Approximating all isolated solutions to polynomial systems using homotopy continuation methods

Jan Verschelde

Department of Math, Stat & CS

University of Illinois at Chicago

Chicago, IL 60607-7045, USA

e-mail: **jan@math.uic.edu**

web: **www.math.uic.edu/~jan**

RAAG Summer School, Rennes, France.

30 June – 5 July, 2003

Outline of Lecture

1. Polynomial Homotopies and Path Tracking
2. Parameter Continuation
3. Exploiting Product Structures
4. Polyhedral Methods to Exploit Sparsity
5. Software and Applications

Recommended Background Literature

E.L. Allgower and K. Georg: **Numerical Continuation Methods, an Introduction.** Springer 1990. To appear in the SIAM Classics in Applied Mathematics Series.

E.L. Allgower and K. Georg: **Numerical Path Following.** In *Techniques of Scientific Computing (Part 2)*, edited by P.G. Ciarlet and J.L. Lions volume 5 of *Handbook of Numerical Analysis*, pages 3–203. North-Holland, 1997.

A. Morgan: **Solving polynomial systems using continuation for engineering and scientific problems**. Prentice-Hall, 1987.

T.Y. Li: **Solving polynomial systems.** *The Mathematical Intelligencer* 9(3):33–39, 1987.

T.Y. Li: **Numerical solution of multivariate polynomial systems by homotopy continuation methods.** *Acta Numerica* 6:399–436, 1997.

Numerical Homotopy Continuation Methods

If we wish to solve $f(\mathbf{x}) = \mathbf{0}$, then we construct a system $g(\mathbf{x}) = \mathbf{0}$ whose solutions are known. Consider the *homotopy*

$$H(\mathbf{x}, t) := (1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}.$$

By *continuation*, we trace the paths starting at the known solutions of $g(\mathbf{x}) = \mathbf{0}$ to the desired solutions of $f(\mathbf{x}) = \mathbf{0}$, for t from 0 to 1.

homotopy continuation methods are *symbolic-numeric*:
homotopy methods treat polynomials as algebraic objects,
continuation methods use polynomials as functions.

The theorem of Bézout

$f = (f_1, f_2, \ldots, f_n)$

$d_i = \deg(f_i)$

total degree D :

$$D = \prod_{i=1}^{n} d_i$$

$$g(\mathbf{x}) = \begin{cases} \alpha_1 x_1^{d_1} - \beta_1 = 0 \\ \alpha_2 x_2^{d_2} - \beta_2 = 0 \\ \vdots \\ \alpha_n x_n^{d_n} - \beta_n = 0 \end{cases}$$

start system

$\alpha_i, \beta_i \in \mathbb{C}$ random

Theorem: $f(\mathbf{x}) = \mathbf{0}$ *has at most D isolated solutions in $\mathbb{C}^n$, counted with multiplicities.*

Sketch of Proof: $V = \{\ (f, \mathbf{x}) \in \mathbb{P}(\mathcal{H}_D) \times \mathbb{P}(\mathbb{C}^n) \mid f(\mathbf{x}) = \mathbf{0}\ \}$

$\Sigma' = \{(f, \mathbf{x}) \in V \mid \det(D_{\mathbf{x}} f(\mathbf{x})) = 0\}$, $\Sigma = \pi_1(\Sigma'), \pi_1 : V \to \mathbb{P}(\mathcal{H}_D)$

Elimination theory: Σ is variety $\Rightarrow \mathbb{P}(\mathcal{H}_D) - \Sigma$ is connected.

Thus $h(\mathbf{x}, t) = (1-t) g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}$ avoids $\Sigma, \forall t \in [0, 1)$.

Implicitly defined curves

Consider a homotopy $h_k(x(t), y(t), t) = 0$, $k = 1, 2$.

By $\frac{\partial}{\partial t}$ on homotopy: $\frac{\partial h_k}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial h_k}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial h_k}{\partial t}\frac{\partial t}{\partial t} = 0$, $k = 1, 2$.

