Polyhedral Methods to Solve Polynomial Systems I: computing pure dimensional solution sets

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Outline

Solving Polynomial Systems by Polyhedral Methods I

1. Introduction
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2. Solving Binomial Systems
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   - computation of the degree and affine sets
3. Polyhedral Methods for Solution Curves
   - looking for power series as solutions
   - pretropisms and initial forms
4. Application to the Cyclic n-Roots Problem
   - a tropical version of Backelin's Lemma

polynomial systems

Consider $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, a system of equations defined by

- N polynomials $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1})$,
- in n variables $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$.

A polynomial in n variables consists of a vector of nonzero complex coefficients with corresponding exponents in A:

$$f_k(\mathbf{x}) = \sum_{\mathbf{a} \in A_k} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}, \quad c_{\mathbf{a}} \in \mathbb{C} \setminus \{0\}, \quad \mathbf{x}^{\mathbf{a}} = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}.$$

Input data:

1. $A = (A_0, A_1, \ldots, A_{N-1})$ are sets of exponents, the *supports*. For $\mathbf{a} \in \mathbb{Z}^n$, we consider *Laurent* polynomials, $f_k \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ $\Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
2. $\mathbf{c}_A = (\mathbf{c}_{A_0}, \mathbf{c}_{A_1}, \ldots, \mathbf{c}_{A_{N-1}})$ are vectors of complex coefficients. Although A is exact, the coefficients may be approximate.

solving systems – input/output specification

Input data:

1. $A = (A_0, A_1, \ldots, A_{N-1})$ are sets of exponents, the *supports*. For $\mathbf{a} \in \mathbb{Z}^n$, we consider *Laurent* polynomials, $f_k \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ $\Rightarrow$ only solutions with coordinates in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ matter.
2. $\mathbf{c}_A = (\mathbf{c}_{A_0}, \mathbf{c}_{A_1}, \ldots, \mathbf{c}_{A_{N-1}})$ are vectors of complex coefficients. Although A is exact, the coefficients may be approximate.

Output data: *What do we want to know?*

1. For dimension zero:
   1. the number of isolated solutions, match a generic bound?
   2. approximations for the coordinates, regular or singular?
2. For positive dimension:
   1. the dimensions and the degrees of the solution sets?
   2. power series developments as approximations for the solutions?

four influencial publications

- D.N. Bernshteĭn. **The number of roots of a system of equations.** *Functional Anal. Appl.*, 9(3):183–185, 1975.
- J. Richter-Gebert, B. Sturmfels, and T. Theobald. **First steps in tropical geometry.** *Contemporary Mathematics* 377:289–317, AMS, 2005.
- T. Bogart, A.N. Jensen, D. Speyer, B. Sturmfels. and R.R. Thomas: **Computing tropical varieties.** *Journal of Symbolic Computation* 42(1): 54–73, 2007.
- A.N. Jensen, H. Markwig, and T. Markwig. **An algorithm for lifting points in a tropical variety.** *Collectanea Mathematica*, 59(2):129–165, 2008.

Solving Polynomial Systems by Polyhedral Methods I

binomial systems

Definition

A ***binomial system*** has exactly two monomials with nonzero coefficient in every equation.

The binomial equation $c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} - c_{\mathbf{b}}\mathbf{x}^{\mathbf{b}} = 0$, $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, $c_{\mathbf{a}}, c_{\mathbf{b}} \in \mathbb{C} \setminus \{0\}$, has normal representation $\mathbf{x}^{\mathbf{a}-\mathbf{b}} = c_{\mathbf{b}}/c_{\mathbf{a}}$.

A binomial system of N equations in n variables is then defined by an exponent matrix $A \in \mathbb{Z}^{N \times n}$ and a coefficient vector $\mathbf{c} \in (\mathbb{C}^*)^N$: $\mathbf{x}^A = \mathbf{c}$.

Solutions of binomial systems are monomial maps.

1. A unimodular coordinate transformation provides a monomial parametrization for the solution set.
2. Finding all solutions with zero coordinates can happen via a generalized permanent calculation.

an example

Consider as an example for $\mathbf{x}^A = \mathbf{c}$ the system

$$\left\{ \begin{array}{r} x_0^2 x_1 x_2^4 x_3^3 - 1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{array} \right. \quad A = \left[\begin{array}{cccc} 2 & 1 & 4 & 3 \\ 1 & 1 & 1 & 1 \end{array} \right]^T \quad \mathbf{c} = \left[\begin{array}{c} 1 \\ 1 \end{array} \right].$$

As basis of the null space of A we can for example take $\mathbf{u} = (-3, 2, 1, 0)$ and $\mathbf{v} = (-2, 1, 0, 1)$.

