Numerical Computations in Algebraic Geometry

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Outline

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2. Zero Dimensional Solving
   - homotopy continuation methods
   - deflating isolated singularities
3. Numerical Irreducible Decomposition
   - witness sets and cascades of homotopies
   - monodromy grouping certified by linear traces
4. Numerical Primary Decomposition

Numerically Solving Polynomial Systems

problems and methods in numerical algebraic geometry

Numerical algebraic geometry applies numerical analysis to problems in algebraic geometry.

Algebraic geometry studies solutions of polynomial systems, therefore the main problem is to solve polynomial systems.

Numerical analysis is concerned with the efficiency and accuracy of mathematical algorithms using floating-point arithmetic.

Three levels of solving a polynomial system:

1. focus on the isolated complex solutions
2. output contains also positive dimensional solution sets
3. a numerical primary decomposition

Why use numerical analysis?

costs and benefits of the numerical approach

Computational algebraic geometry mainly uses symbolic computing:

- resultants and Gröbner bases are well developed,
- already excellent software tools in computer algebra,
- exact computations are more trustworthy.

Two benefits of numerical analysis:

- accepts approximate input, offers sensitivity analysis

  many “real-world” applications have approximate input data
  approximating algebraic numbers leads to speedups as well
- pleasingly parallel algorithms for high performance computing

  with Python possible to do interactive parallel computing

Three References

most relevant for this talk

1. **Tien-Yien Li:** Numerical solution of polynomial systems by homotopy continuation methods. In Volume XI of *Handbook of Numerical Analysis*, pp. 209–304, 2003.
2. **Andrew J. Sommese and Charles W. Wampler:** The Numerical Solution of Systems of Polynomials Arising in Engineering and Science. World Scientific, 2005.
3. **Anton Leykin:** Numerical Primary Decomposition. `arXiv:0801.3105v2 [math.AG] 29 May 2008`. To appear in the proceedings of ISSAC 2008.

Zero Dimensional Solving

specification of input and output

Input: $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_n) \in (\mathbb{C}[\mathbf{x}])^n$.

The polynomial system $f(\mathbf{x}) = \mathbf{0}$ has as many equations as unknowns.

In the output of a numerical zero dimensional solver
we ***expect*** to find two types of solutions in $\mathbb{C}^n$:

1. regular: Jacobian matrix is of full rank,
2. multiple: some isolated solutions coincide.

But we may find also two other types of solutions:

1. at infinity: if we have ***fewer*** solutions than expected,
2. on component: if we have ***more*** solutions than expected.

Solving Simpler Systems

to get to the expected number of isolated complex solutions

How many isolated complex solutions do we expect?

Apply the method of degeneration:

1. Bézout:
   $\longrightarrow$ deform equations into products of linear equations
2. Bernshteĭn, Kushnirenko, Khovanskiĭ:
   $\longrightarrow$ deform system into initial binomial systems

Embed the target problem into a family of similar problems.

Solve the simpler systems and follow paths of solutions starting at the solutions of the simpler systems to the solutions of the target problem.

Tracking Solution Paths

using predictor-correct methods

Given is a family of systems: $h(\mathbf{x}, t) = \mathbf{0}$, a homotopy.

A typical form of a homotopy to solve $f(\mathbf{x}) = \mathbf{0}$ is

$$h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0},$$

where $g(\mathbf{x}) = \mathbf{0}$ is a simpler *good* system.
The parameter t is an artificial continuation parameter.

Three key algorithmic ingredients:

1. Newton's method
2. predictor-corrector methods
3. endgames to deal with singularities

Puiseux Series in Endgames

Solving $f(\mathbf{x}) = \mathbf{0}$ using start system $g(\mathbf{x}) = \mathbf{0}$:

$$h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}, \quad t \rightarrow 1.$$

Puiseux series of $\mathbf{x}(t)$: $h(\mathbf{x}(t), t) \equiv \mathbf{0}$:

$$\left\{ \begin{array}{ll} x_i(s) = b_i s^{v_i}(1 + O(s)) & b_i \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, v_i \in \mathbb{Z} \\ t(s) = 1 - s^\omega & s \rightarrow 0, t \rightarrow 1 \end{array} \right.$$

Fractional powers if the winding number $\omega > 1$.