Set $\Delta x := \frac{\partial x}{\partial t}$, $\Delta y := \frac{\partial y}{\partial t}$, and $\frac{\partial t}{\partial t} = 1$.

$$
\begin{array}{ll}
\text{Increment} & t := t + \Delta t \\
\text{Solve} & \left[\begin{array}{cc} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{array}\right] \left[\begin{array}{c} \Delta x \\ \Delta y \end{array}\right] = - \left[\begin{array}{c} \frac{\partial h_1}{\partial t} \\ \frac{\partial h_2}{\partial t} \end{array}\right] (Newton) \\
\text{Update} & \left\{\begin{array}{l} x := x + \Delta x \\ y := y + \Delta y \end{array}\right.
\end{array}
$$

Predictor-Corrector Methods

loop

1. predict $\begin{cases} t_{k+1} := t_k + \Delta t \\ \mathbf{x}^{(k+1)} := \mathbf{x}^{(k)} + \Delta\mathbf{x} \end{cases}$

2. correct with Newton

3. if convergence

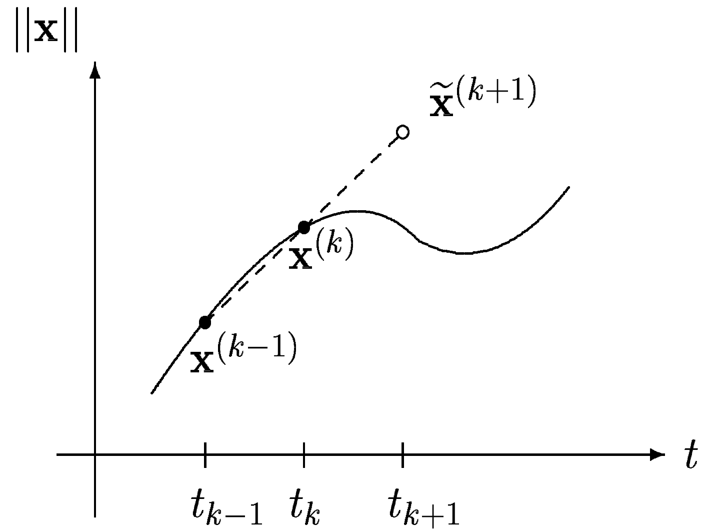
    then enlarge Δt

        continue with $k+1$

    else reduce Δt

        back up and restart at k

until $t = 1$.



$$\widetilde{\mathbf{x}}^{(k+1)} := \mathbf{x}^{(k)} + \lambda\big(\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\big)$$

Complexity of Homotopy Methods

- For bounds on #Newton steps in linear homotopy, see

L. Blum, F. Cucker, M. Shub, and S. Smale: **Complexity and Real Computation**. Springer 1998.

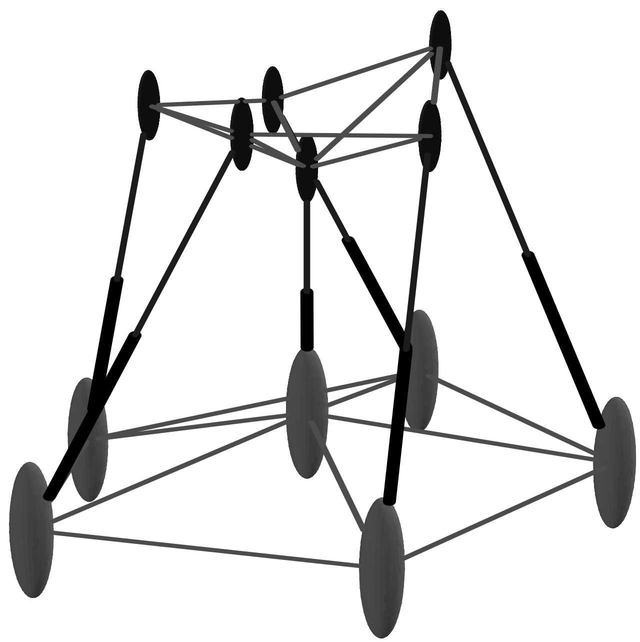
M. Shub and S. Smale: **Complexity of Bezout's theorem V: Polynomial Time**. *Theoretical Computer Science* 133(1):141–164, 1994.

On average, we can find an approximate zero in polynomial time.