The vectors $\mathbf{u}$ and $\mathbf{v}$ are tropisms for a two dimensional algebraic set.

Placing $\mathbf{u}$ and $\mathbf{v}$ in the first two rows of a matrix M, extended so $\det(M) = 1$, we obtain a coordinate transformation, $\mathbf{x} = \mathbf{y}^M$:

$$M = \left[\begin{array}{rccc} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \quad \left\{ \begin{array}{l} x_0 = y_0^{-3} y_1^{-2} y_2 \\ x_1 = y_0^2 y_1 y_3 \\ x_2 = y_0 \\ x_3 = y_1. \end{array} \right.$$

monomial transformations

By construction, as $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$:

$$MA = \begin{bmatrix} -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 4 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} = B.$$

The corresponding monomial transformation $\mathbf{x} = \mathbf{y}^M$ performed on $\mathbf{x}^A = \mathbf{c}$ yields $\mathbf{y}^{MA} = \mathbf{y}^B = \mathbf{c}$, eliminating the first two variables:

$$\begin{cases} y_2^2 y_3 - 1 = 0 \\ y_2 y_3 - 1 = 0. \end{cases}$$

Solving this reduced system gives values z_2 and z_3 for y_2 and y_3. Leaving y_0 and y_1 as parameters t_0 and t_1 we find as solution

$$(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1).$$

unimodular coordinate transformations

Definition

A ***unimodular coordinate transformation*** $\mathbf{x} = \mathbf{y}^M$ is determined by an invertible matrix $M \in \mathbb{Z}^{n \times n}$: $\det(M) = \pm 1$.

For a d dimensional solution set of a binomial system:

1. The null space of A gives d tropisms, stored in the rows of a d-by-n-matrix B.
2. Compute the Smith normal form S of B: $UBV = S$.
3. There are three cases:
   1. $U = I \Rightarrow M = V^{-1}$
   2. If $U \neq I$ and S has ones on its diagonal, then extend U^{-1} with an identity matrix to form M.
   3. Compute the Hermite normal form H of B and let D be the diagonal elements of H, then $M = \left[\begin{array}{cc} D^{-1} \cdot B \\ \mathbf{0} & I \end{array} \right]$.

Solving Polynomial Systems by Polyhedral Methods I

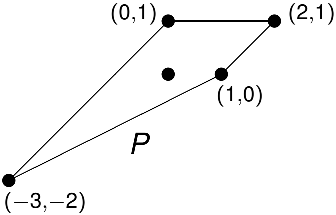
computation of the degree

To compute the degree of $(x_0 = z_2 t_0^{-3} t_1^{-2}, x_1 = z_3 t_0^2 t_1, x_2 = t_0, x_3 = t_1)$ we use two random linear equations:

$$\left\{ \begin{array}{l} c_{10}x_0 + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14} = 0 \\ c_{20}x_0 + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + c_{24} = 0 \end{array} \right.$$

after substitution:

$$\left\{ \begin{array}{l} c'_{10}t_0^{-3}t_1^{-2} + c'_{11}t_0^2 t_1 + c_{12}t_0 + c_{13}t_1 + c_{14} = 0 \\ c'_{20}t_0^{-3}t_1^{-2} + c'_{21}t_0^2 t_1 + c_{22}t_0 + c_{23}t_1 + c_{24} = 0 \end{array} \right.$$



Theorem (Koushnirenko's Theorem)

If all n polynomials in $\mathbf{f}$ *share the same Newton polytope P, then the number of isolated solutions of* $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ *in* $(\mathbb{C}^*)^n \leq$ *the volume of P.*

As the area of the Newton polygon equals 8, the surface has degree 8.

an application: adjacent minors

All adjacent 2-by-2 minors of a general 2-by-4 matrix:

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ x_{13}x_{24} - x_{23}x_{14} = 0 \end{cases}$$

which has

- one 5-dimensional solution of degree four, with all its components different from zero; and
- two affine solutions, each of degree two.