Observe how the leading exponents v_i determine $x_i(s)$:

$$v_i < 0 : x_i(s) \rightarrow \infty, \quad v_i = 0 : x_i(s) \rightarrow b_i, \quad v_i > 0 : x_i(s) \rightarrow 0.$$

Compute v_i: $\log(|x_i(s_1)|) = \log(|b_i|) + v_i \log(s_1)$,
$\log(|x_i(s_2)|) = \log(|b_i|) + v_i \log(s_2)$, extrapolate.

Solutions of Initial Forms

Substitute $x_i(s) = b_i s^{v_i}(1 + O(s))$ into $f(\mathbf{x}) = \mathbf{0}$.

$$f_k(\mathbf{x}) = \sum_{\mathbf{a} \in A_k} c_\mathbf{a} \mathbf{x}^\mathbf{a}, \quad A_k \text{ is support of } f_k.$$

Substitute $x_i(s)$ into $\mathbf{x}^\mathbf{a} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$:

$$\prod_{i=1}^{n} (x_i(s))^{a_i} = \prod_{i=1}^{n} (b_i)^{a_i} s^{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n} (1 + O(s)).$$

As $s \rightarrow 0$, those monomials that matter are those for which $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \langle \mathbf{a}, \mathbf{v} \rangle$ is minimal.

$$\mathrm{in}_\mathbf{v}(f_k) = \sum_{\substack{\mathbf{a} \in A_k \\ \langle \mathbf{a}, \mathbf{v} \rangle \text{ is minimal}}} c_\mathbf{a} \mathbf{x}^\mathbf{a} \text{ vanishes at } (b_1, b_2, \ldots, b_n).$$

A Simple Example

but a difficult one for basic numerical methods

$$f(x,y) = \begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \end{cases}$$

$(0,0)$ is an isolated root of multiplicity 3

Randomization or Embedding:

$$\begin{cases} x^2 + \gamma_1 y^2 = 0 \\ xy + \gamma_2 y^2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} x^2 + \gamma_1 z = 0 \\ xy + \gamma_2 z = 0 \\ y^2 + \gamma_3 z = 0, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ are random numbers and z is a slack variable,

raises the multiplicity from 3 to 4!

Deflation for Isolated Singular Solutions

restoring quadratic convergence of Newton's method

input: $f(\mathbf{x}) = \mathbf{0}$ a polynomial system;
$\mathbf{x}^*$ an approximate solution: $f(\mathbf{x}^*) \approx \mathbf{0}$

- **Stage 1**: recondition the problem

  output: $G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ an extension to f;
  $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a **regular** solution: $G(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0} \Rightarrow f(\mathbf{x}^*) = \mathbf{0}$.

- **Stage 2**: compute the multiplicity

  input: $g(\mathbf{x}) = \mathbf{0}$ a "good" system for $\mathbf{x}^*$
  Newton converges quadratically starting from $\mathbf{x}^*$.

  Apply algorithms of [Dayton & Zeng, 2005]
  or [Bates, Peterson & Sommese, 2006].

output: a quadratically convergent method to refine $\mathbf{x}^*$
and the multiplicity structure of $\mathbf{x}^*$

Deflation Operator Dfl reduces to Corank One

Consider $f(\mathbf{x}) = \mathbf{0}$, N equations in n unknowns, $N \geq n$.

Suppose $\mathrm{Rank}(A(\mathbf{z}_0)) = R < n$ for $\mathbf{z}_0$ an isolated zero of $f(\mathbf{x}) = 0$.

