Polyhedral Methods for Algebraic Curves

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Outline

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   - solving polynomial systems numerically
   - software for polynomial homotopy continuation
   - polyhedral methods for sparse systems

2. Tropisms for Algebraic Curves
   - binomial systems
   - asymptotics of witness sets
   - tropisms and computing mixed volumes

3. Some Preliminary Computations
   - the cyclic 8-roots problem
   - the cyclic 12-roots problem

Numerical Algebraic Geometry

solving polynomial systems numerically

Given a polynomial system: $f(\mathbf{x}) = \mathbf{0}$,

$$f = (f_1, f_2, \ldots, f_N), \quad \mathbf{x} = (x_1, x_2, \ldots, x_n), \quad f_k \in \mathbb{C}[\mathbf{x}].$$

The input coefficients of the polynomials are approximate.

Homotopy continuation methods:

homotopy: embed system in a family of systems, e.g.:

$$h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + t\, f(\mathbf{x}) = \mathbf{0}, \quad t \in [0, 1]$$

deforming a "good" system g into the given f.

continuation: track paths $\mathbf{x}(t)$ defined by the homotopy $h(\mathbf{x}, t) = \mathbf{0}$ using numerical predictor-corrector methods. For generic g, singularities occur only at end.

Software Systems

Starring in alphabetical order:

- **Bertini**, first released in Fall 2006, by D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler.
- **HOM4PS-2.0** by T.L. Lee, T.Y. Li, and C.H. Tsai (2007), extends **HOM4PS** by T. Gao and T.Y. Li.
- **PHoMpara** by T. Gunji, S. Kim, K. Fujisawa, and M. Kojima (2006) is a parallel version of **PHoM** by T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa and T. Mizutani (2004).
- **POLSYS_GLP** is Algorithm 857 of ACM TOMS (2006) by H.-J. Su, J.M. McCarthy, M. Sosonkina, and L.T. Watson extends **HOMPACK90** by L.T. Watson, M. Sosonkina, R.C. Melville, A.P. Morgan, and H.F. Walker (1997) and **HOMPACK** by L.T. Watson, S.C. Billups, and A.P. Morgan (1987).

Anton Leykin is bringing numerical algebraic geometry into Macaulay2.

PHCpack

a package for polynomial homotopy continuation methods

PHC = Polynomial Homotopy Continuation

- Version 1.0 archived as Algorithm 795 by ACM TOMS (1999)
- Pleasingly parallel implementations
  - + **Yusong Wang** of Pieri homotopies (HPSEC'04)
  - + **Anton Leykin** of monodromy factorization (HPSEC'05)
  - + **Yan Zhuang** of polyhedral homotopies (HPSEC'06)
  - + **Yun Guan** of diagonal homotopies (HPCS'08)
- Interactive Parallel Computing:
  - + **Yun Guan:** PHClab, experiments with MPITB in Octave
  - + **Kathy Piret:** bindings with Python, use of sockets

Release v2.3.42 extends **phcpy** and a preliminary **PHCwulf.py**.

Positive Dimensional Solution Sets

Slicing a positive dimensional solution set with enough general hyperplanes reduces to isolated solutions.

Definition

Given a system $f(\mathbf{x}) = \mathbf{0}$, we represent a component of $f^{-1}(\mathbf{0})$ of dimension k and degree d by ***a witness set*** which consists of f and

- k general hyperplanes L to cut the dimension; and
- d generic points in $f^{-1}(\mathbf{0}) \cap L$.

Using a flag of linear spaces, defined by an decreasing sequence of subsets of the k general hyperplanes,

$$L = L_k \supset L_{k-1} \supset \cdots \supset L_1 \supset L_0 = \emptyset,$$

we move solutions with nonzero slack values to generic points on lower dimensional components, using a cascade of homotopies.

Example of a Homotopy in the Cascade

To compute numerical representations of the twisted cubic and the four isolated points, as given by the solution set of one polynomial system, we use the following homotopy:

$$H(\mathbf{x}, z_1, t) = \left[\begin{array}{c} \left[\begin{array}{c} (x_1^2 - x_2)(x_1 - 0.5) \\ (x_1^3 - x_3)(x_2 - 0.5) \\ (x_1 x_2 - x_3)(x_3 - 0.5) \end{array} \right] + t \left[\begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{array} \right] z_1 \\ t\,(c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3) + z_1 \end{array} \right] = \mathbf{0}$$

At $t = 1$: $H(\mathbf{x}, z_1, t) = \mathcal{E}(f)(\mathbf{x}, z_1) = \mathbf{0}$.
At $t = 0$: $H(\mathbf{x}, z_1, t) = f(\mathbf{x}) = \mathbf{0}$.

