Overview of Numerical Algebraic Geometry

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Overview

- Homotopy continuation applies symbolic-numeric algorithms, exploiting sparsity to compute a generically sharp root count, tracking solution paths using predictor-corrector methods.
- When encountering singular solutions, we recondition the problem adding sufficiently many equations to deflate the multiplicity down to the regular case.
- For positive dimensional solution sets, a numerical irreducible decomposition classifies the sets according to their dimension and breaks the pure dimensional sets into irreducible factors.

Problem Statement

Input: $f(\mathbf{x}) = 0$, a system of N polynomials in n unknowns;
the coefficients of f are approximate complex numbers.

Output: a numerical irreducible decomposition of $f^{-1}(\mathbf{0})$;
over $\mathbb{C}^n$, $N = 1$: absolute factorization.

Structure of talk (organization of some topics):

1. Assume $N = n$ and compute all isolated solutions.

2. Deflation to handle isolated singular solutions.

3. Reduce general case to fully determined case.

Symbolic-Numeric Algorithms

Synergy between symbolic computation and numerical analysis necessary to solve polynomial systems from real applications.

Symbolic: exploit structure and recondition

+ a priori estimates of complexity of the problem
+ find the right equations to capture the roots

Numeric: balance between efficiency and accuracy

+ numerically stable deformation methods
+ condition numbers report quality of output

Computer Algebra cares about implementations and users.

Homotopy Continuation Methods

Solve $f(\mathbf{x}) = \mathbf{0}$ in two stages:

1. The homotopy $h(\mathbf{x}, t) = \gamma(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$, $\gamma \in \mathbb{C}$, defines solution paths $\mathbf{x}(t)$, for t going from 0 to 1.

   $g(\mathbf{x}) = \mathbf{0}$ is a start system with the same structure as f. All solutions of $g(\mathbf{x}) = \mathbf{0}$ are isolated and regular.

2. Continuation methods apply predictor-corrector techniques to track the solution paths defined by the homotopy $h(\mathbf{x}, t) = \mathbf{0}$. Singularities do not occur for $t < 1$ for a generic choice of γ.

Knowing the right #paths is critical to the performance!

The Gamma Trick

Consider the homotopy $h(\mathbf{x},t) = \gamma(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$, $\gamma \in \mathbb{C}$.
All solutions of $g(\mathbf{x}) = \mathbf{0}$ are isolated and regular.

1. **Singular** solutions of $h(\mathbf{x},t) = \mathbf{0}$ satisfy

$$H(\mathbf{x},t) = \begin{cases} h(\mathbf{x},t) = \mathbf{0} \\ \det(J_h(\mathbf{x},t)) = 0 \end{cases} \quad J_h \text{ is the Jacobian of } h.$$

2. Embed $(\mathbf{x},t) \in \mathbb{C}^n \times \mathbb{C}$ into projective space: $(\mathbf{z},t) \in \mathbb{P}^n \times \mathbb{C}$.
   Apply the **main theorem of elimination theory** to $H^{-1}(\mathbf{0})$,
   eliminating $\mathbf{z}$, i.e.: apply $\pi : \mathbb{P}^n \times \mathbb{C} \rightarrow \mathbb{C} : (\mathbf{z},t) \mapsto t$.
   Then $\pi(H^{-1}(\mathbf{0}))$ is an algebraic set, defined by $p(t) = 0$.

3. Because all solutions of $g(\mathbf{x}) = \mathbf{0}$ are isolated and regular,
   $p(0) \neq 0$. So there are **only finitely many** singularities.

For a generic choice of γ, $H(\mathbf{x},t) = \mathbf{0}$ has no solutions for $t \in [0,1)$.

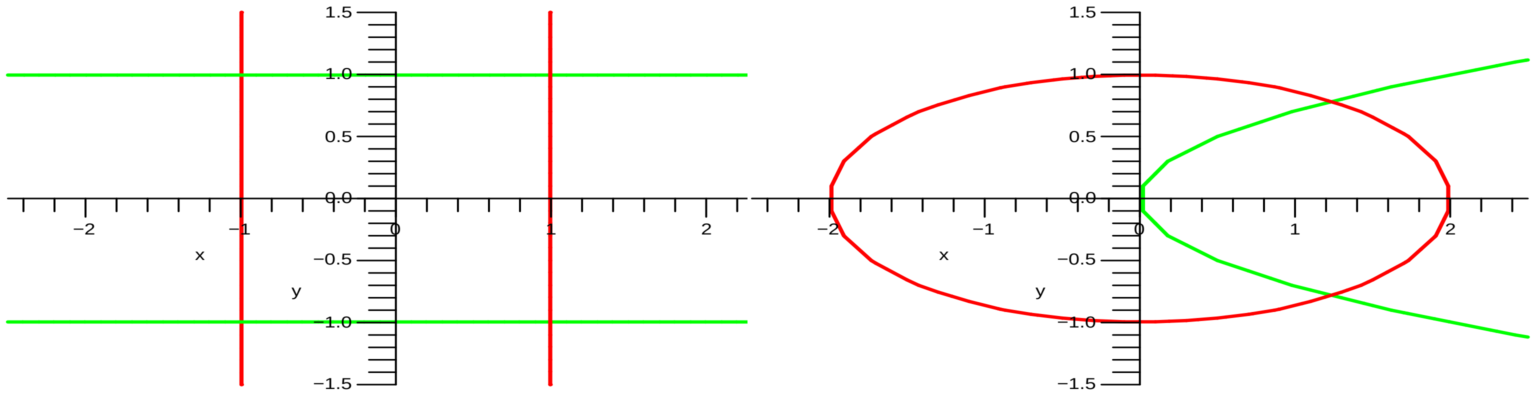
a synthetic proof of Bézout's theorem

Mario Pieri: **Sopra un teorema di geometria ad n dimensioni.** *Giornale di Matematiche di Battaglini* 26:251-254, 1888.

- Consider two varieties V and W in n-space, $\dim(V) + \dim(W) = n$, assume $\#(V \cap W) < \infty$. To show that in general, $\#(V \cap W) = \deg(V) \times \deg(W)$, use induction on n.
- Applying the principle of correspondence (Chasles), replace one hypersurface of degree d by a product of d hyperplanes and apply d times the induction hypothesis (true for $n - 1$).

Elena Anne Marchisotto and James T. Smith: *The Legacy of Mario Pieri in Geometry and Arithmetic.* Birkhäuser, 2007.