Choose $\mathbf{h} \in \mathbb{C}^{R+1}$ and $B \in \mathbb{C}^{n\times(R+1)}$ at random.
Introduce $R + 1$ new multiplier variables $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{R+1})$.

$$\mathbf{Dfl}(f)(\mathbf{x}, \boldsymbol{\lambda}) := \left\{ \begin{array}{rcl} f(\mathbf{x}) & = & \mathbf{0} \\ A(\mathbf{x})B\boldsymbol{\lambda} & = & \mathbf{0} \\ \mathbf{h}\boldsymbol{\lambda} & = & 1 \end{array} \right. \qquad \begin{array}{c} \mathrm{Rank}(A(\mathbf{x})) = R \\ \Downarrow \\ \mathrm{corank}(A(\mathbf{x})B) = 1 \end{array}$$

The operator **Dfl** is used recursively if necessary, note:
(1) # times bounded by multiplicity
(2) symbolic implementation easy, but leads to expression swell
(3) exploiting the structure for evaluation is efficient

The Simple Example – with deflation

reconditioning the multiple solution

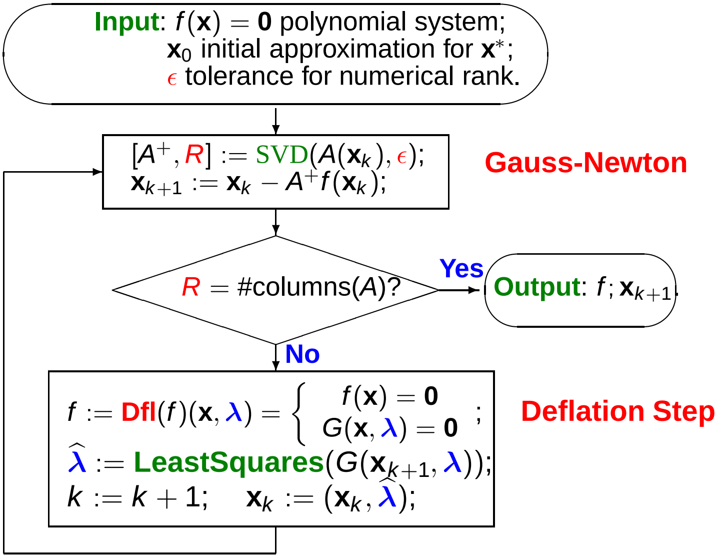
$$f(x,y) = \begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \end{cases} \quad J_f(x,y) = \begin{bmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{bmatrix} \quad \begin{array}{c} \mathbf{z}_0 = (0,0), m = 3 \\ \mathrm{Rank}(J_f(\mathbf{z}_0)) = 0 \end{array}$$

A nontrivial linear combination of the columns of $J_f(\mathbf{z}_0)$ is zero.

$$F(x,y,\lambda_1) = \begin{cases} f(x,y) & = & 0 \\ \begin{bmatrix} 2x & 0 \\ y & x \\ 0 & 2y \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \lambda_1 & = & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, & \text{random } b_{11}, b_{21} \\ h_1 \lambda_1 & = & 1, & \text{random } h_1 \in \mathbb{C} \end{cases}$$

The system $F(x,y,\lambda_1) = 0$ has $(0, 0, \lambda_1^*)$ as **regular** zero!

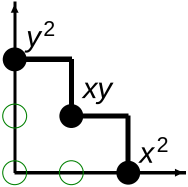
Newton's Method with Deflation



Multiplicity of an Isolated Zero via Duality

- **Analogy with Univariate Case:** z_0 is $\boldsymbol{m}$-fold zero of $f(x) = 0$:
$$\underbrace{f(z_0) = 0, \frac{\partial f}{\partial x}(z_0) = 0, \frac{\partial^2 f}{\partial x^2}(z_0) = 0, \ldots, \frac{\partial^{m-1} f}{\partial x^{m-1}}(z_0) = 0}.$$
$\boldsymbol{m}$ = number of linearly independent polynomials annihilating z_0
- The dual space $\boldsymbol{D_0}$ at $\mathbf{z}_0$ is spanned by $\boldsymbol{m}$ linear independent differentiation functionals annihilating $\mathbf{z}_0$.