As t goes to 0, the hyperplane is removed and z_1 goes to 0.

A.J. Sommese and C.W. Wampler: *the numerical solution of systems of polynomials arising in engineering and science.* World Scientific, 2005.

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As t goes to 0, the hyperplane is removed and z_1 goes to 0.

A.J. Sommese and C.W. Wampler: *the numerical solution of systems of polynomials arising in engineering and science.* World Scientific, 2005.

Newton Polytopes and Mixed Volumes

recognizing the sparse structure of a polynomial system

Most polynomials have few nonzero coefficients:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \neq 0, \quad \mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

The *support* A of f spans the Newton polytope $P = \mathrm{ConvHull}(A)$.
$\mathcal{P} = (P_1, P_2, \ldots, P_n)$ collects the Newton polytopes of a system f.

$N_{\mathbf{c}} =$ the number of solutions for generic coefficients $\mathbf{c}$.

Bernshteĭn's theorem (1975): $N_{\mathbf{c}}$ depends only on $\mathcal{P}$.

In particular: $N_{\mathbf{c}} = V(\mathcal{P})$, the mixed volume of $\mathcal{P}$.
Special case: $P = P_1 = P_2 = \cdots = P_n$: $N_{\mathbf{c}} = n!\mathrm{volume}(P)$.

Bernshteĭn's First Theorem

Mixed volumes relate volume to surface area:

$$V_n(P_1, P_2, \ldots, P_n) = \sum_{\mathbf{v}} p_1(\mathbf{v})\, V_{n-1}(\partial_{\mathbf{v}} P_2, \ldots, \partial_{\mathbf{v}} P_n),$$

$\mathbf{v} \in \mathbb{Z}^n$, $\gcd(\mathbf{v}) = 1$, $p_1(\mathbf{v}) = \min_{\mathbf{x} \in P_1} \langle \mathbf{x}, \mathbf{v} \rangle$ is a support function
$\partial_{\mathbf{v}} P_k = \{ \mathbf{x} \in P_k \mid \langle \mathbf{x}, \mathbf{v} \rangle = p_k(\mathbf{v}) \}$ is a face of P_k.

Theorem (Bernshteĭn Theorem A 1975)

The number of roots of a generic system equals the mixed volume of its Newton polytopes. For any system, the mixed volume bounds the number of isolated solutions in $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

D.N. Bernshteĭn: The number of roots of a system of equations.
Functional Anal. Appl. 9(3):183–185, 1975.

Witness Sets for Sparse Systems

Squaring up systems either by adding slack variables
or adding random combinations of the extra equations
$\longrightarrow$ call blackbox solver for isolated solutions.

Some problems with witness sets:

- Mixed volume increases when adding extra hyperplane.

  Examples: cyclic 8 roots: 2,560 $\rightarrow$ 4,176,
  cyclic 12 roots: 500,352 $\rightarrow$ 983,952.

- Exploiting symmetry with witness sets?
- Users require certificates of computations.

Systems with few monomials

- have few isolated roots
- need fewer data to represent solution curves

A Tropical View

joint with Danko Adrovic

Three observations:

1. If there is a positive dimensional solution set, then it stretches out to infinity. ***We tropicalize the polynomials in the system.***
2. Bernshteǐn 2nd theorem: solutions at infinity are solutions of systems supported on faces of the Newton polytopes. ***Tropisms identify those faces.***
3. Solutions at infinity give the leading coefficients of Puiseux series. ***The next term in a Puiseux series gives a certificate.***

Tropisms and the first terms of a Puiseux series expansion give apriori certificates for a solution component.