Product Deformations



$$\gamma \left(\underbrace{\left\{ \begin{array}{r} x^2 - 1 = 0 \\ y^2 - 1 = 0 \end{array} \right.}_{\text{start system}} \right) (1-t) \quad + \quad \left(\underbrace{\left\{ \begin{array}{r} x^2 + 4y^2 - 4 = 0 \\ 2y^2 - x = 0 \end{array} \right.}_{\text{target system}} \right) t, \quad \gamma \in \mathbb{C}$$

Some Introductions and Surveys

I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky: **Discriminants, Resultants and Multidimensional Determinants**. Birkhäuser, 1994.

L. Blum, F. Cucker, M. Shub, and S. Smale: **Complexity and Real Computation**. Springer-Verlag, 1998.

B. Sturmfels: **Solving Systems of Polynomial Equations**. AMS, 2002.

T.Y. Li.: **Numerical solution of polynomial systems by homotopy continuation methods**. In F. Cucker, editor, *Handbook of Numerical Analysis. Volume XI. Special Volume: Foundations of Computational Mathematics*, pages 209–304. North-Holland, 2003.

A.J. Sommese and C.W. Wampler: **The Numerical Solution of Systems of Polynomials Arising in Engineering and Science**. World Scientific, 2005.

Three Kinds of Homotopies

natural parameter: $f(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$, $\boldsymbol{\lambda} \in \mathbb{C}^m$

the homotopy is fixed, the problem is to detect and handle singularities efficiently and accurately

coefficient parameter: $f(\mathbf{x}, \boldsymbol{\lambda}_0(1-t) + \boldsymbol{\lambda}_1 t) = \mathbf{0}$

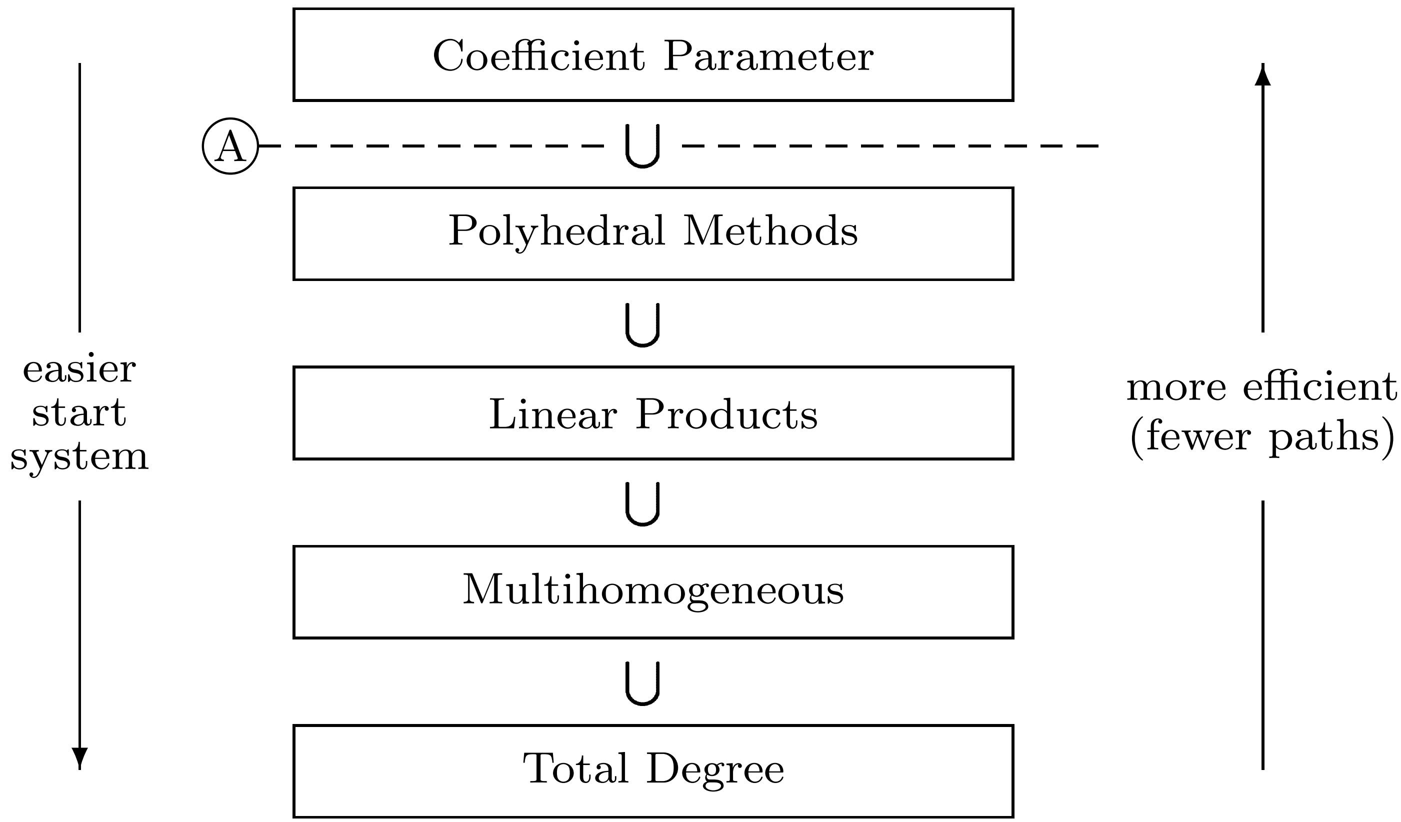
move from $t = 0$ at generic instance $\boldsymbol{\lambda}_0$
to $t = 1$, a specific instance $\boldsymbol{\lambda}_1$ of the parameters $\boldsymbol{\lambda}$

because $\boldsymbol{\lambda}_0$ is generic, singular solutions occur only as $t \rightarrow 1$

artificial parameter: $\gamma(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}$

the system $g(\mathbf{x}) = \mathbf{0}$ is a start system with all its solutions regular, resembling the structure of f

A Hierarchy of Homotopies



Below line A: solving start systems is done automatically.

Above line A: start system has generic values for the parameters.

Multihomogeneous version of Bézout's theorem

Consider $A\mathbf{x} = \lambda\mathbf{x}$, $A \in \mathbb{C}^{n\times n}$. plain Bézout's theorem: $D = 2^n$

Add a hyperplane $c_1x_1 + c_2x_2 + \cdots + c_nx_n + c_0 = 0$ for unique $\mathbf{x}$.

