The Method of Gauss-Newton to Compute Power Series Solutions of Polynomial Homotopies

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Outline

1. introduction
   - problem statement
   - motivation

2. methods for the regular case
   - linearization
   - series for cyclic 8-roots

3. methods for the singular case
   - Viviani's curve
   - scenarios
   - Apollonius circles

Gauss-Newton for Power Series

problem statement

Our focus is on algebraic curves:

- A polynomial homotopy is a family of polynomial systems.
- At least one of the variables in the homotopy is a parameter.
- Our polynomial homotopies have algebraic curves as solutions.

The input to our problem is

1. a polynomial homotopy $\mathbf{f}(x_0, x_1, x_2, \ldots, x_n) = \mathbf{0}$; and
2. a solution for $x_0 = 0$: $\mathbf{f}(0, z_1, z_2, \ldots, z_n) = \mathbf{0}$.

The output is a solution in the form of a power series:

$$\left\{ \begin{array}{l} x_0 = t^{v_0} \\ x_i = z_i t^{v_i}(1 + O(t)), i = 1, 2, \ldots, n. \end{array} \right.$$

We want to compute the terms of the series with Newton's method.

Gauss-Newton for Power Series

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motivation

Why do we want to do this?

Two good reasons:

1. The solution paths of a polynomial homotopy are algebraic curves.

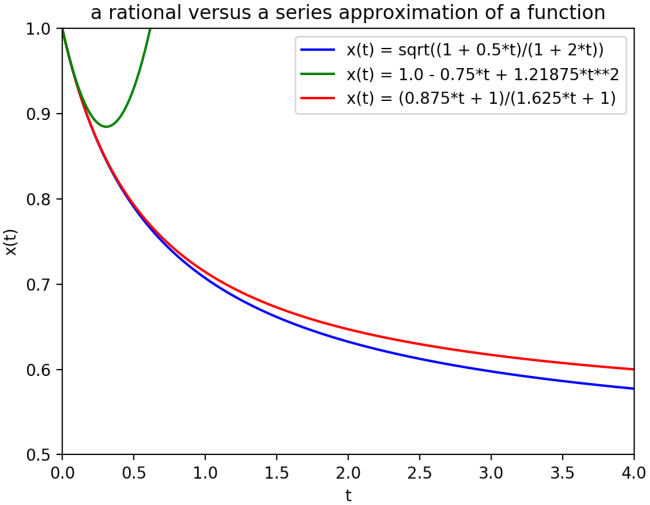
   We want to approximate those solution paths better to improve the accuracy and the reliability of numerical path trackers.

2. Solving a problem may require starting at a singular solution.

   Typically we move from a generic instance to a specific instance, but the introduction of randomness may destroy all structure.

Padé Approximants

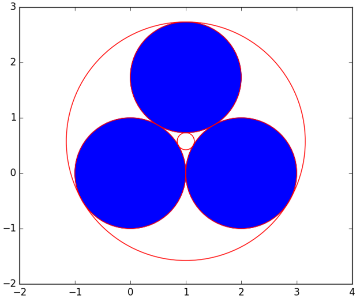
Consider the homotopy: $(1-t)(x^2-1)+t(3x^2-3/2)=0$.

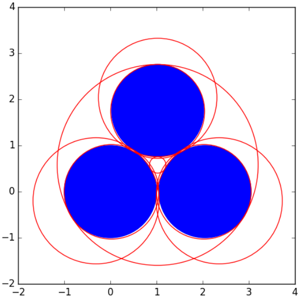


the circles of Apollonius

Given three circles, find all circles which touch all three given circles.

If the three given circles touch each other, then the solutions are the given circles (with multiplicity two) and other two regular circles.





numerical analysis and symbolic computation

- E. L. Allgower and K. Georg: Introduction to Numerical Continuation Methods. Volume 45 of *Classics in Applied Mathematics*, SIAM, 2003.
- A. Morgan: Solving polynomial systems using continuation for engineering and scientific problems. Volume 57 of *Classics in Applied Mathematics*, SIAM, 2009.