Consider again $f(x,y) = \begin{cases} x^2 = 0 \\ xy = 0 \\ y^2 = 0 \end{cases}$



The **multiplicity of $\mathbf{z}_0 = (0,0)$ is 3** because

$$D_0 = \mathrm{span}\{\partial_{00}[\mathbf{z}_0], \partial_{10}[\mathbf{z}_0], \partial_{01}[\mathbf{z}_0]\}, \text{with } \partial_{ij}[\mathbf{z}_0] = \frac{1}{i!j!}\,\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(\mathbf{z}_0).$$

Computing the Multiplicity Structure

following B.H. Dayton and Z. Zeng, ISSAC 2005

Looking for differentiation functionals $d[\mathbf{z}_0] = \sum_{\mathbf{a}} c_{\mathbf{a}} \partial_{\mathbf{a}}[\mathbf{z}_0]$,

$$\text{with} \quad \partial_{\mathbf{a}}[\mathbf{z}_0](p) = \frac{1}{a_1! a_2! \cdots a_n!} \left(\frac{\partial^{a_1+a_2+\cdots+a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}} p \right) (\mathbf{z}_0).$$

Membership criterium for $d[\mathbf{z}_0]$:

$$\boldsymbol{d[\mathbf{z}_0] \in D_0 \Leftrightarrow d[\mathbf{z}_0](pf_i) = 0, \forall p \in \mathbb{C}[\mathbf{x}], i = 1, 2, \ldots, N.}$$

To turning this criterium into an **algorithm**, observe:

1. since $d[\mathbf{z}_0]$ is linear, restrict p to $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$; and
2. limit degrees $k_1 + k_2 + \cdots + k_n \leq a_1 + a_2 + \cdots + a_n$, as $\mathbf{z}_0 = \mathbf{0}$ vanishes trivially if not annihilated by $\partial_{\mathbf{a}}$.

Computing the Multiplicity Structure – An Example

$f_1 = x_1 - x_2 + x_1^2$, $f_2 = x_1 - x_2 + x_2^2$ following B.H. Dayton and Z. Zeng

		$\vert a\vert=0$	$\vert a\vert=1$		$\vert a\vert=2$			$\vert a\vert=3$			
		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}	∂_{30}	∂_{21}	∂_{12}	∂_{03}
	f_1	0	1	–1	1	0	0	0	0	0	0
S_1	f_2	0	1	–1	0	0	1	0	0	0	0
	$x_1 f_1$	0	0	0	1	–1	0	1	0	0	0
	$x_1 f_2$	0	0	0	1	–1	0	0	0	1	0
	$x_2 f_1$	0	0	0	0	1	–1	0	1	0	0
S_2	$x_2 f_2$	0	0	0	0	1	–1	0	0	0	1
	$x_1^2 f_1$	0	0	0	0	0	0	1	–1	0	0
	$x_1^2 f_2$	0	0	0	0	0	0	1	–1	0	0
	$x_1 x_2 f_1$	0	0	0	0	0	0	0	1	–1	0
	$x_1 x_2 f_2$	0	0	0	0	0	0	0	1	–1	0
	$x_2^2 f_1$	0	0	0	0	0	0	0	0	1	–1
S_3	$x_2^2 f_2$	0	0	0	0	0	0	0	0	1	–1

Nullity(S_2) = Nullity(S_3) $\Rightarrow$ stop algorithm

$D_0 = \mathrm{span}\{ \partial_{00}, \partial_{10} + \partial_{01}, -\partial_{10} + \partial_{20} + \partial_{11} + \partial_{02} \} \Rightarrow$ multiplicity = 3

Numerical Irreducible Decomposition

specification of input and output

Input: $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_N) \in (\mathbb{C}[\mathbf{x}])^N$.

The #equations N may differ from n, the #unknowns.

A numerical irreducible decomposition consists of

1. for every dimension: a description of sets of solutions,
2. for every pure dimensional solution set: its decomposition in irreducible components.

The output is summarized in the following numbers

1. for every dimension: the degree of the solution set,
2. for every pure dimensional solution set: the degrees of all irreducible components, and their multiplicities.

Positive Dimensional Solution Sets

represented numerically by witness sets

Given a system $f(\mathbf{x}) = \mathbf{0}$, we represent
a component of $f^{-1}(\mathbf{0})$ of dimension k and degree d by

- k general hyperplanes L to cut the dimension; and
- d generic points in $f^{-1}(\mathbf{0}) \cap L$.

Witness set representations reduce to isolated solutions,
with continuation methods we sample solution sets.

Using a flag of linear spaces, defined by an decreasing sequence of subsets of the k general hyperplanes,

$$\mathbb{C}^n \supset L_{n-1} \supset \cdots \supset L_1 \supset L_0 = \emptyset,$$

we move solutions with nonzero slack values to generic points on lower dimensional components, using a cascade of homotopies.