Embed in multi-projective space: $\mathbb{P} \times \mathbb{P}^n$, separating λ from $\mathbf{x}$.

$\{\lambda\}$	$\{x_1, x_2\}$
1	1
1	1
0	1

degree table

$\iff$

$\{\lambda\}$	$\{x_1, x_2\}$
$\lambda + \gamma_1$	$\alpha_0 + \alpha_1x_1 + \alpha_2x_2$
$\lambda + \gamma_2$	$\beta_0 + \beta_1x_1 + \beta_2x_2$
1	$c_0 + c_1x_1 + c_2x_2$

linear-product start system

The root count $B = 1\cdot 1\cdot 1 + 1\cdot 1\cdot 1 + 0\cdot 1\cdot 1$ is a permanent.

A. Morgan and A. Sommese: A homotopy for solving general polynomial systems that respects m-homogeneous structures. *Appl. Math. Comput.*, 24(2):101–113, 1987.

linear-product start systems

$$f(\mathbf{x}) = \begin{cases} x_1 x_2^2 + x_1 x_3^3 - c x_1 + 1 = 0 & c \in \mathbb{C} \\ x_2 x_1^2 + x_2 x_3^2 - c x_2 + 1 = 0 & \\ x_3 x_1^2 + x_3 x_2^2 - c x_3 + 1 = 0 & D = 27 \end{cases}$$

$$\begin{array}{llllc} \{x_1\} & \{x_2, x_3\} & \{x_2, x_3\} & \text{symmetric} & \\ \{x_2\} & \{x_1, x_3\} & \{x_1, x_3\} & \text{supporting} & B = 21 \\ \{x_3\} & \{x_1, x_2\} & \{x_1, x_2\} & \text{set structure} & \end{array}$$

Choose 7 random complex numbers $c_1, c_2, \ldots, c_7$ and create

$$g(\mathbf{x}) = \begin{cases} (x_1 + c_1)(c_2 x_2 + c_3 x_3 + c_4)(c_5 x_2 + c_6 x_3 + c_7) = 0 \\ (x_2 + c_1)(c_2 x_1 + c_3 x_3 + c_4)(c_5 x_1 + c_6 x_3 + c_7) = 0 \\ (x_3 + c_1)(c_2 x_1 + c_3 x_2 + c_4)(c_5 x_1 + c_6 x_2 + c_7) = 0 \end{cases}$$

8 generating solutions

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i,\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \operatorname{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	desired properties
$L(f) = L(f_2, f_1, \ldots, f_n)$	invariant under permutations
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	shift invariant
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	monotone increasing
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	unimodular invariant
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	root count of product is sum of root counts

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properties of $L(f)$	$V(\mathcal{P})$ mixed volume
invariant under permutations	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
shift invariant	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
monotone increasing	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
unimodular invariant	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
root count of product is sum of root counts	$V(P_{11} + P_{12}, \ldots, P_n) = V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

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$L(f)$ root count in $(\mathbb{C}^*)^n$	$V(\mathcal{P})$ mixed volume
$L(f) = L(f_2, f_1, \ldots, f_n)$	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
$L(f) = L(f_1 \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
$L(f_{11} f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

exploit sparsity $\boxed{L(f) = V(\mathcal{P})}$ 1st theorem of Bernshteĭn

The Theorems of Bernshteĭn

Theorem A: The number of roots of a generic system equals the mixed volume of its Newton polytopes.

Theorem B: Solutions at infinity are solutions of systems supported on faces of the Newton polytopes.

D.N. Bernshteĭn: **The number of roots of a system of equations.** *Functional Anal. Appl.*, 9(3):183–185, 1975.

Structure of proofs: First show Theorem B, looking at power series expansions of diverging paths defined by a linear homotopy starting at a generic system. Then show Theorem A, using Theorem B with a homotopy defined by *lifting* the polytopes.

Some References

J. Canny and J.M. Rojas: **An optimal condition for determining the exact number of roots of a polynomial system.** In ***Proceedings of ISSAC 1991***, pages 96–101. ACM, 1991.

J. Verschelde, P. Verlinden, and R. Cools: **Homotopies exploiting Newton polytopes for solving sparse polynomial systems.** ***SIAM J. Numer. Anal.*** 31(3):915–930, 1994.

B. Huber and B. Sturmfels: **A polyhedral method for solving sparse polynomial systems**. ***Math. Comp.*** 64(212):1541–1555, 1995.

I.Z. Emiris and J.F. Canny: **Efficient incremental algorithms for the sparse resultant and the mixed volume**. ***J. Symbolic Computation*** 20(2):117–149, 1995.

Bernshteĭn's first theorem

Let $g(\mathbf{x}) = \mathbf{0}$ have the same Newton polytopes $\mathcal{P}$ as $f(\mathbf{x}) = \mathbf{0}$, but with randomly choosen complex coefficients.

I.	Compute $V_n(\mathcal{P})$:		II.	Solve $g(\mathbf{x}) = \mathbf{0}$:	
I.1	lift polytopes	$\Leftrightarrow$	II.1	introduce parameter t	
I.2	mixed cells	$\Leftrightarrow$	II.2	start systems	
I.3	volume of mixed cell	$\Leftrightarrow$	II.3	path following	

III. Coefficient-parameter continuation to solve $f(\mathbf{x}) = \mathbf{0}$:

$$h(\mathbf{x}, t) = \gamma(1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}, \quad \text{for } t \text{ from 0 to 1.}$$

#isolated solutions in $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, of $f(\mathbf{x}) = \mathbf{0}$ is bounded by the mixed volume of the Newton polytopes of f.

Software for Solving with Homotopies I

L.T. Watson, M. Sosonkina, R.C. Melville, A.P. Morgan, and H.F. Walker: **HOMPACK90: A suite of Fortran 90 codes for globally convergent homotopy algorithms.** ***ACM Trans. Math. Softw.***, 23(4):514–549, 1997.

J. Verschelde: **Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.** ***ACM Trans. Math. Softw.***, 25(2):251–276, 1999.
Version 2.3.31 is available via `http://www.math.uic.edu/~jan`.

