Computing Isolated Singular Solutions of Polynomial Systems using Newton's Method with Deflation

Jan Verschelde

Department of Math, Stat & CS
University of Illinois at Chicago
Chicago, IL 60607-7045, USA

Email: jan@math.uic.edu

URL: http://www.math.uic.edu/~jan

Symposium in honor of Tien-Yien Li's 60th Birthday

10-12 May 2005

Joint work with Anton Leykin and Ailing Zhao.

Computing Singular Isolated Roots

(Outline of the Talk)

1. Problem: Newton **fails** for singular roots.
   Our goal is to **restore quadratic convergence**.
2. **Deflation Algorithm**: add linear combinations of derivatives.
   We rely on **only one tolerance** to determine the rank.
3. Why it works: **#deflations $<$ multiplicity**.
   The deflation reduces #monomials under the staircase.
4. Implementation and Examples: **Reconditioning**.
   We use a **directed acyclic graph** of derivative operators.

Singularities are keeping us in business

numerical analysis: bifurcation points and endgames

Rall (1966); Reddien (1978); Decker-Keller-Kelley (1983);
Griewank-Osborne (1981); Hoy (1989);
Deuflard-Friedler-Kunkel (1987); Kunkel (1989, 1996);
Morgan-Sommese-Wampler (1991); Li-Wang (1993, 1994);
Govaerts (2000).

computer algebra: standard bases (SINGULAR)

Mora (1982); Greuel-Pfister (1996)

numerical polynomial algebra: multiplicity structure

Möller-Stetter (1995); Mourrain (1997);
Stetter-Thallinger (1998); Dayton-Zeng (2005)

deflation: Ojika-Watanabe-Mitsui (1983); Lecerf (2003)

A Motivating Example: cyclic 9-roots

The system

$$f(\mathbf{x}) = \begin{cases} f_i = \displaystyle\sum_{j=0}^{8} \prod_{k=1}^{i} x_{(k+j) \bmod 9} = 0, \quad i = 1, 2, \ldots, 8 \\ f_9 = x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 - 1 = 0 \end{cases}$$

has 333×18 **isolated regular** zeros, 164 **isolated 4-fold** zeros, and 6 **cubic 2-dimensional** irreducible solution components.

Newton's method with 64 decimal places, tolerance is 10^{-60}:

regular : 4 iterations (**quadratic convergence**)

4-fold : 79 iterations ($>$ 1 **step for one correct decimal place**)

about 20 times slower to reach same magnitude of residual ...

Multiplicity of an Isolated Zero

An isolated zero of multiplicity m occurs in numerical analysis as a cluster of m (ill-conditioned) regular zeros.

Problem: geometrical significance for overdetermined systems?
$\longrightarrow$ perturbed overdetermined system has no zeros!

Analogy with Univariate Case: z_0 is m-fold zero of $f(x) = 0$:

$$\underbrace{f(z_0) = 0, \frac{\partial f}{\partial x}(z_0) = 0, \frac{\partial^2 f}{\partial x^2}(z_0) = 0, \ldots, \frac{\partial^{m-1} f}{\partial x^{m-1}}(z_0) = 0}\qquad .$$

m = number of linearly independent polynomials annihilating z_0

The dual space D_0 at $\mathbf{z}_0$ is spanned by m linear independent differentiation functionals annihilating $\mathbf{z}_0$.

D_0 is **the multiplicity structure** of the m-fold zero $\mathbf{z}_0$.

A Simple Example

Consider

$$f(x,y) = \begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \end{cases} \qquad \mathbf{z}_0 = (0,0).$$

The **multiplicity of $\mathbf{z}_0$ is 3** because

$$D_0 = \mathrm{span}\{\partial_{00}[\mathbf{z}_0], \partial_{10}[\mathbf{z}_0], \partial_{01}[\mathbf{z}_0]\}$$

with

$$\partial_{ij}[\mathbf{z}_0] = \frac{1}{i!j!}\ \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(\mathbf{z}_0).$$

Solving means to compute $\mathbf{z}_0$ ***and*** D_0.

Newton's Method for Overdetermined Systems

Singular Value Decomposition of N-by-n Jacobian matrix J_f:

$$J_f = U\Sigma V^T, \qquad U \text{ and } V \text{ are orthogonal: } U^TU = I_N, V^TV = I_n,$$

and singular values $\boldsymbol{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n}$ as the only nonzero elements on the diagonal of the N-by-n matrix Σ $(N > n)$.

The **condition number** $\mathrm{cond}(J_f(\mathbf{z})) = \frac{\sigma_1}{\sigma_n}$.

$\mathrm{Rank}(J_f(\mathbf{z})) = R \iff \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_R, 0, \ldots, 0)$.