Three references:

- *Lattice Walks and Primary Decomposition*, by Diaconis, Eisenbud, and Sturmfels, 1998.
- *Combinatorics of binomial primary decomposition*, by Dickenstein, Matusevich, and Miller, 2010.
- *Affine solution sets of sparse polynomial systems*, by Herrero, Jeronimo, and Sabia, 2012.

affine solution sets

An incidence matrix M of a bipartite graph:

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{l} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \end{array} \right. \qquad M[\mathbf{x}^{\mathbf{a}}, x_k] = \left\{ \begin{array}{lll} 1 & \text{if} & a_k > 0 \\ 0 & \text{if} & a_k = 0. \end{array} \right.$$

Meaning of $M[\mathbf{x}^{\mathbf{a}}, x_k] = 1$: $x_k = 0 \Rightarrow \mathbf{x}^{\mathbf{a}} = 0$.

The matrix linking monomials to variables is

$$M[\mathbf{x}^{\mathbf{a}}, x_k] = \left[\begin{array}{c|cccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \hline x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right].$$

Observe: overlapping columns x_{12} with x_{22} gives all ones.

enumerating all candidate affine solution sets

Apply row expansion on the matrix

$$M[\mathbf{x^a}, x_k] = \left[\begin{array}{c|cccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \hline x_{11}x_{22} & 1 & 0 & 0 & 0 & 1 & 0 \\ x_{21}x_{12} & 0 & 1 & 0 & 1 & 0 & 0 \\ x_{12}x_{23} & 0 & 1 & 0 & 0 & 0 & 1 \\ x_{22}x_{13} & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right].$$

- Selecting 1 means setting the corresponding variable to zero.
- Monomials must be considered in pairs: if one monomial in an equation vanishes, then so must the other one.
- For all affine sets, we must skip pairs of rows, preventing from certain variables to be set to zero.
- To decide whether one candidate set C_1 belongs to another set C_2, we construct the defining equations $I(C_1)$ and $I(C_2)$ and apply $C_1 \subseteq C_2 \Leftrightarrow I(C_1) \supseteq I(C_2)$.

running times in seconds on adjacent minors

n	2^{n-1}	#maps	search	witness
3	4	2	0.00	0.03
4	8	3	0.00	0.16
5	16	5	0.00	0.68
6	32	8	0.00	2.07
7	64	13	0.01	7.68
8	128	21	0.01	28.10
9	256	34	0.02	71.80
10	512	55	0.05	206.01
11	1024	89	0.10	525.46
12	2048	144	0.24	—
13	4096	233	0.57	—
14	8192	377	1.39	—
15	16384	610	3.33	—
16	32768	987	8.57	—
17	65536	1597	21.36	—
18	131072	2584	55.95	—
19	262144	4181	140.84	—
20	524288	6765	372.62	—
21	1048576	10946	994.11	—

- For a general 2-by-n matrix, consider adjacent 2-by-2 minors.
- The degree of the solution set is 2^{n-1}, for $2n$ variables.
- The number of irreducible factors is the n-th Fibonacci number. This number equals #maps.
- Times are in seconds, on 1 core of a 3.49GHz Linux workstation. Times are $<$ 1,000 seconds.
- The "witness" column times the computation of 2^{n-1} generic points.
- The "search" column times the computation of all monomial maps.
- Conclusion: it works!

Solving Polynomial Systems by Polyhedral Methods I

the cyclic 4-roots problem

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{array} \right.$$

- The cyclic n-roots system is a benchmark in computer algebra.
- Its solutions are important in the study of biunimodular vectors.
- Haagerup: for prime p, there are $\left(\begin{array}{c} 2p-2 \\ p-1 \end{array} \right)$ isolated roots.
- Backelin: for $n = \ell m^2$, there are infinitely many cyclic n-roots.
- Björck and Saffari conjecture: if n is not divisible by a square, then the set of cyclic n-roots is finite.

power series as solutions

If we expect that

$$\mathbf{f}(\mathbf{x}) = \left\{ \begin{array}{c} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 = 0 \\ x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_3x_0x_1 = 0 \\ x_0x_1x_2x_3 - 1 = 0 \end{array} \right.$$

has a solution curve, then we look for solutions of the form

$$x_0 = t^{v_0}, x_1 = c_1 t^{v_1}(1 + O(t)), x_2 = c_2 t^{v_2}(1 + O(t)), x_3 = c_3 t^{v_3}(1 + O(t)),$$

where

- the leading exponents v_0, v_1, v_2, and v_3 are integer numbers; and
- the leading coefficients c_1, c_2, and c_3 are *nonzero* complex coefficients.

conditions on the exponents

We can compute the exponents of the power series solutions, *before* computing the coefficients.