T. Gao, T.Y. Li, and M. Wu: **Algorithm 846: MixedVol: a software package for mixed-volume computation.** ***ACM Trans. Math. Softw.***, 31(4):555–560, 2005.
HOM4PS is available via `http://www.math.msu.edu/~li`.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani: **PHoM – a polyhedral homotopy continuation method for polynomial systems.** ***Computing***, 73(4):55–77, 2004.
Available via `http://www.is.titech.ac.jp/~kojima`.

Software for Solving with Homotopies II

T. Gunji, S. Kim, K. Fujisawa, and M. Kojima: PHoMpara – parallel implementation of the Polyhedral Homotopy continuation Method for polynomial systems. *Computing* 77(4):387–411, 2006.

H.-J. Su, J.M. McCarthy, M. Sosonkina, and L.T. Watson: Algorithm 857: POLSYS_GLP: A parallel general linear product homotopy code for solving polynomial systems of equations. *ACM Trans. Math. Softw.* 32(4):561–579, 2006.

T. Mizutani and A. Takeda: DEMiCs: A software package for computing the mixed volume via dynamic enumeration of all mixed cells. *IMA Volume on Software for Algebraic Geometry*.

D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler: Software for numerical algebraic geometry: a paradigm and progress towards its implementation. *IMA Volume on Software for Algebraic Geometry*.
Bertini is available at `http://www.nd.edu/~sommese/bertini`.

Singularities are keeping us in business

numerical analysis: bifurcation points and endgames

Rall (1966); Reddien (1978); Decker-Keller-Kelley (1983);
Griewank-Osborne (1981); Hoy (1989);
Deuflard-Friedler-Kunkel (1987); Kunkel (1989, 1996);
Morgan-Sommese-Wampler (1991); Li-Wang (1993, 1994);
Allgower-Schwetlick (1995); Pönisch-Schnabel-Schwetlick (1999);
Allgower-Böhmer-Hoy-Janovský (1999); Govaerts (2000)

computer algebra: standard bases (SINGULAR)

Mora (1982); Greuel-Pfister (1996); Marinari-Möller-Mora (1993)

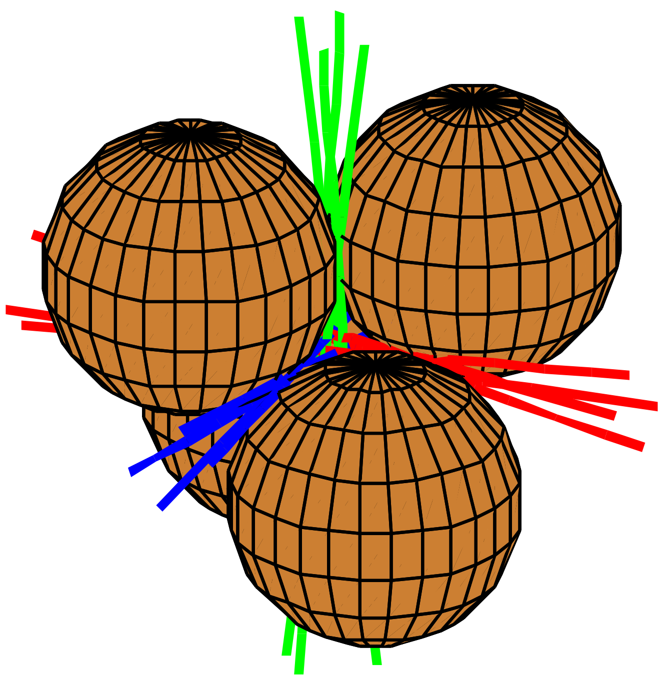
numerical polynomial algebra: multiplicity structure

Möller-Stetter (1995); Mourrain (1997);
Stetter-Thallinger (1998); Dayton-Zeng (2005)

deflation: Ojika-Watanabe-Mitsui (1983); Lecerf (2003)

Twelve lines tangent to four spheres

Frank Sottile and Thorsten Theobald: Lines tangents to $2n-2$ spheres in $\mathbb{R}^n$
Trans. Amer. Math. Soc. 354
pages 4815-4829, 2002.



Problem:
Given 4 spheres,
find all lines tangent
to all 4 given spheres.

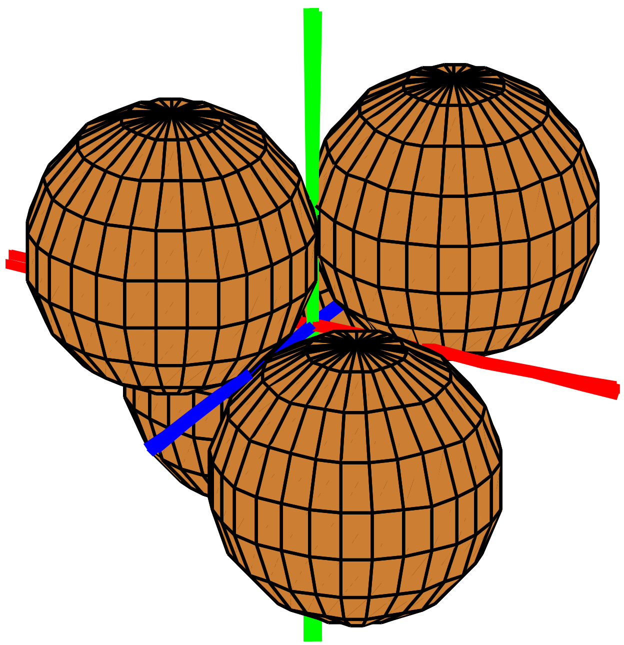
Observe:
12 solutions in groups of 4.

Twelve lines tangent to four spheres

Frank Sottile and Thorsten Theobald: Lines tangents to $2n-2$ spheres in $\mathbb{R}^n$

Trans. Amer. Math. Soc. 354

pages 4815-4829, 2002.



Problem:

Given 4 spheres,

find all lines tangent

to all 4 given spheres.

Observe:

3 lines of multiplicity 4.