At a **multiple root** $\mathbf{z}_0$: $\mathrm{Rank}(J_f(\mathbf{z}_0)) = R < n$.

Close to $\mathbf{z}_0$, $\mathbf{z} \approx \mathbf{z}_0 : \sigma_{R+1} \approx 0$, or $\boldsymbol{|\sigma_{R+1}| < \epsilon}$, ϵ is tolerance.

Moore-Penrose inverse: $J_f^+ = V\Sigma^+U^T$, with $R = \mathrm{Rank}(J_f)$,

and $\Sigma^+ = \mathrm{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_R}, 0, \ldots, 0)$.

Then $\Delta\mathbf{z} = -J_f(\mathbf{z})^+ f(\mathbf{z})$ is the least squares solution.

Dedieu-Shub (1999); Li-Zeng (2005)

Newton with Deflation – Simple Example revisited

$$f(x,y) = \begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \end{cases} \qquad J_f(x,y) = \begin{bmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{bmatrix} \qquad \begin{array}{l} \mathbf{z}_0 = (0,0), m = 3 \\ \mathrm{Rank}(J_f(\mathbf{z}_0)) = 0 \end{array}$$

A nontrivial linear combination of the columns of $J_f(\mathbf{z}_0)$ is zero.

$$G(x,y,\lambda_1,\lambda_2) = \begin{cases} f(x,y) & = & 0 \\ \begin{bmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ c_1\lambda_1 + c_2\lambda_2 & = & 1, \qquad \text{random } c_1, c_2 \in \mathbb{C} \end{cases}$$

The system $G(x,y,\lambda_1,\lambda_2) = 0$ has $(0, 0, \lambda_1^*, \lambda_2^*)$ as **regular** zero!

Deflation Operator Dfl reduces to Corank One

Suppose $\mathrm{Rank}(J_f(\mathbf{z}_0)) = R$ for $\mathbf{z}_0$ an isolated zero of $f(\mathbf{x}) = 0$.

Choose $\mathbf{h} \in \mathbb{C}^{R+1}$ and $B \in \mathbb{C}^{n \times (R+1)}$ at random.

Introduce $R+1$ new multiplier variables $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{R+1})$.

$$Dfl(f)(\mathbf{x}, \boldsymbol{\lambda}) := \left\{ \begin{array}{rcl} f(\mathbf{x}) & = & \mathbf{0} \\ J_f(\mathbf{x}) B \boldsymbol{\lambda} & = & \mathbf{0} \\ \mathbf{h}\boldsymbol{\lambda} & = & 1 \end{array} \right. \qquad \begin{array}{c} \mathrm{Rank}(J_f(\mathbf{x})) = R \\ \Downarrow \\ \mathrm{corank}(J_f(\mathbf{x})B) = 1 \end{array}$$

Theorem (Anton Leykin, JV, Ailing Zhao):

The number of deflations needed to restore the quadratic convergence of Newton's method converging to an isolated solution is strictly less than the multiplicity.

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

$[J_f^+, R] :=$ SVD$(J_f(\mathbf{x}_k), \epsilon)$;
$\mathbf{x}_{k+1} := \mathbf{x}_k - J_f^+ f(\mathbf{x}_k)$;

$R = \#\text{columns}(J_f)$?

Yes → Output: $f; \mathbf{x}_{k+1}$.

No ↓

$$f := Dfl(f)(\mathbf{x}, \boldsymbol{\lambda}) = \left\{ \begin{array}{c} f(\mathbf{x}) = \mathbf{0} \\ G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0} \end{array} \right. ;$$

$\widehat{\boldsymbol{\lambda}} :=$ LeastSquares$(G(\mathbf{x}_{k+1}, \boldsymbol{\lambda}))$;
$k := k + 1; \quad \mathbf{x}_k := (\mathbf{x}_k, \widehat{\boldsymbol{\lambda}})$;

cyclic 9-roots revisited

Recall:

$$f(\mathbf{x}) = \begin{cases} f_i = \displaystyle\sum_{j=0}^{8} \prod_{k=1}^{i} x_{(k+j) \bmod 9} = 0, \quad i = 1, 2, \ldots, 8 \\ f_9 = x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 - 1 = 0 \end{cases}$$

has 164 solutions of multiplicity 4.

One deflation suffices to restore quadratic convergence.