J. Maurer: Puiseux expansion for space curves. *Manuscripta Math.* 32: 91–100, 1980.

Binomial Systems

A binomial system has exactly two monomials with nonzero coefficient in every equation. We consider $n-1$ equations in n variables.
For example ($n=3$):

$$\left\{ \begin{array}{r} x_1 x_2^2 x_3 - 2 x_1^2 x_2^3 x_3 = 0 \\ 3 x_1^2 x_2^2 x_3^5 + 9 x_1 x_2 x_3 = 0 \end{array} \right. \equiv \left\{ \begin{array}{ccccr} x_1^{-1} & x_2^{-1} & & = & 2 \\ x_1^2 & x_2 & x_3^4 & = & -3 \end{array} \right. \quad \begin{array}{c} \text{normal} \\ \text{form} \end{array}$$

The normal form removes trivial solutions with zero components.
Collecting the exponents of the normal form in a matrix:

$$A = \left[\begin{array}{rrr} -1 & -1 & 0 \\ 2 & 1 & 4 \end{array} \right], \quad \mathrm{rank}(A) = n-1, \quad \mathbf{v} = \left[\begin{array}{r} 4 \\ -4 \\ -1 \end{array} \right] : A\mathbf{v} = \mathbf{0}.$$

We look for a solution $x_k = c_k t^{v_k}$, $c_k \in \mathbb{C}^*$, $k = 1, 2, \ldots, n$.

Unimodular Coordinate Transformations

$$\begin{cases} x_1^{-1} \; x_2^{-1} & = \; 2 \\ x_1^2 \; x_2 \; x_3^4 & = -3 \end{cases} \quad A = \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 4 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} +4 \\ -4 \\ -1 \end{bmatrix} : A\mathbf{v} = \mathbf{0}$$

Use **v** to define a unimodular matrix M ($\det(M) = 1$):

$$M = \begin{bmatrix} +4 & 0 & 1 \\ -4 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad AM = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{cases} x_1 = y_1^{+4} y_3 \\ x_2 = y_1^{-4} y_2 \\ x_3 = y_1^{-1} \end{cases}$$

After applying the coordinate transformation, defined by M:

$$\begin{cases} y_2^{-1} \; y_3^{-1} & = \; 2 \\ y_2 \; y_3^2 & = -3 \end{cases} \quad \begin{matrix} (y_2 = -\frac{1}{12}, y_3 = -6) \\ \text{is solution} \end{matrix} \quad \begin{cases} x_1 = -6t^{+4} \\ x_2 = -\frac{1}{12} t^{-4} \\ x_3 = t^{-1} \end{cases}$$

We found a curve of degree 8.

the Degree of a Curve

We use a random hyperplane to compute the degree:

$$\left\{ \begin{array}{rcl} x_1^{-1} \ x_2^{-1} & = & 2 \\ x_1^2 \ \ x_2 \ \ x_3^4 & = & -3 \\ c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} x_1 = -6t^{+4} \\ x_2 = -\frac{1}{12} t^{-4} \\ x_3 = t^{-1} \end{array} \right.$$

The coefficients c_0, c_1, c_2, and c_3 are random complex numbers.
After substitution: $c_0 + c_1(-6t^{+4}) + c_2(-\frac{1}{12}t^{-4}) + c_3 t^{-1}$
and clearing denominators, we find a polynomial in t of degree 8.

Given the leading term of the Puiseux series expansion for an algebraic curve, the degree of the curve follows from the tropism, i.e.: the leading exponents of the Puiseux series.

Tropisms and Initial Forms

Definition

Consider $f(\mathbf{x}) = \mathbf{0}$ with Newton polytopes $(P_1, P_2, \ldots, P_N)$. A ***tropism*** is a vector perpendicular to one edge of each P_i, for $i = 1, 2, \ldots, N$.

The edges perpendicular to a tropism are Newton polytopes of *an initial form system* which may have solutions in $(\mathbb{C}^*)^n$.

Definition

Let $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ and $f = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$. Denote the inner product by $\langle \cdot, \cdot \rangle$.

