Newton’s Method with Deflation for Isolated Singularities

Jan Verschelde and Ailing Zhao

email: jan@math.uic.edu
azhaol@math.uic.edu

University of Illinois at Chicago, Department of Mathematics, Statistics and Computer Science, 851 S. Morgan (Mail Code 249), IL 60607-7045, USA

1. A Modified Deflation Method
A singular root \( x^* \) of a square system \( F(x) = 0 \) satisfies
\[
F(x) = 0 \quad \text{det}(A(x)) = 0,
\]
which forms the basis of deflation, as adapted and modified in [1], [2], and [3].

In theory, \( \text{det}(A(x)) = 0 \) (or maximal minors) could be used as new equations.

But: 1. high in degree: expression swell; and moreover
2. numerically unstable: \( \text{det}(A(x)) \neq \text{det}(\mathcal{R}(x)) \) for all \( x \in \mathbb{C} \).

Three steps to set up new equations:
1. Obtain \( R = \text{Rank}(A(x)) \) for \( x \neq x^* \) using Singular Value Decomposition (SVD).
2. Generate a random vector \( \alpha \in \mathbb{C}^{n-1} \) and a random matrix \( \beta \in \mathbb{C}^{n \times (n-1)} \).
3. Let \( C = \beta^T / \| \beta \|, \quad C = \begin{pmatrix} x_1 & \ldots & x_k \end{pmatrix}, \quad \text{adding } R + 1 \text{ new unknowns}.
\[
\text{det}(A(x^*)) = 0 \quad \Longleftrightarrow \quad \text{det}(A(x^*) + \alpha \beta^T / \| \beta \|) = 0
\]
\[
\Rightarrow \exists \lambda_1, \lambda_2, \ldots, \lambda_{n-k} : G(x, \lambda) = 0
\]
where \( \alpha_1, \alpha_2, \ldots, \alpha_{n-k} : G(x, \lambda) = 1 \).

The random \( \alpha \) and \( \beta \) guarantee there is a unique solution for \( \lambda \).
Add \( G(x, \lambda) \) instead of \( \text{det}(A(x)) = 0 \) to the system \( F(x) = 0 \).

References