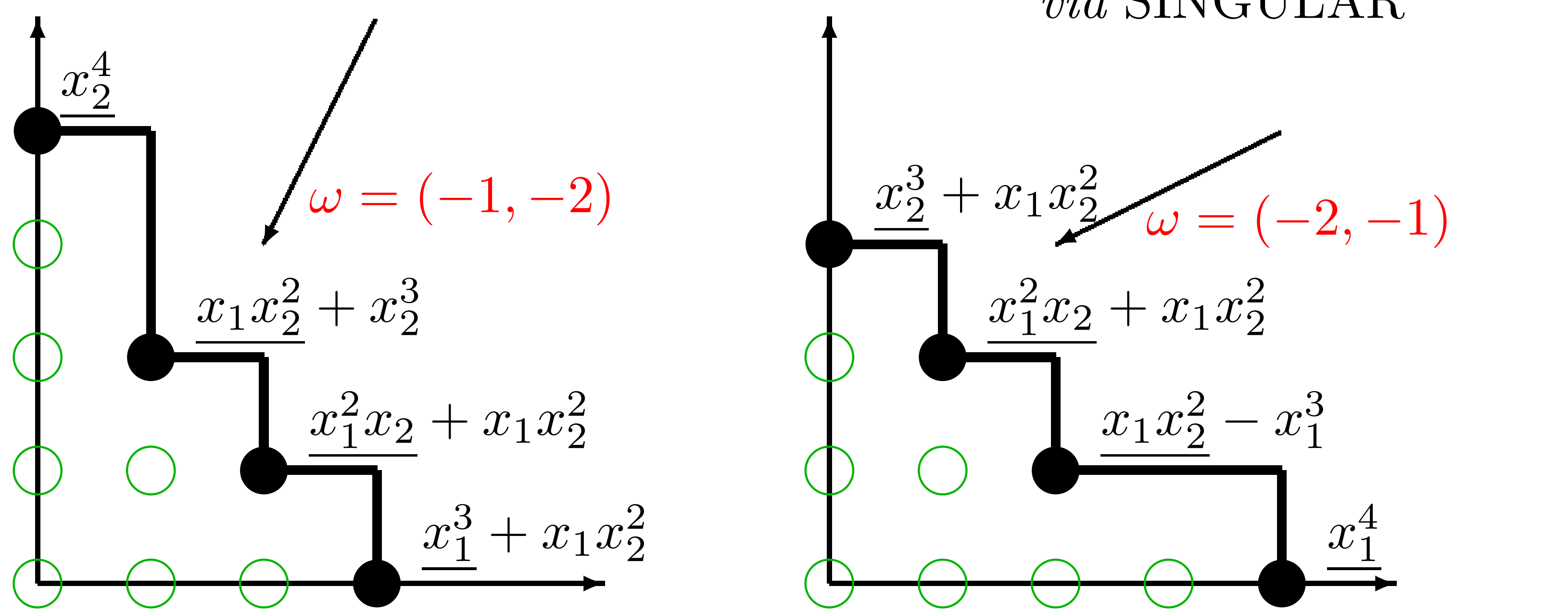
The **condition number** drops from 1.8E+9 to 5.6E+2.

$\rightarrow$ **deflation <u>reconditions</u> the system**

Two Staircases with Different Local Ordering

Example: $I = \langle x_1^3 + x_1x_2^2, x_1x_2^2 + x_2^3, x_1^2x_2 + x_1x_2^2\rangle$ in the ring $\mathbb{Q}[x_1, x_2]$, $\mathbf{x}^* = \mathbf{0}$, ω defines the monomial order.

via SINGULAR



●: monomials generating $\mathbf{in}_\omega(I)$ ○: standard monomials

$$\boxed{\textbf{\#standard monomials = multiplicity of } \mathbf{x}^* = \mathbf{7}}$$

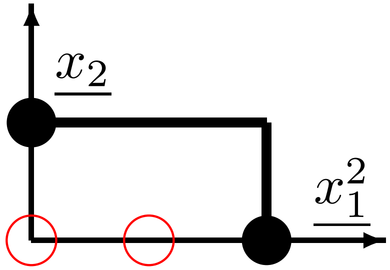
Standard Bases and Dual Space

Consider $\begin{cases} x_1^2 + 2x_2^2 - 2x_2 = 0 \\ x_1x_2^2 - x_1x_2 = 0 \\ x_2^3 - 2x_2^2 + x_2 = 0 \end{cases}$ from Möller-Stetter (1995).