The ***initial form of f in the direction*** $\mathbf{v}$ is

$$\mathrm{in}_{\mathbf{v}}(f) = \sum_{\substack{\mathbf{a} \in A \\ \langle \mathbf{a}, \mathbf{v} \rangle = m}} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad \text{with} \quad m = \min\{ \langle \mathbf{a}, \mathbf{v} \rangle \mid \mathbf{a} \in A \}$$

Bernshteĭn's Second Theorem 1975

rephrased in the tropical language

Theorem (Bernshteĭn Theorem B 1975)

Consider $f(\mathbf{x}) = \mathbf{0}$, $f = (f_1, f_2, \ldots, f_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
Denote by $\mathcal{P}$ the tuple of Newton polytopes of f.
If for all tropisms $\mathbf{v}$: $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ has no solutions in $(\mathbb{C}^*)^n$,
then $f(\mathbf{x}) = \mathbf{0}$ has exactly as many isolated solutions in $(\mathbb{C}^*)^n$
as the mixed volume of $\mathcal{P}$.

- No tropisms $\mathbf{v}$: $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$ has roots in $(\mathbb{C}^*)^n$
  $\Rightarrow$ no solutions at infinity.

Solutions at infinity are roots of $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \mathbf{0}$.

Solving the cyclic 4-roots System

$$f(\mathbf{x}) = \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases}$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \begin{cases} x_2 + x_4 = 0 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0 \\ x_2x_3x_4 + x_4x_1x_2 = 0 \\ x_1x_2x_3x_4 - 1 = 0 \end{cases} \qquad \begin{cases} x_1 = y_1^{+1} \\ x_2 = y_1^{-1}y_2 \\ x_3 = y_1^{+1}y_3 \\ x_4 = y_1^{-1}y_4 \end{cases}$$

The system $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{y}) = \mathbf{0}$ has two solutions.
We find two solution curves: $(t, -t^{-1}, -t, t^{-1})$ and $(t, t^{-1}, -t, -t^{-1})$.

Asymptotics of Witness Sets

Deform a witness set in two stages:

1. move to a hyperplane in special position:

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ (c_0 + c_1 x_1 + \cdots + c_n x_n)(1-t) + (c_0 + c_1 x_1) t = 0 \end{cases}$$

2. rename $c_0 + c_1 x_1 = 0$ into $x_1 = \gamma$ and move γ to zero:

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ x_1 - \gamma t = 0 \end{cases} \quad \begin{cases} x_1 = t \\ x_2 = c_2 t^{v_2}(1 + O(t)) \\ \vdots \\ x_n = c_n t^{v_n}(1 + O(t)) \end{cases}$$

as $t \to 0$, the tropism is the direction of the solution path.

Move into Special Position

first deformation in the asymptotics of witness sets

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ (c_0 + c_1 x_1 + \cdots + c_n x_n)(1-t) + (c_0 + c_1 x_1)t = 0 \end{cases}$$

Some paths will diverge, e.g.: $f(x_1, x_2) = x_1 x_2 - 1$.

Claims:

1. Same result as solving $h(\mathbf{x}, 1) = \mathbf{0}$ directly.
   Exceptional case: curve belongs to $x_1 = c$, $c \in \mathbb{C}$.
2. In general, we recover lost data with tropisms.
   For example we represent the solution to $x_1 x_2 - 1 = 0$:

$$\begin{cases} x_1 = t \\ x_2 = t^{-1} \end{cases}$$

and the tropism is $\mathbf{v} = (1, -1)$. *Note: $v_1 > 0$.*

Polyhedral Endgames

after renaming $c_0 + c_1 x_1 = 0$ into $x_1 = \gamma$:

$$h(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) = \mathbf{0} \\ x_1 - \gamma t = 0 \end{cases} \quad \text{for } t \text{ going from } 1 \text{ to } 0.$$

Notes:

1. Paths run off to infinity for $v_k < 0$, recall hyperbola.
2. We can apply polyhedral endgames to compute tropisms.

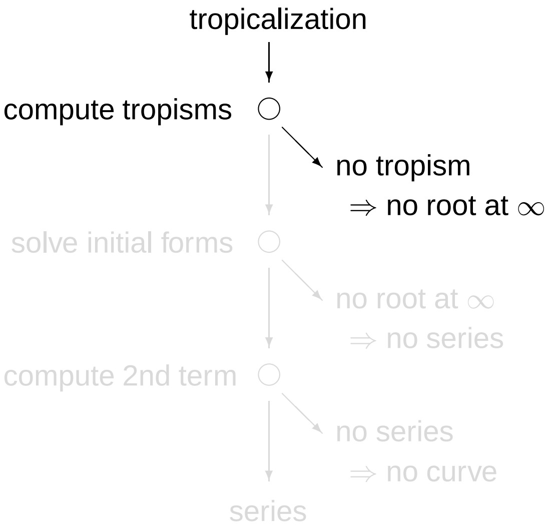
Claim: $\deg(f^{-1}(\mathbf{0})) = \displaystyle\sum_{\mathbf{v} \textit{ tropism}} \deg((\mathrm{in}_{\mathbf{v}}(f))^{-1}(\mathbf{0}))$.