An Input Polynomial System

```
x0**2 + x1**2 + x2**2 - 1;
x0*x3 + x1*x4 + x2*x5;
x3**2 + x4**2 + x5**2 - 0.25;
x3**2 + x4**2 - 2*x2*x4 + x2**2 + x5**2 + 2*x1*x5 + x1**2 - 0.25;
x3**2 + 1.73205080756888*x2*x3 + 0.75*x2**2 + x4**2 - x2*x4 + 0.25*x2**2
+ x5**2 - 1.73205080756888*x0*x5 + x1*x5
+ 0.75*x0**2 - 0.86602540378444*x0*x1 + 0.25*x1**2 - 0.25;
x3**2 - 1.63299316185545*x1*x3 + 0.57735026918963*x2*x3
+ 0.66666666666667*x1**2 - 0.47140452079103*x1*x2 + 0.08333333333333*x2**2
+ x4**2 + 1.63299316185545*x0*x4 - x2*x4 + 0.66666666666667*x0**2
- 0.81649658092773*x0*x2 + 0.25*x2**2
+ x5**2 - 0.57735026918963*x0*x5 + x1*x5 + 0.08333333333333*x0**2
- 0.28867513459481*x0*x1 + 0.25*x1**2 - 0.25;
```

Original formulation as polynomial system: **Cassiano Durand**.

Centers of the spheres at the vertices of a tetrahedron: **Thorsten Theobald**.

Algebraic numbers sqrt(3), sqrt(6), etc. approximated by double floats.

The system has 6 isolated solutions, each of multiplicity 4.

Solutions at the End of Continuation

Two solutions in a **cluster**: (real and imaginary parts)

```
solution 1 :
  x0 :  -7.07106803165780E-01    3.77452918725401E-08
  x1 :  -4.08248430737360E-01   -1.83624917064964E-07
  x2 :   5.77350143082334E-01   -8.36140714113780E-08
  x3 :  -2.50000000000000E-01   -1.57896818458518E-16
  x4 :   4.33012701892221E-01   -9.11600174682333E-17
  x5 :   9.56878363411174E-08    1.54062878745083E-07
solution 2 :
  x0 :  -7.07106794356709E-01   -1.29682370414209E-07
  x1 :  -4.08248217029256E-01    1.11010906008961E-07
  x2 :   5.77350304985648E-01   -8.03312536501087E-08
  x3 :  -2.50000000000001E-01   -1.74789416181029E-16
  x4 :   4.33012701892220E-01   -1.00914936462574E-16
  x5 :  -6.07788020445124E-08   -1.39412292964849E-07
```

this is the **input** to our deflation algorithm

Newton's Method for Overdetermined Systems

Singular Value Decomposition of N-by-n Jacobian matrix J_f:

$J_f = U\Sigma V^T$, $\quad U$ and V are orthogonal: $U^T U = I_N, V^T V = I_n$,

and singular values $\boldsymbol{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n}$ as the only nonzero elements on the diagonal of the N-by-n matrix Σ $(N > n)$.

The **condition number** $\mathrm{cond}(J_f(\mathbf{z})) = \frac{\sigma_1}{\sigma_n}$.

$\mathrm{Rank}(J_f(\mathbf{z})) = R \Longleftrightarrow \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_R, 0, \ldots, 0)$.

At a **multiple root** $\mathbf{z}_0$: $\mathrm{Rank}(J_f(\mathbf{z}_0)) = R < n$.

Close to $\mathbf{z}_0$, $\mathbf{z} \approx \mathbf{z}_0 : \sigma_{R+1} \approx 0$, or $\boldsymbol{|\sigma_{R+1}| < \epsilon}$, ϵ is tolerance.

Moore-Penrose inverse: $J_f^+ = V\Sigma^+ U^T$, with $R = \mathrm{Rank}(J_f)$,

and $\Sigma^+ = \mathrm{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \ldots, \frac{1}{\sigma_R}, 0, \ldots, 0)$.

Then $\Delta\mathbf{z} = -J_f(\mathbf{z})^+ f(\mathbf{z})$ is the least squares solution.

Dedieu-Shub (1999); Li-Zeng (2005)

Deflation Operator Dfl reduces to Corank One

Consider $f(\mathbf{x}) = \mathbf{0}$, N equations in n unknowns, $N \geq n$.

Suppose $\mathrm{Rank}(A(\mathbf{z}_0)) = R < n$ for $\mathbf{z}_0$ an isolated zero of $f(\mathbf{x}) = 0$.

Choose $\mathbf{h} \in \mathbb{C}^{R+1}$ and $B \in \mathbb{C}^{n \times (R+1)}$ at random.

Introduce $R+1$ new multiplier variables $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{R+1})$.

$$\mathbf{Dfl}(f)(\mathbf{x}, \boldsymbol{\lambda}) := \left\{ \begin{array}{rcl} f(\mathbf{x}) & = & \mathbf{0} \\ A(\mathbf{x})B\boldsymbol{\lambda} & = & \mathbf{0} \\ \mathbf{h}\boldsymbol{\lambda} & = & 1 \end{array} \right. \qquad \begin{array}{c} \mathrm{Rank}(A(\mathbf{x})) = R \\ \Downarrow \\ \mathrm{corank}(A(\mathbf{x})B) = 1 \end{array}$$

Compared to the deflation of Ojika, Watanabe, and Mitsui:
(1) we do not compute a maximal minor of the Jacobian matrix;
(2) we only add new equations, we never replace equations.

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

$$[A^+, R] := \mathrm{SVD}(A(\mathbf{x}_k), \epsilon);$$
$$\mathbf{x}_{k+1} := \mathbf{x}_k - A^+ f(\mathbf{x}_k);$$

Gauss-Newton

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

$[A^+, R] := \mathrm{SVD}(A(\mathbf{x}_k), \epsilon)$;
$\mathbf{x}_{k+1} := \mathbf{x}_k - A^+ f(\mathbf{x}_k)$;

Gauss-Newton

$R = \#\mathrm{columns}(A)?$

Yes

Output: $f; \mathbf{x}_{k+1}$.

Newton's Method with Deflation

Input: $f(\mathbf{x}) = \mathbf{0}$ polynomial system;
$\mathbf{x}_0$ initial approximation for $\mathbf{x}^*$;
ϵ tolerance for numerical rank.

$[A^+, R] := \mathrm{SVD}(A(\mathbf{x}_k), \epsilon)$;
$\mathbf{x}_{k+1} := \mathbf{x}_k - A^+ f(\mathbf{x}_k)$;

Gauss-Newton

$R = \#\mathrm{columns}(A)$?

Yes

Output: $f; \mathbf{x}_{k+1}$.

No

$$f := \mathbf{Dfl}(f)(\mathbf{x}, \boldsymbol{\lambda}) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ G(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0} \end{cases};$$

$\widehat{\boldsymbol{\lambda}} :=$ **LeastSquares**$(G(\mathbf{x}_{k+1}, \boldsymbol{\lambda}))$;
$k := k + 1$; $\quad \mathbf{x}_k := (\mathbf{x}_k, \widehat{\boldsymbol{\lambda}})$;

Deflation Step

12 Lines Tangent to 4 Spheres revisited

Continuation methods find 24 solutions, clustered in groups of 4.