For example, for cyclic 4-roots, substituting the series

$$x_0 = t^{v_0}, x_1 = c_1 t^{v_1}(1 + O(t)), x_2 = c_2 t^{v_2}(1 + O(t)), x_3 = c_3 t^{v_3}(1 + O(t)),$$

into the last equation:

$$x_0 x_1 x_2 x_3 - 1 = 0$$

leads to

$$c_1 c_2 c_3 t^{v_0 + v_1 + v_2 + v_3}(1 + O(t)) - 1 = 0.$$

For all coefficients to be nonzero: $c_1 c_2 c_3 - 1 = 0$ is equivalent to

$$v_0 + v_1 + v_2 + v_3 = 0.$$

Solving Polynomial Systems by Polyhedral Methods I

initial forms

Substituting $x_k = c_k t^{v_k}(1 + O(t))$ into $f(\mathbf{x}) = \displaystyle\sum_{\mathbf{a} \in A} c_\mathbf{a} \mathbf{x}^\mathbf{a}$ leads to

$$
\begin{aligned}
& f(x_0 = t^{v_0}, x_1 = c_1 t^{v_1}(1 + O(t)), \ldots, x_{n-1} = c_{n-1} t^{v_{n-1}}(1 + O(t))) \\
= & \sum_{\mathbf{a} \in A} (t^{v_0})^{a_0} (c_1 t^{v_1}(1 + O(t)))^{a_1} \cdots (c_{n-1} t^{v_{n-1}}(1 + O(t)))^{a_{n-1}} \\
= & \sum_{\mathbf{a} \in A} (c_1^{a_1} \cdots c_{n-1}^{a_{n-1}}) t^{\overbrace{v_0 a_0 + v_1 a_1 + \cdots + v_{n-1} a_{n-1}}^{= \langle \mathbf{v}, \mathbf{a} \rangle}} (1 + O(t))
\end{aligned}
$$

Definition

For $\mathbf{v} \neq \mathbf{0}$, and a polynomial f supported on A, ***the initial form*** $\mathrm{in}_\mathbf{v}(f)$ is

$$
\mathrm{in}_\mathbf{v}(f)(\mathbf{x}) = \sum_{\langle \mathbf{v}, \mathbf{a} \rangle = m_\mathbf{v}(A)} c_\mathbf{a} \mathbf{x}^\mathbf{a}, \quad m_\mathbf{v}(A) = \min_{\mathbf{a} \in \mathbf{A}} \langle \mathbf{v}, \mathbf{a} \rangle.
$$

curves of cyclic 4-roots

$$\mathbf{f}(\mathbf{x}) = \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 = 0 \\ x_0 x_1 x_2 + x_1 x_2 x_3 + x_2 x_3 x_0 + x_3 x_0 x_1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0 \end{cases}$$

One tropism $\mathbf{v} = (+1, -1, +1, -1)$ with $\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{z}) = \mathbf{0}$:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(\mathbf{x}) = \begin{cases} x_1 + x_3 = 0 \\ x_0 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_0 = 0 \\ x_1 x_2 x_3 + x_3 x_0 x_1 = 0 \\ x_0 x_1 x_2 x_3 - 1 = 0. \end{cases}$$

We look for solutions of the form

$$(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1}).$$

solving the initial form system

Substitute $(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$:

$$\mathrm{in}_{\mathbf{v}}(\mathbf{f})(x_0 = t^{+1}, x_1 = z_1 t^{-1}, x_2 = z_2 t^{+1}, x_3 = z_3 t^{-1})$$

$$= \left\{ \begin{array}{c} (1 + z_2)t^{+1} = 0 \\ z_1 + z_1 z_2 + z_2 z_3 + z_3 = 0 \\ (z_1 z_2 + z_3 z_1)t^{+1} = 0 \\ z_1 z_2 z_3 - 1 = 0. \end{array} \right.$$

We find two solutions: $(z_1 = -1, z_2 = -1, z_3 = +1)$ and $(z_1 = +1, z_2 = -1, z_3 = -1)$.