Except for:

- solution components with zero coordinates are at infinity
- curves in $x_1 = c$ plane require special care

Computing a Series Expansion

a staggered approach to find a certificate for a solution curve



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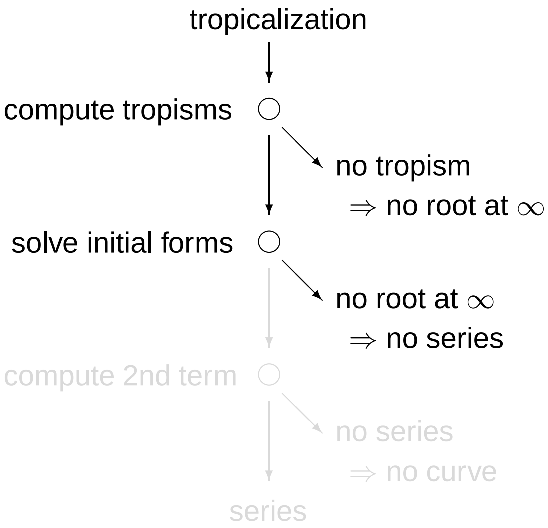

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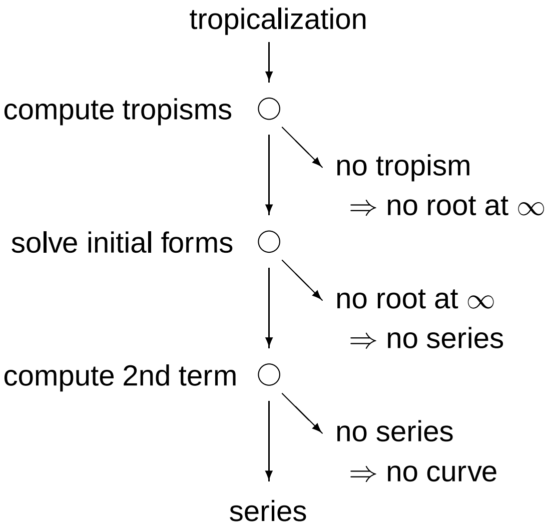

Computing Tropisms

- Gfan: tropical intersections

  A.N. Jensen: Computing Gröbner fans and tropical variaties in Gfan IMA Volume 148, pages 33–46, Springer 2008.

  Released under the GNU GPL license, is a component of Sage.

- MixedVol: computes mixed volumes

  T. Gao, T.Y. Li, and M. Wu: Algorithm 846: MixedVol: a software package for mixed-volume computation. *ACM Trans. Math. Softw.* 31(4):555–560, 2005.

  MixedVol is available in PHCpack since version 2.3.13.

  Interested in the relationship between the generic number of isolated solutions and the degrees of the solution curves.

Lifting for Mixed Volumes

Let $A = (A_1, A_2, \ldots, A_n)$ be the supports of $f(\mathbf{x}) = \mathbf{0}$.

Compute the mixed volume in three stages:

1. Lift $\mathbf{a} \in A$ using lifting function ω: $\omega(A_i) \subseteq \mathbb{R}^{n+1}$.
2. Facets on the lower hull of the Minkowski sum $\displaystyle\sum_{i=1}^{n} \omega(A_i)$ spanned by one edge of each of $\omega(A_i)$ define *mixed cells* C.
3. The mixed volume is $$V(A) = \sum_{\substack{C \subseteq A \\ C \text{ is mixed}}} \mathrm{Vol}(C).$$

By duality, mixed cells are defined by inner normals perpendicular to edges of the polytopes. These normals are tropisms.