The rank at all solutions is 4, corank is 2.

One deflation suffices to restore quadratic convergence.

An average **condition number** drops from 3.4E+8 to 1.1E+2.

We can compute the solutions
with **accuracy close to machine precision**,
on a system with approximate coefficients,
given with double float precision.

A Bound on the Number of Deflations

Theorem (Anton Leykin, JV, Ailing Zhao):

The number of deflations needed to restore the quadratic convergence of Newton's method converging to an isolated solution is strictly less than the multiplicity.

Theoretical Computer Science, 359(1-3):111–122, 2006.

Duality Analysis of Barry H. Dayton and Zhonggang Zeng:

(1) tighter bound on number of deflations; and

(2) special case algorithms, for corank = 1.

B.H. Dayton and Z. Zeng: Computing the multiplicity structure in solving polynomial systems.
In *Proceedings of ISSAC2005*, pages 116–123. ACM, 2005.

Numerical Irreducible Decomposition

input: $f(\mathbf{x}) = \mathbf{0}$ a polynomial system with $\mathbf{x} \in \mathbb{C}^n$

- **Stage 1**: represent the k-dimensional solutions Z_k, $k = 0, 1, \ldots$

  output: sequence $[W_0, W_1, \ldots, W_{n-1}]$ of ***witness sets***
  $W_k = (E_k, E_k^{-1}(\mathbf{0}) \setminus J_k)$, $\deg Z_k = \#(E_k^{-1}(\mathbf{0}) \setminus J_k)$
  $E_k = f + k$ random hyperplanes, $J_k =$ "junk"

- **Stage 2**: decompose Z_k, $k = 0, 1, \ldots$ into irreducible factors

  output: $W_k = \{W_{k1}, W_{k2}, \ldots, W_{kn_k}\}$, $k = 1, 2, \ldots, n-1$
  n_k irreducible components of dimension k

output: a numerical irreducible decomposition of $f^{-1}(\mathbf{0})$
is a sequence of partitioned witness sets

Computing Witness Sets for $f^{-1}(\mathbf{0})$

Witness set $W_k = (E_k, E_k^{-1}(\mathbf{0}) \setminus J_k)$ for $Z_k \subset f^{-1}(\mathbf{0})$, $k = \dim Z_k$,
consists of $E_k = f + k$ random hyperplanes
and its solutions, $\#(E_k^{-1}(\mathbf{0}) \setminus J_k) = \deg Z_k$.

- **top down**: use a cascade of homotopies
  - \+ benefits from existing blackbox solver
  - − requires top dimension on input
- **bottom up**: with an equation-by-equation solver
  - \+ requires no guess for top dimension
  - − performance depends on order of equations

Example of a Homotopy in the Cascade

To compute numerical representations of the twisted cubic and the four isolated points, as given by the solution set of one polynomial system, we use the following homotopy:

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{ccc} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] & + & t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) & + & z_1 \end{array} \right] = \mathbf{0}$$

At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$.

At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes from 1 to 0, the hyperplane is removed from the system, and z_1 is forced to zero.

A Cascade of Homotopies

Denote $\mathcal{E}_i$ as an embedding of $f(\mathbf{x}) = \mathbf{0}$ with i random hyperplanes and i slack variables $\mathbf{z} = (z_1, z_2, \ldots, z_i)$.

Theorem (Sommese - Verschelde): *J. Complexity* 16(3):572–602, 2000

1. Solutions with $(z_1, z_2, \ldots, z_i) = \mathbf{0}$ contain $\deg W$ generic points on every i-dimensional component W of $f(\mathbf{x}) = \mathbf{0}$.
2. Solutions with $(z_1, z_2, \ldots, z_i) \neq \mathbf{0}$ are regular; and

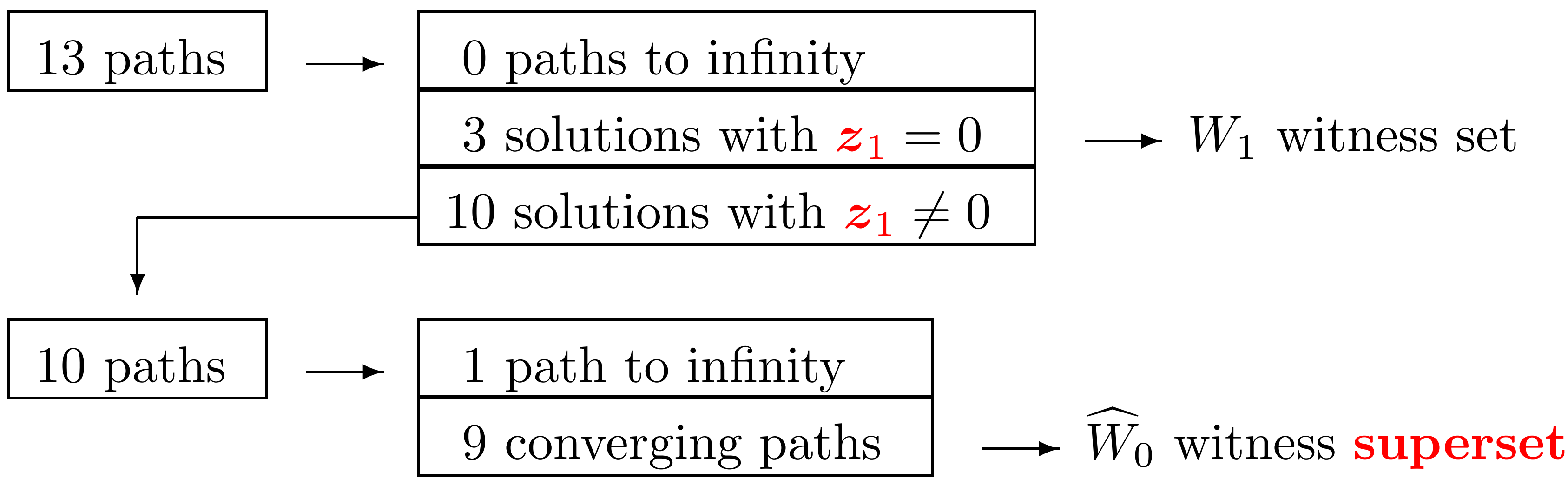
solution paths defined by

$$H_i(\mathbf{x}, \mathbf{z}, t) = t\mathcal{E}_i(\mathbf{x}, \mathbf{z}) + (1-t)\begin{pmatrix} \mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) \\ z_i \end{pmatrix} = \mathbf{0}$$

starting at $t = 1$ with all solutions with $z_i \neq 0$ reach at $t = 0$ all isolated solutions of $\mathcal{E}_{i-1}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.