- In practice:
  1. fix #Newton steps to force quadratic convergence;
  2. rerun clustered paths with same discretization of t.
- Robust step control by interval methods, see

R.B. Kearfott and Z. Xing: **An interval step control for continuation methods.** *SIAM J. Numer. Anal.* 31(3): 892–914, 1994.

A Case Study: Stewart-Gough Platforms



end plate, the platform

is connected by legs to

a stationary base

Forward Displacement Problem:

Given: position of base and leg lengths.

Wanted: position of end plate.

Literature on Stewart-Gough platforms

M. Raghavan: **The Stewart platform of general geometry has 40 configurations.** *ASME J. Mech. Design* 115:277–282, 1993.

J.C. Faugère and D. Lazard: **Combinatorial classes of parallel manipulators.** *Mech. Mach. Theory* 30(6):765–776, 1995.

M.L. Husty: **An algorithm for solving the direct kinematics of general Stewart-Gough Platforms.** *Mech. Mach. Theory,* 31(4):365–380, 1996.

C.W. Wampler: **Forward displacement analysis of general six-in-parallel SPS (Stewart) platform manipulators using soma coordinates.** *Mech. Mach. Theory* 31(3): 331–337, 1996.

P. Dietmaier: **The Stewart-Gough platform of general geometry can have 40 real postures.** In *Advances in Robot Kinematics: Analysis and Control*, ed. by J. Lenarcic and M.L. Husty, pages 1–10. Kluwer 1998.

J.P. Merlet: **Parallel Robots**. Kluwer Academic Publishers, 2000.

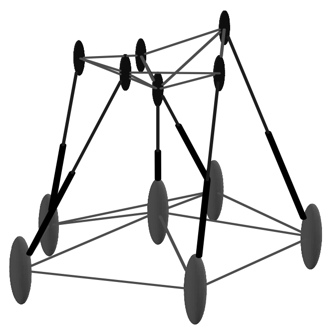
Coefficient-Parameter Homotopies

- Study how solutions change when parameters vary.
- Key Idea:
  1. solve system once for a generic choice of the parameters;
  2. use homotopy to move from generic to specific instance.
- Works for nested parameter spaces (Charles Wampler).

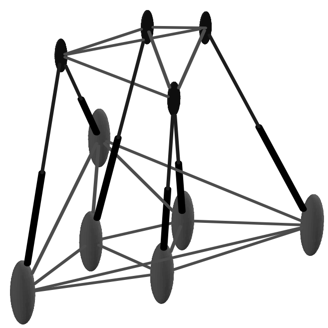
For the theory, see

A.P. Morgan and A.J. Sommese: **Coefficient-parameter polynomial continuation.** *Appl. Math. Comput.*, 29(2):123–160, 1989.

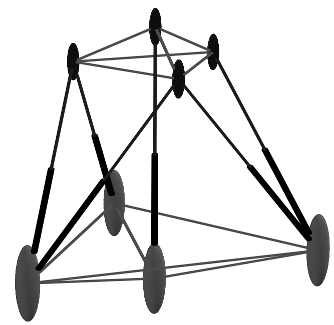
A family of Stewart-Gough platforms



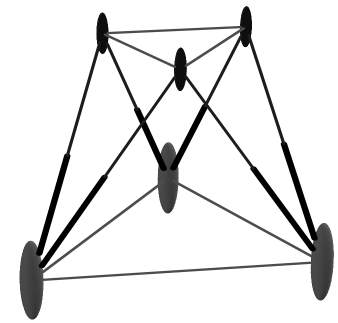
6-6, 40 solutions



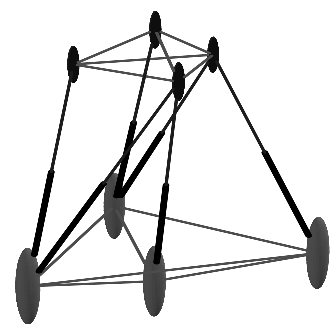
4-6, 32 solutions



4-4a, 16 solutions



3-3, 16 solutions



4-4b, 24 solutions

thanks to Charles Wampler

Multihomogeous version of Bézout's theorem

Consider the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$, $A \in \mathbb{C}^{n\times n}$.