B. Huber and B. Sturmfels: A polyhedral method for solving sparse polynomial systems. *Math. Comp.* 64(212): 1541–1555, 1995.

the Polyhedral Homotopy Idea

We lift polynomials introducing a parameter t:

$$f(\mathbf{x}) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \quad \rightarrow \quad \widehat{f}(\mathbf{x}, t) = \sum_{\mathbf{a} \in A} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} t^{\omega}(\mathbf{a})$$

using the same lifting function ω as before. (Note: $c_{\mathbf{a}}$ is random.)

Now we look for solution curves of the form

$$\begin{cases} x_1 = t \\ x_2 = c_2 t^{v_2}(1 + O(t)) \\ \quad \vdots \\ x_n = c_n t^{v_n}(1 + O(t)) \end{cases}$$

Idea: since $x_1 = t$, use $\omega = \deg(\mathbf{x}^{\mathbf{a}}, x_1) = a_1$ as lifting.

Squaring with Slack Variables

Let $f(\mathbf{x}) = \mathbf{0}$ have n equations in n variables.
But, if we use $t = x_1$ on f, then too few variables.

We use a slack variable z in the lifting:

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \rightarrow z^r x_2^{a_2} \cdots x_n^{a_n} t^{a_1}$$

where r is some random exponent.
Now we can apply our mixed volume calculators.

Claim: inner normals $\mathbf{v}$ with $\mathbf{v}_z = 0$ are tropisms.

Note: take $t = x_1$, $z = t$, and we get: $t^r x_2^{a_2} \cdots x_n^{a_n} x_1^{a_i} = \mathbf{x}^{\mathbf{a}} t^r$.

is random lifting used in mixed volume computation

Note: we get tropisms only for *proper* algebraic curves.

Preliminary Implementation and Results

- Modification of random lifting of `MixedVol` in PHCpack.
  $\longrightarrow$ computing of tropisms will be new feature for v2.3.45
  (the next release of PHCpack)

- Capable of handling 8-dimensional problems:
  - cyclic 8-roots system
    is one of the widely used benchmark systems
  - Griffis-Duffy platforms
    are architecturally singular Stewart-Gough mechanisms

- Using mixed volume calculators is effective
  but certainly not the most efficient way.
  The shape of the original Newton determines complexity.

the cyclic 8-roots system

a well known benchmark problem

a system of 8 equations in 8 unknowns:

$$f(\mathbf{z}) = \begin{cases} z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 = 0 \\ z_0 z_1 + z_1 z_2 + z_2 z_3 + z_3 z_4 + z_4 z_5 + z_5 z_6 + z_6 z_7 + z_7 z_0 = 0 \\ i = 3, 4, \ldots, 7 : \displaystyle\sum_{j=0}^{7} \prod_{k=j}^{i} z_{k \bmod 8} = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{cases}$$

J. Backelin: "Square multiples n give infinitely many cyclic n-roots". Reports, Matematiska Institutionen, Stockholms Universitet, 1989.
$n = 8$ has 4 as divisor, $4 = 2^2$, so infinitely many roots

how to verify numerically?

Tropisms for cyclic 8 roots

The program found 23 tropisms:

3	1	1	1	1	1	1	1	1	1	1	1
-1	1	-1	-1	-1	0	1	0	0	0	0	1
-1	-1	1	0	-1	0	1	0	0	0	-1	-3
-1	-1	-1	1	1	0	-3	-1	-1	-1	0	1
3	1	1	-1	1	-1	1	0	0	1	0	1
-1	1	-1	-1	-1	1	1	0	1	-1	0	1
-1	-1	1	1	-1	-1	1	1	-1	0	1	-3
-1	-1	-1	-1	1	0	-3	-1	0	0	-1	1

1	1	1	1	1	1	1	1	1	1	1
0	0	0	-1	-1	-1	-1	-3	-1	-1	-1
-1	-1	-1	0	0	1	0	1	1	1	1
1	1	1	0	0	0	1	1	0	0	-1
0	-1	-1	1	0	0	0	1	0	-1	0
-1	1	0	0	1	0	0	-3	-1	0	1
0	-1	0	0	0	-1	-1	1	0	0	0
0	0	0	-1	-1	0	0	1	0	0	-1

