intersect power series with a hypersurface.
   On a system with N equations in n variables:
   1. Compute power series for the first $N - 1$ equations.
   2. Intersect the power series with the Nth equation.

Will this work?

- Assuming Noether position for the curves defined by the first $N - 1$ equations, all isolated solutions can be computed.
- Curves are computed as a byproduct of the intersection.
- Intersecting the Nth equation also with the two dimensional surfaces defined by the first $N - 1$ equations gives curves.

Polyhedral Cascades

Cascading homotopies in a numerical irreducible decomposition compute generic points at all dimensions, in a top-down fashion, starting at the top dimension.

The computation of a tropical prevariety returns the maximal cones, i.e.: those that are not contained in any other polyhedral cones.

The second theorem of Bernshtein: diverging paths in a coefficient-parameter homotopy go to solutions of initial form systems.

Starting at the top dimensional cones, running polyhedral homotopies with power series leads to two cases:

1. converging paths give initial coefficients of power series;
2. diverging paths lead to lower dimensional cones, recurse.