$\mathbf{z}_0 = (0,0)$

$\boldsymbol{m_0 = 2}$

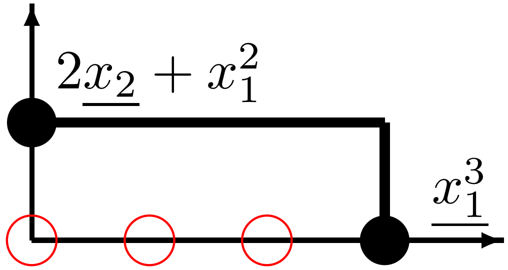
$D_0 = \mathrm{span}\{\partial_{00}, \partial_{10}\}$



$\mathbf{z}_1 = (0,1)$ (shift to (0,0))

$\boldsymbol{m_1 = 3}$

$D_1 = \mathrm{span}\{\partial_{00}, \partial_{10}, 2\partial_{20} - \partial_{01}\}$

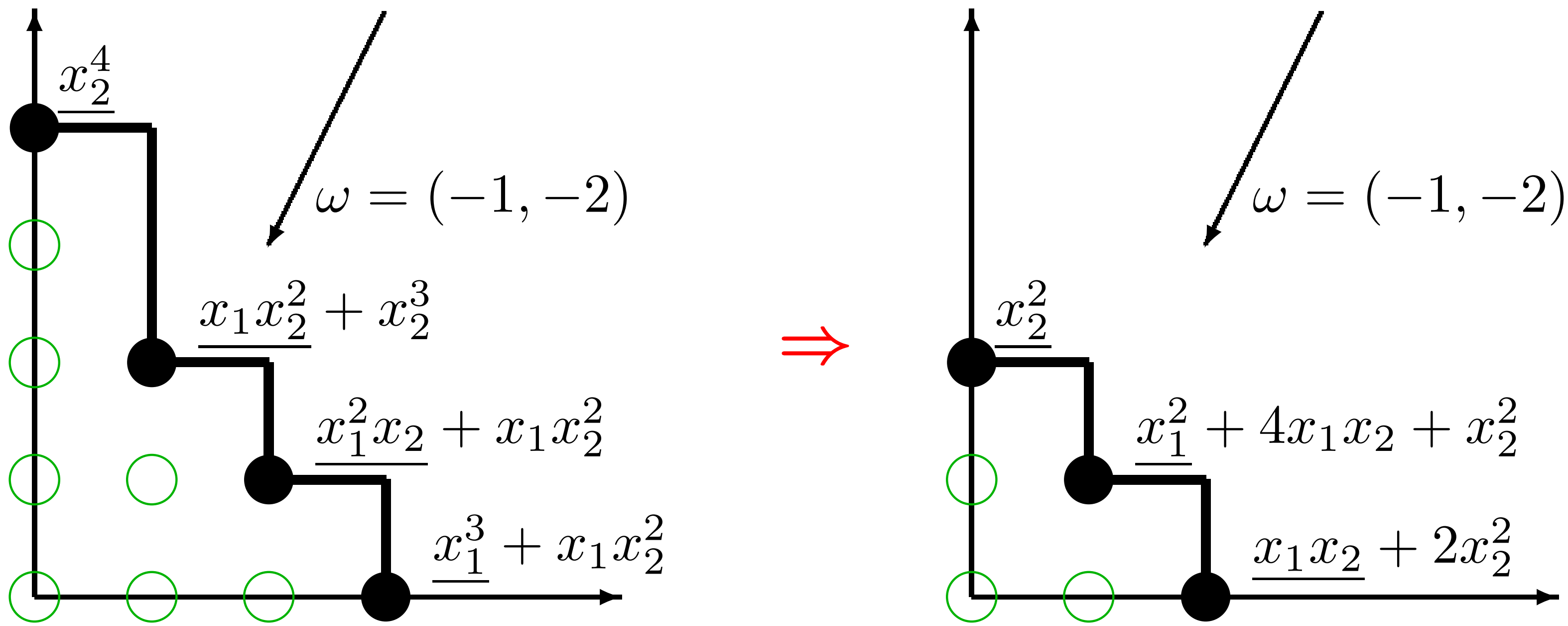


$$D[I] = D_0 \cup D_1$$

Effect of Deflation on the Staircase

$$I = \langle f_1 = x_1^3 + x_1x_2^2, f_2 = x_1x_2^2 + x_2^3, f_3 = x_1^2x_2 + x_1x_2^2 \rangle, \boldsymbol{\lambda} = (1,1).$$

$J = \langle f_1, f_2, f_3, \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial x_2}, \frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \frac{\partial f_3}{\partial x_1} + \frac{\partial f_3}{\partial x_2} \rangle$ is a **deflation** of I.



●: monomials generating $\mathbf{in}_\omega(I)$ ○: standard monomials

$\boldsymbol{m = 7}$ ⟶ $\boldsymbol{m = 3}$

deflation

One Deflation Step with fixed $\boldsymbol{\lambda}$

- Assume $\mathrm{corank}(A(\mathbf{x}^*)) = 1$.
  (reduce to this case with random combinations of columns)
- Let $\boldsymbol{\lambda} \in \ker(A(\mathbf{x}^*))$, $\boldsymbol{\lambda} \neq \mathbf{0}$,
  then for $g_i(\mathbf{x}) = \boldsymbol{\lambda} \cdot \nabla f_i = \displaystyle\sum_{j=1}^{n} \lambda_j \frac{\partial f_i}{\partial x_j}(x)$, we have: $g_i(\mathbf{x}^*) = \mathbf{0}$.