Add one general hyperplane $\displaystyle\sum_{i=1}^{n} c_i x_i + c_0 = 0$ for unique $\mathbf{x}$.

Bézout's theorem: $D = 2^n \leftrightarrow$ at most n solutions

Embed in multi-projective space: $\mathbb{P} \times \mathbb{P}^n$, separating λ from $\mathbf{x}$.

$\{\lambda\}$	$\{x_1, x_2\}$
1	1
1	1
0	1

degree table

$\Longleftrightarrow$

$\{\lambda\}$	$\{x_1, x_2\}$
$\lambda + \gamma_1$	$\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$
$\lambda + \gamma_2$	$\beta_0 + \beta_1 x_1 + \beta_2 x_2$
1	$c_0 + c_1 x_1 + c_2 x_2$

linear-product start system

The root count $B = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 1$ is a permanent.

linear-product start systems

$$f(\mathbf{x}) = \begin{cases} x_1 x_2^2 + x_1 x_3^3 - c x_1 + 1 = 0 & c \in \mathbb{C} \\ x_2 x_1^2 + x_2 x_3^2 - c x_2 + 1 = 0 & \\ x_3 x_1^2 + x_3 x_2^2 - c x_3 + 1 = 0 & D = 27 \end{cases}$$

$$\begin{array}{llllc} \{x_1\} & \{x_2, x_3\} & \{x_2, x_3\} & \quad \text{symmetric} & \\ \{x_2\} & \{x_1, x_3\} & \{x_1, x_3\} & \quad \text{supporting} & B = 21 \\ \{x_3\} & \{x_1, x_2\} & \{x_1, x_2\} & \quad \text{set structure} & \end{array}$$

Choose 7 random complex numbers $c_1, c_2, \ldots, c_7$ and create

$$g(\mathbf{x}) = \begin{cases} (x_1 + c_1)(c_2 x_2 + c_3 x_3 + c_4)(c_5 x_2 + c_6 x_3 + c_7) = 0 \\ (x_2 + c_1)(c_2 x_1 + c_3 x_3 + c_4)(c_5 x_1 + c_6 x_3 + c_7) = 0 \\ (x_3 + c_1)(c_2 x_1 + c_3 x_2 + c_4)(c_5 x_1 + c_6 x_2 + c_7) = 0 \end{cases}$$

8 generating solutions

Papers on Exploiting Product Structures

A. Morgan and A. Sommese: **A homotopy for solving general polynomial systems that respects m-homogeneous structures.** *Appl. Math. Comput.* 24(2):101–113, 1987.

T.Y. Li, T. Sauer, and J.A. Yorke: **The random product homotopy and deficient polynomial systems.** *Numer. Math.* 51(5):481–500, 1987.

J. Verschelde and A. Haegemans: **The *GBQ*-Algorithm for constructing start systems of homotopies for polynomial systems.** *SIAM J. Numer. Anal.* 30(2):583–594, 1993.

C.W. Wampler: **An efficient start system for multi-homogeneous polynomial continuation.** *Numer. Math.* 66(4):517–523, 1994.

A.P. Morgan, A.J. Sommese, and C.W. Wampler: **A product-decomposition theorem for bounding Bézout numbers.** *SIAM J. Numer. Anal.* 32(4):1308–1325, 1995.

T.Y. Li, T. Wang, and X. Wang: **Random product homotopy with minimal BKK bound.** In *The Mathematics of Numerical Analysis*, ed. by J. Renegar, M. Shub, and S. Smale, pages 503–512, AMS, 1996.