Two space curves $\left(t, -t^{-1}, -t, t^{-1}\right)$ and $\left(t, t^{-1}, -t, -t^{-1}\right)$ satisfy the entire cyclic 4-roots system.

overview of our polyhedral methods

- finding pretropisms and solving initial forms

  Initial forms with at least two monomials in every equation define the intersection points of the solution set with the coordinate hyperplanes.

- unimodular coordinate transformations

  Via the Smith normal form we obtain nice representations for solutions at infinity.
  Solutions of binomial systems are monomial maps.

- computing terms of power series

  Although solutions to any initial forms may be monomial maps, in general we need more terms in the power series expansion to distinguish between

  - a positive dimensional solution set, and
  - an isolated solution at infinity.

Solving Polynomial Systems by Polyhedral Methods I

a tropical version of Backelin's Lemma

Lemma (Tropical Version of Backelin's Lemma, CASC 2013)

For $n = m^2\ell$, where $\ell \in \mathbb{N} \setminus \{0\}$ and ℓ is no multiple of k^2, for $k \geq 2$, there is an $(m-1)$-dimensional set of cyclic n-roots, represented exactly as

$$
\begin{array}{rcl}
x_{km+0} & = & u^k t_0 \\
x_{km+1} & = & u^k t_0 t_1 \\
x_{km+2} & = & u^k t_0 t_1 t_2 \\
 & \vdots & \\
x_{km+m-2} & = & u^k t_0 t_1 t_2 \cdots t_{m-2} \\
x_{km+m-1} & = & \gamma u^k t_0^{-m+1} t_1^{-m+2} \cdots t_{m-3}^{-2} t_{m-2}^{-1}
\end{array}
$$

for $k = 0, 1, 2, \ldots, \ell m - 1$, free parameters $t_0, t_1, \ldots, t_{m-2}$, constants $u = e^{\frac{i2\pi}{m\ell}}$, $\gamma = e^{\frac{i\pi\beta}{m\ell}}$, with $\beta = (\alpha \mod 2)$, and $\alpha = m(m\ell - 1)$.

one polyhedral cone for cyclic 16-roots

For cyclic 16-roots, there is a *three* dimensional solution set.
The rays in the tropical version of Backelin's Lemma are

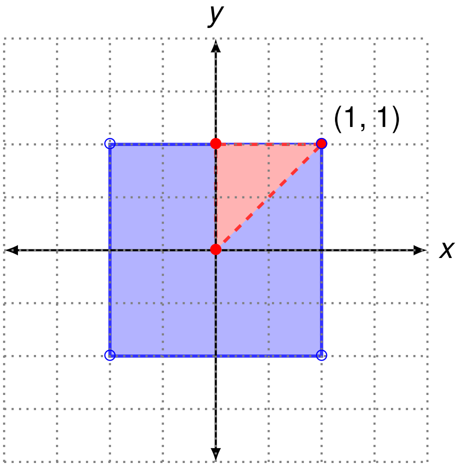
$$
\begin{array}{l}
(1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3),\\
(0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2, 0, 1, 1, -2),\\
(0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1, 0, 0, 1, -1).
\end{array}
$$

Based on the tropical prevariety (PASCO 2017), the cone spanned by the rays above are contained in the relative interior of a cone spanned by the following *four* rays:

$$
\begin{array}{l}
(1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3),\\
(-1, -1, 3, -1, -1, -1, 3, -1, -1, -1, 3, -1, -1, -1, 3, -1),\\
(1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1),\\
(-1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1).
\end{array}
$$

one polyhedral cone for cyclic 16-roots – continued

Every ray in the interior of the two polyhedral cones defines the *same* initial form system.



conclusions

Sparse polynomial systems may have sparse solution sets.

- Solution sets of binomial systems are monomial maps.

- Interested in a solution set of dimension d?
  $\rightarrow$ Examine the initial form systems defined by the cones in the tropical prevariety of dimension d.

- Solution curves are represented by power series:
  - The leading powers of the series define initial form systems.
  - The leading coefficients of the power series are solutions of the initial form systems.

- Encouraging results for the cyclic n-roots problem.