Homotopies for positive dimensional solution components of polynomial systems – exercise session

Jan Verschelde

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1. The equations defining the adjacent 2-by-2-minors of a general 2-by-4-matrix are

```
3 8
 x11*x22 - x21*x12;
 x12*x23 - x22*x13;
 x13*x24 - x23*x14;
```

So we are looking at a system of 3 polynomials of the 8 undeterminates of a general 2-by-4-matrix. We save this file under the name `minors`.

(a) As we have three equations in eight unknowns, we expect the solution set to be of dimension five. Call `phc -c` and select the first option. We will create an embedding of the minors system. Save this new system under the name `minors_e5`. When `phc` asks for the expected top dimension, type in 5. We consider general slices, so no restrictions to a subspace should be made. Look at the file `minors_e5` for the slack variables and the added hyperplanes.

(b) Solve the embedded system with phc. We can use the total degree homotopy, or for convenience, just the blackbox solver, typing `phc -b minors_e5 output`. The output file is only relevant for diagnostic purposes (e.g., how long it took). If all went well, the file `minors_e5` now contains a witness set to represent the five dimensional solution components of the system. What is the degree of the five dimensional solution set?

(c) To factor the 5-dimensional solution component into irreducible components, we call `phc -f` and select the second option from the menu. The input file is of course `minors_e5` and a good name for the output file is `minors_e5.fac`. There are two factorization methods. Try them both (thus calling `phc -f` once more), and compare the results and timings. How many irreducible factors do you find? What are the degrees of the factors? Which method was fastest?

(d) The system we considered is the first of a whole family of adjacent 2-by-2-minors of a general 2-by-n-matrix. In using `phc` for larger values of n, the total degree homotopy is recommended to find the witness sets (just call `phc` without any option on the embedded system). For components of larger degrees, say a couple of hundreds, or exceeding one thousand, the monodromy method (first option in the menu of factorization methods) is superior above the method which does the plain combinatorial enumeration of all possible factors.

2. The cyclic 4-roots problem

```
4
 x1 + x2 + x3 + x4;
 x1*x2 + x2*x3 + x3*x4 + x4*x1;
 x1*x2*x3 + x2*x3*x4 + x3*x4*x1 + x4*x1*x2;
 x1*x2*x3*x4 - 1;
```

has a one dimensional solution set. Save this system under the name `cyclic4`.

(a) Call `phc -c` to generate an embedding of cyclic4. A good name for the file to store the embedded system is `cyclic4_e1`, since the top dimension is one.

(b) Compute a witness set by solving `cyclic4_e1`, calling the blackbox solver like `phc -b cyclic4_e1 output`. After the computation, look into the file `cyclic4_e1` for the witness points. What is the degree of the one dimensional component?

(c) To run the cascade of homotopy, call `phc -c` again, choosing the second option, with the system `cyclic4_e1` on input. If we look again in the file `cyclic4_e1`, we see that two lists of solutions: one with vectors of dimension five (but with slack variable equal to zero), while the solution vectors of the second list are of dimension 4. The first list is a witness set for the one dimensional solution component. You should see the degree of the one dimensional solution component better now.

(d) With `phc -f` we can factor the one dimensional solution set into irreducible components, selecting option 2 from the first menu of `phc -f` and giving the file `cyclic4_e1` on input. Give the low degree of the solution set, the second factorization method will be fastest. How many irreducible factors does the program find? What are the degrees of the irreducible factors?

(e) We still have to deal with the second list of solutions found as output of the cascade. Save this second list on a separate file (say `cyclic4tsols`). The first line of this solution list should be `16 4`, respectively the number of test solutions and the dimension of each solution. Then call `phc -f` and select the first item of the menu, as we are going to filter this solution list, in search for isolated solution of the original system. We need to give two input files. The first file is `cyclic4_e1` which contains a witness set for the one dimensional solution set. The second file is the list of test solutions (we saved this list in `cyclic4tsols`). The output file lists several diagnostics and intermediate results but from the last lines we should see how many of the test solutions were found to lie on the one dimensional solution set. How many isolated solutions does the system have?

3. There are special factorization routines in phc for one single polynomial in several variables. As example, consider

```
3
 x**6 - x**5*y + 2*x**5*z - x**4*y**2 - x**4*y*z+x**3*y**3 - 4*x**3*y**2*z
 + 3*x**3*y*z**2 - 2*x**3*z**3 + 3*x**2*y**3*z - 6*x**2*y**2*z**2
 + 5*x**2*y*z**3 - x**2*z**4 + 3*x*y**3*z**2 - 4*x*y**2*z**3
 + 2*x*y*z**4+y**3*z**3 - y**2*z**4;
```

and save this polynomial in the file poly. Do not forget to type the 3 at the start of the file as this is the number of variables.

   (a) Choose option 6 from the main menu of phc -f. How many factors does this polynomial have? What are the multiplicities of the factors?

   (b) Factor the polynomial again with the other factorization method. Compare the results (which ought to be the same) and the execution times (which should be different).

4. The twisted cubic in the input format of phc is

```
 2 3
   y - x^2;
   z - x^3;
```

We save this system in the file twisted and perform a numerical elimination:

   (a) Call phc -c and choose the first option to generate an embedding for the system we saved in the file twisted. A good name for the output file is twisted_e1. We type in 1 when phc prompts us for the top dimension. The numerical elimination we wish to perform (i.e.: the finding of $y - x^2 = 0$ requires us to restrict the slices to a 2-dimensional subspace $\mathbb{C}[x, y]$. We type 2 when phc prompts us for the dimensional of the subspace.
   In some applications we must give the variables a new order because a k-dimensional subspace will use the first k variables. For this system, that the order of the variables is such that y precedes x does not matter.

   (b) Type phc -b twisted_e1 twisted_e1.phc to compute a witness set for the cubic with this special slice. Look in the file twisted_e1 to verify whether there are two (and not three) witness points.

   (c) To find the equation $y - x^2$ numerically, we call phc -f and choose for the fifth option in the menu. The order of the variables is fine and since the degree is only two, we can type in any number ≤ 16 for the working precision. Look at the end of the output file for a numerical representation of the elimination of z from the twisted cubic.