<u>**Theorem**</u>:

The augmented system $\left\{ \begin{array}{l} f_1 = f_2 = \cdots = f_N = 0 \\ g_1 = g_2 = \cdots = g_N = 0 \end{array} \right.$

has $\mathbf{x}^*$ as isolated root of lower multiplicity.

Proposition: Suppose $m > 1$ and let $g \in \mathcal{B}$, a reduced standard basis of I with respect to a local monomial ordering $\leq$, such that $g = x_i^d$ + lower order terms, for $i \in \{1, 2, \dots, n\}$ and $d > 1$. Then $I' = I + \langle \frac{\partial g}{\partial x_i} \rangle$ is a **deflation** of I.

Lemma: Take a nonzero vector $\boldsymbol{\lambda} \in \ker A(\mathbf{0}) \subset \mathbb{C}^n$ and let $\mathbf{x} = T(\mathbf{y})$ be a linear coordinate transformation such that

$$y_i = \lambda_i x_1 + \sum_{j=2}^{n} \mu_{ij} x_j, \qquad \text{for } i = 1, 2, \dots, n,$$

where $\mathbf{y} = (y_1, \dots, y_n)$ are the new variables and $[\boldsymbol{\lambda}, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_n]$ is a nonsingular matrix.

Let $T(I) = \{f(T(\mathbf{y})) \mid f \in I\} = \langle f_1(T(\mathbf{y})), \dots, f_N(T(\mathbf{y})) \rangle$ be the ideal after the change of coordinates.

Then $\partial_1 T(I) = \left\{ \frac{\partial f}{\partial y_1} \mid f \in T(I) \right\}$ leads to a **deflation** of $T(I)$.

One Deflation Step with indeterminate $\boldsymbol{\lambda}$

- Still assuming $\text{corank}(A(\mathbf{x}^*)) = 1$.
- Denote $G(\mathbf{x}, \boldsymbol{\lambda}) = \begin{cases} g_i(\mathbf{x}, \boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \nabla f_i(\mathbf{x}) = \mathbf{0} \\ \langle \mathbf{h}, \boldsymbol{\lambda} \rangle = h_1\lambda_1 + h_2\lambda_2 + \cdots + h_n\lambda_n = 1. \end{cases}$

Theorem:

Let $\mathbf{x}^* \in \mathbb{C}^n$ be an isolated singular root of $f(\mathbf{x}) = \mathbf{0}$ with multiplicity m. There exists a unique $\boldsymbol{\lambda}^*$ such that $\begin{cases} f(\mathbf{x}) = \mathbf{0} \\ G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0} \end{cases}$ has $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ as isolated root of multiplicity strictly less than m.

Proof: Consider $G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ in the local ring $R_* = \mathbb{C}[\mathbf{x}, \boldsymbol{\lambda}]_{(\mathbf{x}^*, \boldsymbol{\lambda}^*)}$.
Because $G(\mathbf{x}, \boldsymbol{\lambda})$ is linear in $\boldsymbol{\lambda}$, specializing $\mathbf{x} = \mathbf{x}^*$ turns $G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ into a linear system with unique solution $\boldsymbol{\lambda}^*$.

Using row operations in R_*, reduce $G(\mathbf{x}, \boldsymbol{\lambda})$ to the form :
$$\left\{ \begin{array}{c} \lambda_1 = a_1(\mathbf{x}) \\ \vdots \\ \lambda_n = a_n(\mathbf{x}) \end{array} \right.$$

where $a_i(\mathbf{x})$ are rational expressions $(a_i(\mathbf{x}^*) = \lambda_i^*)$.

multiplicity of $\mathbf{x}^*$ in $\left\{ \begin{array}{c} f(\mathbf{x}) = 0 \\ G(\mathbf{x}, \boldsymbol{\lambda}) = 0 \end{array} \right.$ $\Leftrightarrow$ multiplicity of $\mathbf{x}^*$ in $\left\{ \begin{array}{c} f(\mathbf{x}) = 0 \\ G(\mathbf{x}, \boldsymbol{\lambda}^*) = 0 \end{array} \right.$

local ring $\mathbb{C}[\mathbf{x}, \boldsymbol{\lambda}]_{(\mathbf{x}^*, \boldsymbol{\lambda}^*)}$ local ring $\mathbb{C}[\mathbf{x}]_{(\mathbf{x}^*)}$