#paths in twisted cubic + 4 isolated points example

The flow chart below summarizes the number of solution paths traced in the cascade of homotopies.



The set $\widehat{W}_0$ contains, in addition to the four isolated roots, also points on the twisted cubic. The points in $\widehat{W}_0$ which lie on the twisted cubic are considered **junk** and must be filtered out.

Absolute Factorization

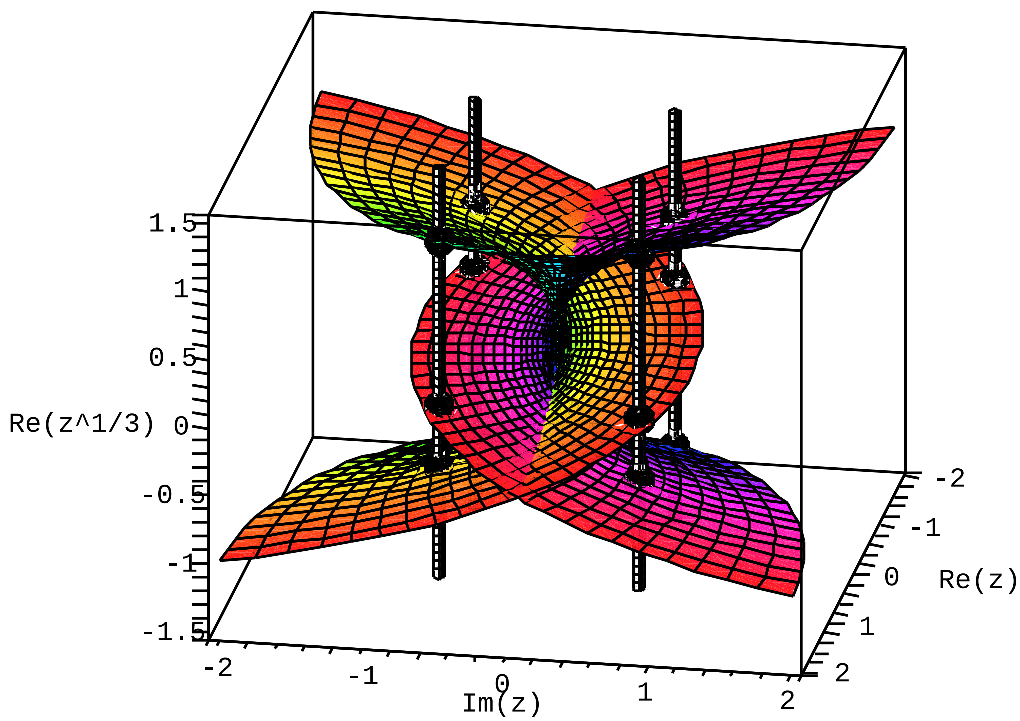
"*Given is a polynomial* $f(x,y) \in \mathbb{Q}[x,y]$ *and* $\epsilon \in \mathbb{Q}$. *Decide in polynomial time in the degree and coefficient size if there is a factorizable* $\hat{f}(x,y) \in \mathbb{C}[x,y]$ *with* $||f - \hat{f}|| \leq \epsilon$, *for a reasonable coefficient vector norm* $||.||$."

Erich Kaltofen (JSC 29, 2000) (originally in Kaltofen, 1992)

Recent Work: new symbolic-numeric algorithms by

- T. Sasaki, T. Saito, T. Hilano (1992); T. Sasaki (2001)
- A. Galligo, D. Rupprecht (2001); G. Chèze, A. Galligo (2003)
- R.M. Corless, M.W. Giesbrecht, M. van Hoeij, I.S. Kotsireas, S.M. Watt (2001)
- A.J. Sommese, J. Verschelde, C.W. Wampler (2001, 2002, 2004)
- S. Gao, E. Kaltofen, J. May, Z. Yang, L. Zhi (2004)
- Z. Zeng, B.H. Dayton (2004)
- A. Poteaux (2007)
- M. van Hoeij, A. Galligo (2007)

The Riemann Surface of $z - w^3 = 0$:



Loop around the singular point (0,0) permutes the points.

Generating Loops by Homotopies

W_L represents a k-dimensional solution set of $f(\mathbf{x}) = \mathbf{0}$, cut out by k random hyperplanes L. For k other hyperplanes K, we move W_L to W_K, using the **homotopy** $\boldsymbol{h}_{L,K,\boldsymbol{\alpha}}(\mathbf{x},t) = 0$, from $t = 0$ to 1:

$$\boldsymbol{h}_{L,K,\boldsymbol{\alpha}}(\mathbf{x},t) = \left(\begin{array}{c} f(\mathbf{x}) \\ \boldsymbol{\alpha}(1-t)L(\mathbf{x}) + tK(\mathbf{x}) \end{array} \right) = \mathbf{0}, \quad \boldsymbol{\alpha} \in \mathbb{C}.$$

The constant $\boldsymbol{\alpha}$ is chosen at random, to avoid singularities, as $t < 1$.

To turn back we generate another random constant $\boldsymbol{\beta}$, and use

$$\boldsymbol{h}_{K,L,\boldsymbol{\beta}}(\mathbf{x},t) = \left(\begin{array}{c} f(\mathbf{x}) \\ \boldsymbol{\beta}(1-t)K(\mathbf{x}) + tL(\mathbf{x}) \end{array} \right) = \mathbf{0}, \quad \boldsymbol{\beta} \in \mathbb{C}.$$

A permutation of points in W_L occurs only among points on the same irreducible component.