Example of a Homotopy in the Cascade

To compute numerical representations of the twisted cubic and the four isolated points, as given by the solution set of one polynomial system, we use the following homotopy:

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{c} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] + t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) + z_1 \end{array} \right] = \mathbf{0}$$

At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$.
At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the system, and z_1 is forced to zero.

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At $t = 1$: $H(\mathbf{x}, \mathbf{z}_1, t) = \mathcal{E}(f)(\mathbf{x}, \mathbf{z}_1) = \mathbf{0}$.
At $t = 0$: $H(\mathbf{x}, \mathbf{z}_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the system, and $\mathbf{z}_1$ is forced to zero.

Computing a Numerical Irreducible Decomposition

input: $f(\mathbf{x}) = \mathbf{0}$ a polynomial system with $\mathbf{x} \in \mathbb{C}^n$

- **Stage 1**: represent the k-dimensional solutions Z_k, $k = 0, 1, \ldots$

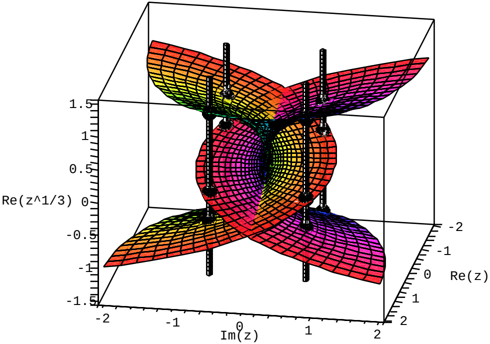
  output: sequence $[W_0, W_1, \ldots, W_{n-1}]$ of ***witness sets***
  $W_k = (E_k, E_k^{-1}(\mathbf{0}) \setminus J_k)$, $\deg Z_k = \#(E_k^{-1}(\mathbf{0}) \setminus J_k)$
  $E_k = f$ + k random hyperplanes, J_k = “junk”

- **Stage 2**: decompose Z_k, $k = 0, 1, \ldots$ into irreducible factors

  output: $W_k = \{W_{k1}, W_{k2}, \ldots, W_{kn_k}\}$, $k = 1, 2, \ldots, n-1$
  n_k irreducible components of dimension k

output: a numerical irreducible decomposition of $f^{-1}(\mathbf{0})$
is a sequence of partitioned witness sets

The Riemann Surface of $z - w^3 = 0$:



Loop around the singular point (0,0) permutes the points.

Generating Loops by Homotopies

W_L represents a k-dimensional solution set of $f(\mathbf{x}) = \mathbf{0}$, cut out by k random hyperplanes L. For k other hyperplanes K, we move W_L to W_K, using the **homotopy** $\boldsymbol{h}_{L,K,\alpha}(\mathbf{x}, t) = 0$, from $t = 0$ to 1:

$$\boldsymbol{h}_{L,K,\alpha}(\mathbf{x}, t) = \left(\begin{array}{c} f(\mathbf{x}) \\ \alpha(1-t)L(\mathbf{x}) + tK(\mathbf{x}) \end{array} \right) = \mathbf{0}, \quad \alpha \in \mathbb{C}.$$

The constant α is chosen at random, to avoid singularities, as $t < 1$. To turn back we generate another random constant β, and use

$$\boldsymbol{h}_{K,L,\beta}(\mathbf{x}, t) = \left(\begin{array}{c} f(\mathbf{x}) \\ \beta(1-t)K(\mathbf{x}) + tL(\mathbf{x}) \end{array} \right) = \mathbf{0}, \quad \beta \in \mathbb{C}.$$

A permutation of points in W_L occurs only among points on the same irreducible component.