Computing the Multiplicity Structure

following B.H. Dayton and Z. Zeng

Looking for differentiation functionals $d[\mathbf{z}_0] = \sum_{\mathbf{a}} c_{\mathbf{a}} \partial_{\mathbf{a}}[\mathbf{z}_0]$,

$$\text{with} \quad \partial_{\mathbf{a}}[\mathbf{z}_0](p) = \frac{1}{a_1! a_2! \cdots a_n!} \left(\frac{\partial^{a_1+a_2+\cdots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}} p \right) (\mathbf{z}_0).$$

Membership criterium for $d[\mathbf{z}_0]$:

$$\boldsymbol{d[\mathbf{z}_0] \in D_0 \Leftrightarrow d[\mathbf{z}_0](p f_i) = 0, \forall p \in \mathbb{C}[\mathrm{x}], i = 1, 2, \ldots, N.}$$

To turning this criterium into an **algorithm**, observe:

1. since $d[\mathbf{z}_0]$ is linear, restrict p to $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$; and
2. limit degrees $k_1 + k_2 + \cdots + k_n \leq a_1 + a_2 + \cdots + a_n$, as $\mathbf{z}_0 = \mathbf{0}$ vanishes trivially if not annihilated by $\partial_{\mathbf{a}}$.

Computing the Multiplicity Structure – An Example

$f_1 = x_1 - x_2 + x_1^2,\ f_2 = x_1 - x_2 + x_2^2$ following B.H. Dayton and Z. Zeng

		$\lvert a\rvert=0$	$\lvert a\rvert=1$		$\lvert a\rvert=2$			$\lvert a\rvert=3$			
		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}	∂_{30}	∂_{21}	∂_{12}	∂_{03}
	f_1	0	1	–1	1	0	0	0	0	0	0
S_1	f_2	0	1	–1	0	0	1	0	0	0	0
	$x_1 f_1$	0	0	0	1	–1	0	1	0	0	0
	$x_1 f_2$	0	0	0	1	–1	0	0	0	1	0
	$x_2 f_1$	0	0	0	0	1	–1	0	1	0	0
S_2	$x_2 f_2$	0	0	0	0	1	–1	0	0	0	1
	$x_1^2 f_1$	0	0	0	0	0	0	1	–1	0	0
	$x_1^2 f_2$	0	0	0	0	0	0	1	–1	0	0
	$x_1 x_2 f_1$	0	0	0	0	0	0	0	1	–1	0
	$x_1 x_2 f_2$	0	0	0	0	0	0	0	1	–1	0
	$x_2^2 f_1$	0	0	0	0	0	0	0	0	1	–1
S_3	$x_2^2 f_2$	0	0	0	0	0	0	0	0	1	–1

Nullity(S_2) = Nullity(S_3) $\Rightarrow$ stop algorithm

$D_0 = \text{span}\{\ \partial_{00}, \partial_{10} + \partial_{01}, -\partial_{10} + \partial_{20} + \partial_{11} + \partial_{02}\ \} \Rightarrow$ multiplicity $= 3$

cyclic 9-roots once more

Recall:

$$f(\mathbf{x}) = \begin{cases} f_i = \displaystyle\sum_{j=0}^{8} \prod_{k=1}^{i} x_{(k+j) \bmod 9} = 0, \quad i = 1, 2, \ldots, 8 \\ f_9 = x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 - 1 = 0 \end{cases}$$

has 164 solutions of multiplicity 4.

Running the algorithm of Dayton and Zeng:

$$\boldsymbol{H[1] = 1, H[2] = 2, H[3] = 1, H[4] = 0,}$$

$$\textbf{with } \boldsymbol{H[i] = \mathrm{\textbf{Nullity}}(S_i) - \mathrm{\textbf{Nullity}}(S_{i-1}), i > 0,}$$

so we compute the multiplicity as 4.