Newton-Hensel iteration is discussed in the following:

- J. Heintz, T. Krick, S. Puddu, J. Sabia, and A. Waissbein: Deformation techniques for efficient polynomial equation solving. *Journal of Complexity* 16(1):70-109, 2000.
- D. Castro, L.M. Pardo, K. Hägele, and J.E. Morais, Kronecker's and Newton's Approaches to Solving: A First Comparison. *Journal of Complexity* 17(1):212-303 2001.
- A. Bompadre, G. Matera, R. Wachenchauzer, and A. Waissbein: Polynomial equation solving by lifting procedures for ramified fibers. *Theoretical Computer Science* 315(2-3):335-369, 2004.

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linearization

Working with truncated power series, computing modulo $O(t^d)$, is doing arithmetic over the field of formal Laurent series $C((t))$.

Linearization: consider $\mathbb{C}^n((t))$ instead of $\mathbb{C}((t))^n$. Instead of a vector of power series, we consider a power series with vectors as coefficients.

Solve $\mathbf{Ax} = \mathbf{b}$, $\mathbf{A} \in \mathbb{C}^{n \times n}((t))$, $\mathbf{b}, \mathbf{x} \in \mathbb{C}^n((t))$.

$$
\begin{array}{rcl}
\mathbf{A} & = & A_0 t^a + A_1 t^{a+1} + \cdots , \\
\mathbf{b} & = & \mathbf{b}_0 t^b + \mathbf{b}_1 t^{b+1} + \cdots \\
\mathbf{x} & = & \mathbf{x}_0 t^{b-a} + \mathbf{x}_1 t^{b-a+1} + \cdots
\end{array}
$$

where $A_i \in \mathbb{C}^{n \times n}$ and $\mathbf{b}_i, \mathbf{x}_i \in \mathbb{C}^n$.

block linear algebra

Computing the first d terms of the solution of $\mathbf{Ax} = \mathbf{b}$:

$$\begin{array}{l} \left(A_0 t^a + A_1 t^{a+1} + A_2 t^{a+2} + \cdots + A_d t^{a+d}\right) \\ \cdot\ \left(\mathbf{x}_0 t^{b-a} + \mathbf{x}_1 t^{b-a+1} + \mathbf{x}_2 t^{b-a+2} + \cdots + \mathbf{x}_d t^{b-a+d}\right) \\ = \mathbf{b}_0 t^b + \mathbf{b}_1 t^{b+1} + \mathbf{b}_2 t^{b+2} + \cdots + \mathbf{b}_d t^{b+d}. \end{array}$$

Written in matrix format:

$$\begin{bmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ A_d & A_{d-1} & A_{d-2} & \cdots & A_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_d \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_d \end{bmatrix}.$$

If A_0 is regular, then solving $\mathbf{Ax} = \mathbf{b}$ is straightforward.

Gauss-Newton for Power Series

biunimodular vectors and cyclic n-roots

$$\left\{ \begin{array}{c} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ i = 2, 3, 4, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\ x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0. \end{array} \right.$$

The system arises in the study of biunimodular vectors.
A vector $\mathbf{u} \in \mathbb{C}^n$ of a unitary matrix A is biunimodular if for $k = 1, 2, \ldots, n$: $|u_k| = 1$ and $|v_k| = 1$ for $\mathbf{v} = A\mathbf{u}$.

- J. Backelin: *Square multiples n give infinitely many cyclic n-roots.* Technical Report, 1989.
- H. Führ and Z. Rzeszotnik. On biunimodular vectors for unitary matrices. *Linear Algebra and its Applications* 484:86–129, 2015.

series developments for cyclic 8-roots

Cyclic 8-roots has solution curves not reported by Backelin.

With Danko Adrovic (ISSAC 2012, CASC 2013): a tropism is $\mathbf{v} = (1, -1, 0, 1, 0, 0, -1, 0)$, the leading exponents of the series.