An Initial Form of the cyclic 8-roots system

For the tropism $\mathbf{v} = (-1, 0, 0, +1, 0, -1, +1, 0)$:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{z}) = \left\{ \begin{array}{r} z_0 + z_5 = 0 \\ z_0 z_1 + z_4 z_5 + z_7 z_0 = 0 \\ z_0 z_1 z_2 + z_7 z_0 z_1 = 0 \\ z_5 z_6 z_7 z_0 + z_7 z_0 z_1 z_2 = 0 \\ z_4 z_5 z_6 z_7 z_0 + z_5 z_6 z_7 z_0 z_1 = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 + z_4 z_5 z_6 z_7 z_0 z_1 + z_5 z_6 z_7 z_0 z_1 z_2 = 0 \\ z_4 z_5 z_6 z_7 z_0 z_1 z_2 + z_7 z_0 z_1 z_2 z_3 z_4 z_5 = 0 \\ z_0 z_1 z_2 z_3 z_4 z_5 z_6 z_7 - 1 = 0 \end{array} \right.$$

Observe: for all $\mathbf{z}^{\mathbf{a}}$: $\langle \mathbf{a}, \mathbf{v} \rangle = -1$,
except for the last equation: $\langle \mathbf{a}, \mathbf{v} \rangle = 0$.

Transforming Coordinates

to eliminate one variable

The tropism $\mathbf{v} = (-1, 0, 0, +1, 0, -1, +1, 0)$ defines a change of coordinates:

$$\left\{ \begin{array}{l} z_0 = x_0^{-1} \\ z_1 = x_0^{0} x_1 \\ z_2 = x_0^{0} x_2 \\ z_3 = x_0^{+1} x_3 \\ z_4 = x_0^{0} x_4 \\ z_5 = x_0^{-1} x_5 \\ z_6 = x_0^{+1} x_6 \\ z_7 = x_0^{0} x_7 \end{array} \right. \quad \mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}) = \left\{ \begin{array}{r} 1 + x_5 = 0 \\ x_1 + x_4 x_5 + x_7 = 0 \\ x_1 x_2 + x_7 x_1 = 0 \\ x_5 x_6 x_7 + x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 + x_5 x_6 x_7 x_1 = 0 \\ x_1 x_2 x_3 x_4 x_5 + x_4 x_5 x_6 x_7 x_1 \\ + x_5 x_6 x_7 x_1 x_2 = 0 \\ x_4 x_5 x_6 x_7 x_1 x_2 + x_7 x_1 x_2 x_3 x_4 x_5 = 0 \\ x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 = 0 \end{array} \right.$$

After clearing x_0, $\mathrm{in}_{\mathbf{v}}(f)$ consists of 8 equations in 7 unknowns.

Solving an overconstrained Initial Form

Choose eight random numbers $\gamma_k \in \mathbb{C}$, $k = 1, 2, \ldots, 8$, to introduce a slack variable s:

$$\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}, s) = \left\{ \begin{array}{r} 1 + x_5 + \gamma_1 s = 0 \\ x_1 + x_4 x_5 + x_7 + \gamma_2 s = 0 \\ x_1 x_2 + x_7 x_1 + \gamma_3 s = 0 \\ x_5 x_6 x_7 + x_7 x_1 x_2 + \gamma_4 s = 0 \\ x_4 x_5 x_6 x_7 + x_5 x_6 x_7 x_1 + \gamma_5 s = 0 \\ x_1 x_2 x_3 x_4 x_5 + x_4 x_5 x_6 x_7 x_1 + x_5 x_6 x_7 x_1 x_2 + \gamma_6 s = 0 \\ x_4 x_5 x_6 x_7 x_1 x_2 + x_7 x_1 x_2 x_3 x_4 x_5 + \gamma_7 s = 0 \\ x_1 x_2 x_3 x_4 x_5 x_6 x_7 - 1 + \gamma_8 s = 0 \end{array} \right.$$

The mixed volume of this system is 25 and is exact.
Among the 25 solutions, there are 8 with $s = 0$.