Linear Traces as Stop Criterium

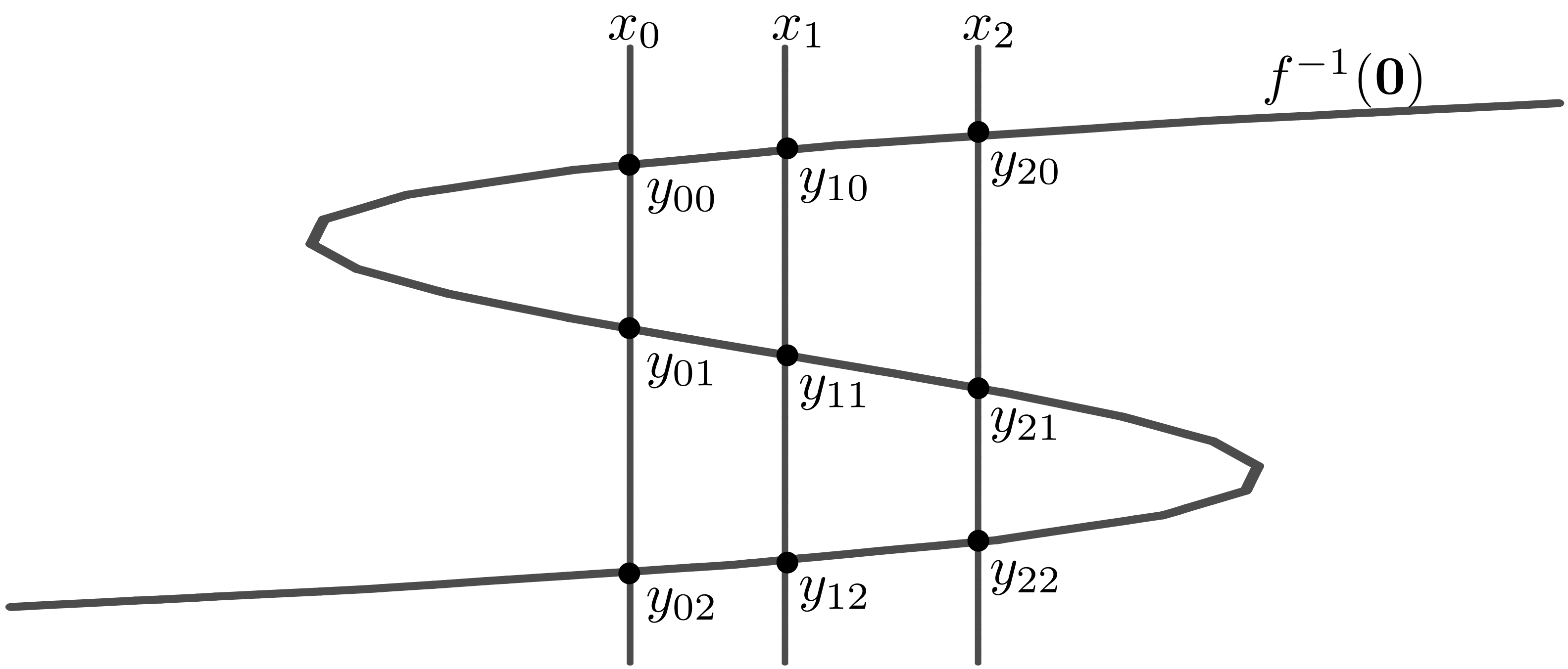
$$\text{Consider} \quad \begin{aligned} f(x, y(x)) &= (y - y_1(x))(y - y_2(x))(y - y_3(x)) \\ &= y^3 - \boldsymbol{t_1(x)} y^2 + t_2(x) y - t_3(x) \end{aligned}$$

We are interested in **the linear trace**: $\boldsymbol{t_1(x) = c_1 x + c_0}$.
Sample the cubic at $x = x_0$ and $x = x_1$. The samples are
$\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$.

$$\text{Solve} \quad \left\{ \begin{array}{l} y_{00} + y_{01} + y_{02} = c_1 x_0 + c_0 \\ y_{10} + y_{11} + y_{12} = c_1 x_1 + c_0 \end{array} \right. \quad \text{to find} \quad c_0, c_1.$$

With t_1 we can predict the sum of the y's for a fixed choice of x.
For example, samples at $x = x_2$ are $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$.
Then, $t_1(x_2) = c_1 x_2 + c_0 = y_{20} + y_{21} + y_{22}$.
If $\neq$, then samples come from irreducible curve of degree > 3.

Linear Traces – an example



Use $\{(x_0, y_{00}), (x_0, y_{01}), (x_0, y_{02})\}$ and $\{(x_1, y_{10}), (x_1, y_{11}), (x_1, y_{12})\}$ to find the linear trace $t_1(x) = c_0 + c_1 x$.

At $\{(x_2, y_{20}), (x_2, y_{21}), (x_2, y_{22})\}$: $c_0 + c_1 x_2 = y_{20} + y_{21} + y_{22}$?

Griffis-Duffy Platforms: Factorization

Case A: One irreducible component of degree 28 (general case).

Case B: Five irreducible components of degrees 6, 6, 6, 6, and 4.

user cpu on 800Mhz	Case A	Case B
witness points	1m 12s 480ms	
monodromy breakup	**33s 430ms**	**27s 630ms**
Newton interpolation	1h 19m 13s 110ms	2m 34s 50ms
32 decimal places used to interpolate polynomial of degree 28		
linear trace	4s 750ms	4s 320ms

Linear traces replace Newton interpolation:

$\Rightarrow$ **time to factor independent of geometry!**

Adjacent minors of a general 2-by-$(n+1)$ matrix

$$n = 3: \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \qquad f(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{cases}$$

P. Diaconis, D. Eisenbud, and B. Sturmfels. **Lattice walks and primary decomposition.** In *Mathematical Essays in Honor of Gian-Carlo Rota*, ed. B.E. Sagan and R.P. Stanley, pages 173–193, Birkhäuser, 1998.

S. Hoşten and J.Shapiro. **Primary decomposition of lattice basis ideals.** *J. Symbolic Computation* 29(4&5): 625–639.

Computational results on adjacent minors

n	d	$\#f$	witness set	#loops	factorization
3	8	3	1.4 s	9	6.8 s
4	16	5	4.5 s	3	9.4 s
5	32	8	23.9 s	4	41.6 s
6	64	13	56.4 s	2	1 m 17.0 s
7	128	21	3 m 39.5 s	4	6 m 42.0 s
8	256	34	8 m 22.6 s	5	16 m 54.7 s
9	512	55	25 m 19.2 s	7	1 h 48 m 52.9 s
10	1024	89	1 h 9 m 27.0 s	5	2 h 9 m 5.1 s

on 1 Ghz PowerBook G4 Mac OS X 10.3.4 with gcc 3.3