The corresponding unimodular coordinate transformation $\mathbf{x} = \mathbf{z}^M$ is

$$M = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} x_0 = z_0 \\ x_1 = z_1 z_0^{-1} \\ x_2 = z_2 \\ x_3 = z_3 z_0 \\ x_4 = z_4 \\ x_5 = z_5 \\ x_6 = z_6 z_0^{-1} \\ x_7 = z_7. \end{array}$$

Solving $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x} = \mathbf{z}^M) = \mathbf{0}$ gives the leading term of the series.

version 2.4.21 of PHCpack and 0.5.0 of phcpy

The source code (GNU GPL License) is availale at `github`.

After 2 Newton steps with `phc -u`, the series for z_1:

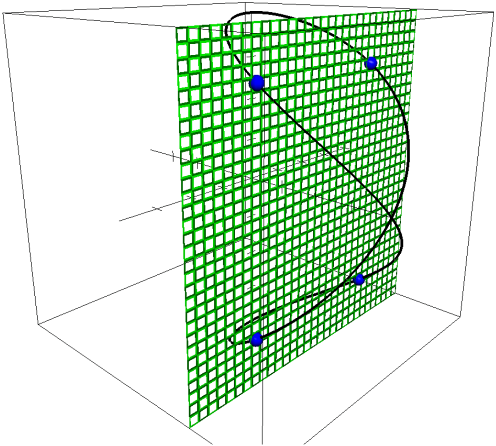
```
(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 2.37676980513323E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

After 3 Newton steps with `phc -u`, the series for z_1:

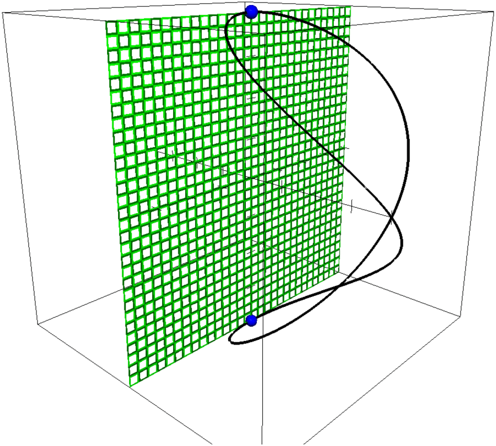
```
( 7.12500000000000E+00 + 7.12500000000000E+00*i)*z0^4
+(-1.52745512076048E-16 - 4.25000000000000E+00*i)*z0^3
+(-1.25000000000000E+00 + 1.25000000000000E+00*i)*z0^2
+( 5.00000000000000E-01 - 1.45255178343636E-17*i)*z0
+(-5.00000000000000E-01 - 5.00000000000000E-01*i);
```

Gauss-Newton for Power Series

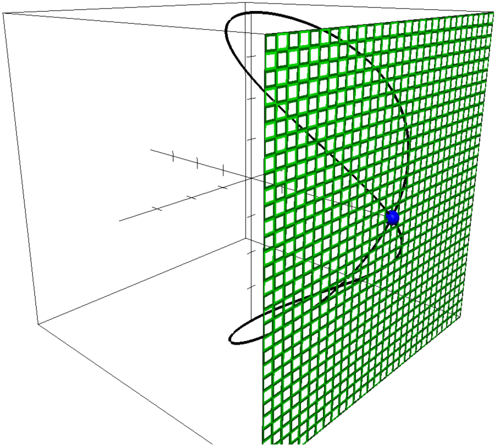
Viviani's curve – the regular case



Viviani's curve – two turning points



Viviani's curve – turning at a crossing point



Gauss-Newton for Power Series

three possible scenarios

We develop a power series for $x_0 = t$.

Geometric interpretation: we cut the curve with the plane perpendicular to the first coordinate axis.

We assume: the curve does not lie in the coordinate plane $x_0 = 0$.

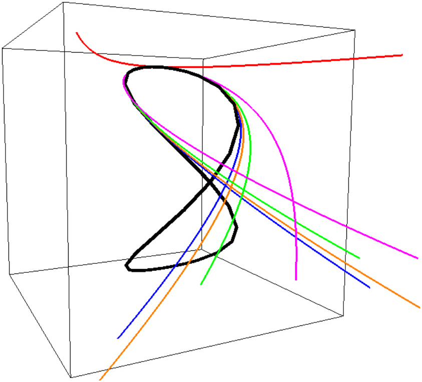
There are three different cases at an intersection point:

1. The plane cuts the curve transversally (regular).
2. The plane touches the curve at the point.
3. The plane intersects at a crossing point.

As in the crossing point of the Viviani curve,
the crossing point may occur at a turning point.

Viviani's curve at a turning point

Viviani's curve expanded around $(0, 0, 2)$:



Viviani's curve at a turning point

Consider:

$$\mathbf{f} = (x_1^2 + x_2^2 + x_3^2 - 4, (x_1 - 1)^2 + x_2^2 - 1), \quad \mathbf{p} = (0, 0, 2).$$

We apply the transformation $x_1 \rightarrow 2t^2$ and start from $\mathbf{z} = (2t, 2)$.

$$\left[\begin{array}{cc} 4t & 4 \\ 4t & 0 \end{array} \right] \Delta \mathbf{z} = - \left[\begin{array}{r} 4t^2 + 4t^4 \\ 4t^4 \end{array} \right].$$

The matrix is invertible over $\mathbb{C}((t))$.

Its inverse begins with negative exponents of t:

$$\left[\begin{array}{cc} 0 & 1/4 \\ 1/4\ t^{-1} & -1/4\ t^{-1} \end{array} \right].$$

linearization

The linearized block form is

$$\left[\begin{array}{cc|cc|cc} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{array}\right] \mathbf{x} = \left[\begin{array}{r} -4 \\ 0 \\ 0 \\ 0 \\ -4 \\ -4 \end{array}\right].$$

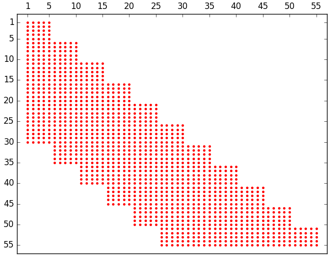
Solving gives the Newton update

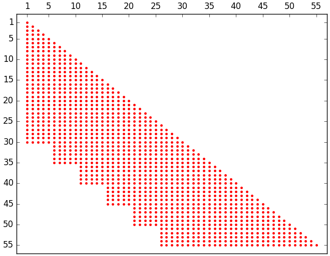
$$\Delta \mathbf{z} = \left[\begin{array}{c} -t^3 \\ -t^2 \end{array}\right].$$

Substituting $\mathbf{z} + \Delta \mathbf{z} = (2t - t^3, 2 - t^2)$ into the Viviani equations gives $(x_1^6 + x_1^4, x_1^6)$, the desired cancellation of terms.

Lower Triangular Echelon Form

The banded block structure of a generic matrix for $n = 5$ at the left, with its lower triangular echelon form at right:





Viviani's curve, continued

The block matrix reduction:

$$\begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 4 & 0 \end{bmatrix}.$$

Gauss-Newton for Power Series

a singular configuration of Apollonius circles

The system is $\mathbf{f}(t, x_1, x_2, r) =$

$$\left\{ \begin{array}{rcl} x_1^2 + 3x_2^2 - r^2 - 2r - 1 & = & 0 \\ x_1^2 + 3x_2^2 - r^2 - 4x_1 - 2r + 3 & = & 0 \\ 3t^2 + x_1^2 - 6tx_2 + 3x_2^2 - r^2 + 6t - 2x_1 - 6x_2 + 2r + 3 & = & 0. \end{array} \right.$$

We examine at the point $(t, x_1, x_2, r) = (0, 1, 1, 1) = \mathbf{p}$.

We obtain

$$\left\{ \begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 1 + 7.464t + 45.017t^2 + 290.992t^3 + \cdots \\ r & = & 1 + 11.196t + 77.971t^2 + 504.013t^3 + \cdots . \end{array} \right.$$

The growth of the coefficients explains why one circle grows large.