The first Term of a Puiseux Expansion

For $f(\mathbf{x}) = \mathrm{in}_{\mathbf{e}} f(x) + O(x_0)$, $\mathbf{e} = (1, 0, 0, 0, 0, 0, 0, 0)$,
we use a solution as the leading term of a Puiseux expansion:

$$\left\{ \begin{array}{ll} x_0 = t^1 & \\ x_1 = (\ 0.5 + 0.5i\ )\ t^0 & +y_1\ t \\ x_2 = (\ 1 + i\ )\ t^0 & +y_2\ t \\ x_3 = (\ -i\ )\ t^0 & +y_3\ t \\ x_4 = (\ -0.5 - 0.5i\ )\ t^0 & +y_4\ t \\ x_5 = (\ -1\ )\ t^0 & +y_5\ t \\ x_6 = (\ i\ )\ t^0 & +y_6\ t \\ x_7 = (\ -1 - i\ )\ t^0 & +y_7\ t \end{array} \right. \qquad i = \sqrt{-1}.$$

Decide whether solution is isolated: substitute series in $f(\mathbf{x}) = \mathbf{0}$
and solve for y_k, $k = 1, 2, \ldots, 7$ in lowest order terms of t.
$\rightarrow$ solve an overdetermined linear system in the coefficients
of the 2nd term of the Puiseux expansion.

The second Term of a Puiseux Expansion

Because we find a nonzero solution for the y_k coefficients, we use it as the second term of a Puiseux expansion:

$$\left\{\begin{array}{lll} x_0 = t^{1} & & \\ x_1 = (\ 0.5 + 0.5i\ )\ t^{0} & + (\ -0.5i\ )\ t & \\ x_2 = (\ 1 + i\ )\ t^{0} & + (\ -i\ )\ t & \\ x_3 = (\ -i\ )\ t^{0} & + (\ 1 - i\ )\ t & \\ x_4 = (\ -0.5 - 0.5i\ )\ t^{0} & + (\ 0.5i\ )\ t & \\ x_5 = (\ -1\ )\ t^{0} & + (\ 0\ )\ t & \\ x_6 = (\ i\ )\ t^{0} & + (\ -1 + i\ )\ t & \\ x_7 = (\ -1 - i\ )\ t^{0} & + (\ i\ )\ t & \end{array}\right. \qquad i = \sqrt{-1}.$$

Substitute series in $f(\mathbf{x})$: result is $O(t^2)$.

the cyclic 12-roots problem

According to J. Backelin, also here infinitely many solutions.

Mixed volume is 500,352 and increases to 983,952
by adding one random hyperplane and slack variable.

Like for cyclic 8, $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$
is a tropism. Mixed volume of $\mathrm{in}_{\mathbf{v}}(f)(\mathbf{x}, s) = \mathbf{0}$ is 49,816.
One of the solutions is

$x_0 = t$	$x_1 = 0.5 - 0.866025403784439i$
$x_2 = -1.0$	$x_3 = -0.5 - 0.866025403784439i$
$x_4 = -0.5 + 0.866025403784439i$	$x_5 = 0.5 + 0.866025403784439i$
$x_6 = -1.0$	$x_7 = -0.5 + 0.866025403784438i$
$x_8 = 1.0$	$x_9 = 0.5 + 0.866025403784438i$
$x_{10} = 0.5 - 0.866025403784439i$	$x_{11} = -0.5 - 0.866025403784439i$

It satisfies not only $\mathrm{in}_{\mathbf{v}}(f)$, but also f itself.

An Exact Solution for cyclic 12-roots

For the tropism $\mathbf{v} = (-1, +1, -1, +1, -1, +1, -1, +1, -1, +1, -1, +1)$:

$$
\begin{array}{ll}
z_0 = t^{-1} & z_1 = t\left(\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right) \\
z_2 = -t^{-1} & z_3 = t\left(-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right) \\
z_4 = t^{-1}\left(-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) & z_5 = t\left(\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) \\
z_6 = -t^{-1} & z_7 = t\left(-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) \\
z_8 = t^{-1} & z_9 = t\left(\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) \\
z_{10} = t^{-1}\left(\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right) & z_{11} = t\left(-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right)
\end{array}
$$

makes the system entirely and exactly equal to zero.

Conclusions

An apriori certificate for a solution component consists of

1. a tropism: leading powers of a Puiseux series,
2. a root at infinity: leading coefficients of the Puiseux series,
3. the next term in the Puiseux series.

The certificate is compact and easy to verify with substitution.

Preprocessing for more costly representations:

- either lifting fibers for a geometric resolution,
- or witness sets in a numerical irreducible decomposition.