Avoiding Expression Swell

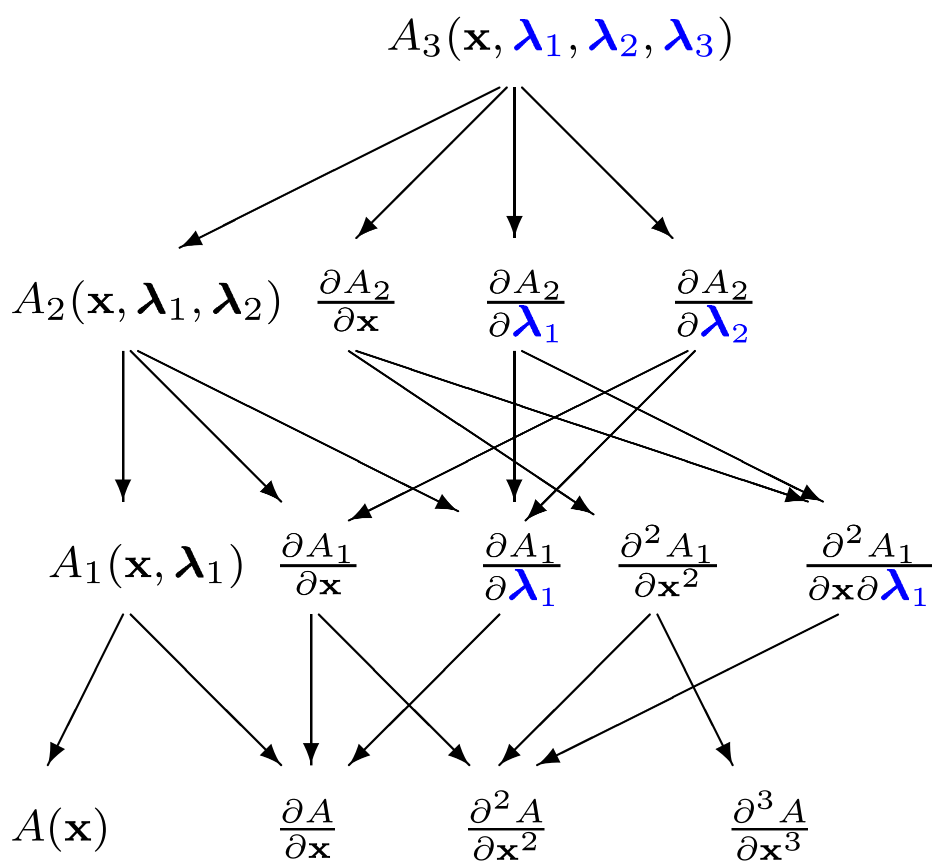
Evaluation of $A(\mathbf{x})B$: for efficiency we must first replace $\mathbf{x}$ by values ***before*** the matrix multiplication.

Triangular block structure of Jacobian matrix: for example:

$$\left[\begin{array}{ccc} A & \mathbf{0} & \mathbf{0} \\ \left(\frac{\partial A}{\partial \mathbf{x}}\right) B^{(1)} \boldsymbol{\lambda}_1 & AB^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{h}^{(1)} & \mathbf{0} \\ \left(\frac{\partial A^{(1)}}{\partial \mathbf{x}}\right) B^{(2)} \boldsymbol{\lambda}_2 & \left(\frac{\partial A^{(1)}}{\partial \boldsymbol{\lambda}_1}\right) B^{(2)} \boldsymbol{\lambda}_2 & A^{(1)} B^{(2)} \boldsymbol{\lambda}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{h}^{(2)} \end{array}\right].$$

Multipliers occur linearly: compute derivatives only with respect to $\mathbf{x}$, not with respect to $\boldsymbol{\lambda}$.

A Directed Acyclic Graph of Derivative Operators



Numerical Results (double float)

System	n	m	D	corank$(A(\mathbf{x}))$	Inverse Condition#	#Digits
baker1	2	2	1	$1 \to 0$	1.7e-08 $\to$ 3.8e-01	$9 \to 24$
cbms1	3	11	1	$3 \to 0$	4.2e-05 $\to$ 5.0e-01	$5 \to 20$
cbms2	3	8	1	$3 \to 0$	1.2e-08 $\to$ 5.0e-01	$8 \to 18$
mth191	3	4	1	$2 \to 0$	1.3e-08 $\to$ 3.5e-02	$7 \to 13$
decker1	2	3	2	$1 \to 1 \to 0$	3.4e-10 $\to$ 2.6e-02	$6 \to 11$
decker2	2	4	3	$1 \to 1 \to 1 \to 0$	4.5e-13 $\to$ 6.9e-03	$5 \to 16$
decker3	2	2	1	$1 \to 0$	4.6e-08 $\to$ 2.5e-02	$8 \to 17$
ojika1	2	3	2	$1 \to 1 \to 0$	9.3e-12 $\to$ 4.3e-02	$5 \to 12$
ojika2	3	2	1	$1 \to 0$	3.3e-08 $\to$ 7.4e-02	$6 \to 14$
ojika3	3	2	1	$1 \to 0$	1.7e-08 $\to$ 9.2e-03	$7 \to 15$
		4	1	$2 \to 0$	6.5e-08 $\to$ 8.0e-02	$6 \to 13$
ojika4	3	3	2	$1 \to 1 \to 0$	1.9e-13 $\to$ 2.4e-04	$6 \to 11$
cyclic9	9	4	1	$2 \to 0$	5.6e-10 $\to$ 1.8e-03	$5 \to 15$

What is Symbolic-Numeric Computing?