Linear Traces as Stop Criterium

Consider

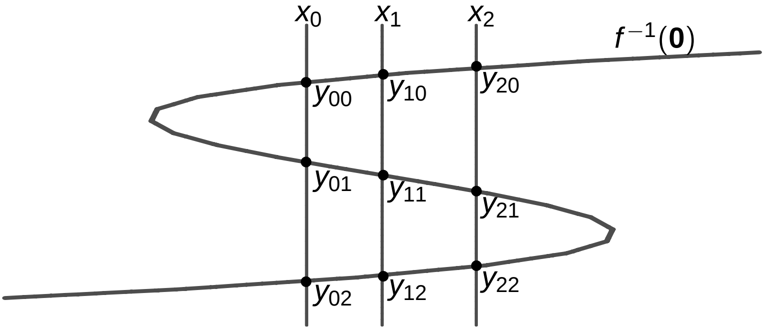
$$\begin{aligned} f(x, y(x)) &= (y - y_1(x))(y - y_2(x))(y - y_3(x)) \\ &= y^3 - \mathbf{t_1(x)} y^2 + t_2(x) y - t_3(x) \end{aligned}$$

We are interested in **the linear trace**: $\boldsymbol{t_1(x) = c_1 x + c_0}$.
Sample the cubic at $x = x_0$ and $x = x_1$. The samples are $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \left\{ \begin{array}{l} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{array} \right. \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x.
For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$.
Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.
If $\neq$, then samples come from irreducible curve of degree > 3.

Linear Traces – an example



Use $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$ to find the linear trace $t_1(x) = c_0 + c_1 x$.

At $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$: $c_0 + c_1 x_2 = y_{20} + y_{21} + y_{22}$?

Numerical Primary Decomposition

specification of input and output

Input: $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $f = (f_1, f_2, \ldots, f_N) \in (\mathbb{C}[\mathbf{x}])^N$.

Consider the ideal I generated by the polynomials in f.

The output of a numerical primary decomposition is
a list of irreducible components:
$\rightarrow$ each component is a solution set of an associated prime ideal
in the primary decomposition of I,

Each component contains

- an indication whether embedded or isolated
  – isolated means: not contained in another component,
- its dimension, degree, and multiplicity structure,
- sufficient information to solve the ideal membership problem.

Deflation reveals Embedded Components

Anton Leykin, ISSAC 2008

Embedded components become visible after deflation.

Let $I = \langle f_1, f_2, \ldots, f_N \rangle$, $\partial_k = \frac{\partial}{\partial x_k}$, $k = 1, 2, \ldots, n$.

The first order deflation matrix of I is

$$A_I^{(1)}(\mathbf{x}) = \begin{bmatrix} f_1 & \partial_1 f_1 & \partial_2 f_1 & \cdots & \partial_n f_1 \\ f_2 & \partial_1 f_2 & \partial_2 f_2 & \cdots & \partial_n f_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_N & \partial_1 f_N & \partial_2 f_N & \cdots & \partial_n f_N \end{bmatrix}$$

Consider $A_I^{(1)}(\mathbf{x})\mathbf{a}^T$, where $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_n)$ in $\mathbb{C}[\mathbf{x}, \mathbf{a}]$.

Compute a numerical irreducible decomposition to the deflated ideal $I^{(1)} = \langle A_I^{(1)}(\mathbf{x})\mathbf{a}^T \rangle$ and apply $(\mathbf{x}, \mathbf{a}) \mapsto \mathbf{x}$ to the result.

An Example

Consider $I = \langle x^2, xy \rangle = \langle x \rangle \cap \langle x^2, y \rangle$.

We see the embedded point $(0,0) = V(\langle x^2, y \rangle)$ after one deflation:

$$A_I^{(1)}(x,y) = \begin{bmatrix} x^2 & 2x & 0 \\ xy & y & x \end{bmatrix}.$$

The deflated ideal is

$$I^{(1)} = \langle x^2, xy, a_1 2x, a_1 y + a_2 x \rangle.$$

Computing $V(I^{(1)})$ via its radical:

$$\sqrt{I^{(1)}} = \langle x, y a_1 \rangle = \langle x, a_1 \rangle \cap \langle x, y \rangle.$$

Summary

a complexity classification

We considered three types of solving:

1. **zero dimensional solving**
   typically one single homotopy
   parallel implementations allow tracking of millions of paths
2. **numerical irreducible decomposition**
   need to add extra linear equations
   different homotopies used after each other
3. **numerical primary decomposition**
   one deflation doubles the dimension
   numerical irreducible decomposition is blackbox