A puristic point of view:

Computer algebra rewrites the problem, producing "easier" equations of the ideal, but **"easier" $\neq$ numerically better**.

Numerical analysis produces approximate numbers for a fixed system of equations, but **many problems are "ill-posed"**.

The synergistic approach:

Symbolic-Numeric Computing rewrites an ill-conditioned numerical problem into a well-conditioned formulation.

works very well in Newton's method with deflation

List of References

E.L. Allgower and K. Georg: **Introduction to Numerical Continuation Methods**, volume 45 of *Classics in Applied Mathematics*. *SIAM*, 2003.

B.H. Dayton and Z. Zeng: **Computing the Multiplicity Structure in Solving Polynomial Systems.** To appear in ISSAC'05.

D.W. Decker, H.B. Keller, and C.T. Kelley: **Convergence rates for Newton's method at singular points.** *SIAM J. Numer. Anal.*, 20(2):296–314, 1983.

J.P. Dedieu and M. Shub: **Newton's method for overdetermined systems of equations.** *Math. Comp.*, 69(231):1099–1115, 1999.

W.J.F. Govaerts: **Numerical Methods for Bifurcations of Dynamical Equilibria.** *SIAM*, 2000.

G.M. Greuel and G. Pfister: **Advances and improvements in the theory of standard bases and syzygies.** *Arch. Math.*, 66:163–196, 1996.

A. Griewank and M.R. Osborne: **Analysis of Newton's method at irregular singularities.** *SIAM J. Numer. Anal.* 20(4):747–773, 1983.

A. Hoy: **An efficiently implementable Gauss-Newton-like method for solving singular nonlinear equations.** *Computing* 41:107–122, 1989.

P. Kunkel: **Efficient computation of singular points.** *IMA J. Numer. Anal.* 9:421–433, 1989.

P. Kunkel: **A tree-based analysis of a family of augmented systems for the computation of singular points.** *IMA J. Numer. Anal.* 16:501–527, 1996.

G. Lecerf: **Quadratic Newton iteration for systems with multiplicity.** *Found. Comput. Math.*, 2:247–293, 2002.

A. Leykin and J. Verschelde: **PHCmaple: A Maple interface to the numerical homotopy algorithms in PHCpack.** In Quoc-Nam Tran, editor, *Proceedings of the Tenth International Conference on Applications of Computer Algebra (ACA'2004)*, pages 139–147, 2004.

A. Leykin, J. Verschelde, and A. Zhao: **Newton's Method with Deflation for Isolated Singularities of Polynomial Systems.** Manuscript, 2004.

T.Y. Li and X. Wang: **Solving real polynomial systems with real homotopies.** *Math. Comp.* 60:669–680, 1993.

T.Y. Li and X. Wang: **Higher order turning points.** *Appl. Math. Comput.*, 64:155–166, 1994.

T.Y. Li and Z. Zeng: **A rank-revealing method with updating, downdating and applications.** *SIAM J. Matrix Anal. Appl.* 26(4):918–946, 2005.

H.M. Möller and H.J. Stetter: **Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems.** *Numer. Math.* 70:311–329, 1995.

F. Mora: **An algorithm to compute the equations of tangent cones.** In J. Calmet, editor, *Computer Algebra. EUROCAM'82, European Computer Algebra Conference. Marseille, France, April 1982*, volume 144 of *Lecture Notes in Computer Science*, pages 158–165. Springer-Verlag, 1982.

A.P. Morgan, A.J. Sommese and C.W. Wampler: **Computing singular solutions to nonlinear analytic systems.** *Numer. Math.* 58(7):669–684, 1991.

B. Mourrain: **Isolated points, duality and residues.** *Journal of Pure and Applied Algebra* 117/118:469–493, 1997.

T. Ojika, S. Watanabe, and T. Mitsui: **Deflation algorithm for the multiple roots of a system of nonlinear equations.** *J. Math. Anal. Appl.* 96:463–479, 1983.

L.B. Rall: **Convergence of the Newton Process to Multiple Solutions.** *Numer. Math.* 9:23–37, 1966.

G.W. Reddien: **On Newton's method for singular problems.** *SIAM J. Numer. Anal.* 15(5):993–996, 1978.

H.J. Stetter and G.H. Thallinger: **Singular Systems of Polynomials.** In *ISSAC'98*, pages 9–16, ACM 1998.

H.J. Stetter: **Numerical Polynomial Algebra.** SIAM, 2004.

J. Verschelde: **Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.** *ACM Transactions on Mathematical Software* 25(2):251–276, 1999. Software available at `http://www.math.uic.edu/~jan`.