Geometric Root Counting

$$f_i(\mathbf{x}) = \sum_{\mathbf{a}\in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

$$c_{i\mathbf{a}} \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

$$f = (f_1, f_2, \ldots, f_n)$$

$$P_i = \mathrm{conv}(A_i)$$

Newton polytope

$$\mathcal{P} = (P_1, P_2, \ldots, P_n)$$

$L(f)$ root count in $(\mathbb{C}^*)^n$	$V(\mathcal{P})$ mixed volume
$L(f) = L(f_2, f_1, \ldots, f_n)$	$V(P_2, P_1, \ldots, P_n) = V(\mathcal{P})$
$L(f) = L(f_1\mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(P_1 + \mathbf{a}, \ldots, P_n) = V(\mathcal{P})$
$L(f) \leq L(f_1 + \mathbf{x}^{\mathbf{a}}, \ldots, f_n)$	$V(\mathrm{conv}(P_1 + \mathbf{a}), \ldots, P_n) \geq V(\mathcal{P})$
$L(f) = L(f_1(\mathbf{x}^{U\mathbf{a}}), \ldots, f_n(\mathbf{x}^{U\mathbf{a}}))$	$V(UP_1, \ldots, UP_n) = V(\mathcal{P})$
$L(f_{11}f_{12}, \ldots, f_n)$ $= L(f_{11}, \ldots, f_n) + L(f_{12}, \ldots, f_n)$	$V(P_{11} + P_{12}, \ldots, P_n)$ $= V(P_{11}, \ldots, P_n) + V(P_{12}, \ldots, P_n)$

exploit sparsity $\boxed{L(f) = V(\mathcal{P})}$ 1st theorem of Bernshteĭn

The Theorems of Bernshteĭn

Theorem A: The number of roots of a generic system equals the mixed volume of its Newton polytopes.

Theorem B: Solutions at infinity are solutions of systems supported on faces of the Newton polytopes.

D.N. Bernshteĭn: **The number of roots of a system of equations.** *Functional Anal. Appl.*, 9(3):183–185, 1975.

Structure of proofs: First show Theorem B, looking at power series expansions of diverging paths defined by a linear homotopy starting at a generic system. Then show Theorem A, using Theorem B with a homotopy defined by *lifting* the polytopes.

Systems, Supports, and Newton Polytopes

$$f = (f_1, f_2) = \left\{ \begin{array}{r} x_1^3 x_2 + x_1 x_2^2 + 1 = 0 \\ x_1^4 + x_1 x_2 + 1 = 0 \end{array} \right.$$

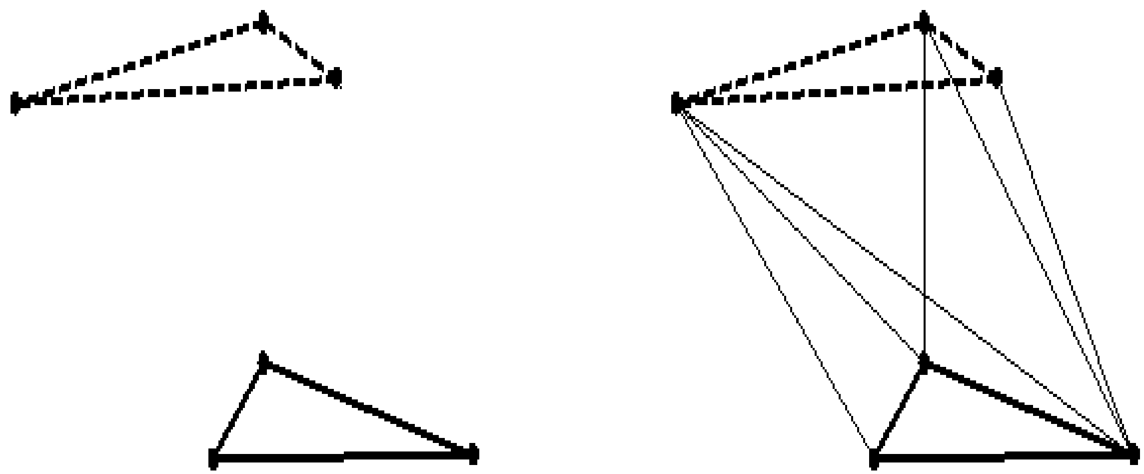
$$\mathcal{A} = (A_1, A_2)$$
$$A_1 = \{(3,1), (1,2), (0,0)\}$$
$$A_2 = \{(4,0), (1,1), (0,0)\}$$

The sparse structure of f is modeled by the tuple $\mathcal{A} = (A_1, A_2)$.

A_1 and A_2 are the *supports* of f_1 and f_2 respectively.

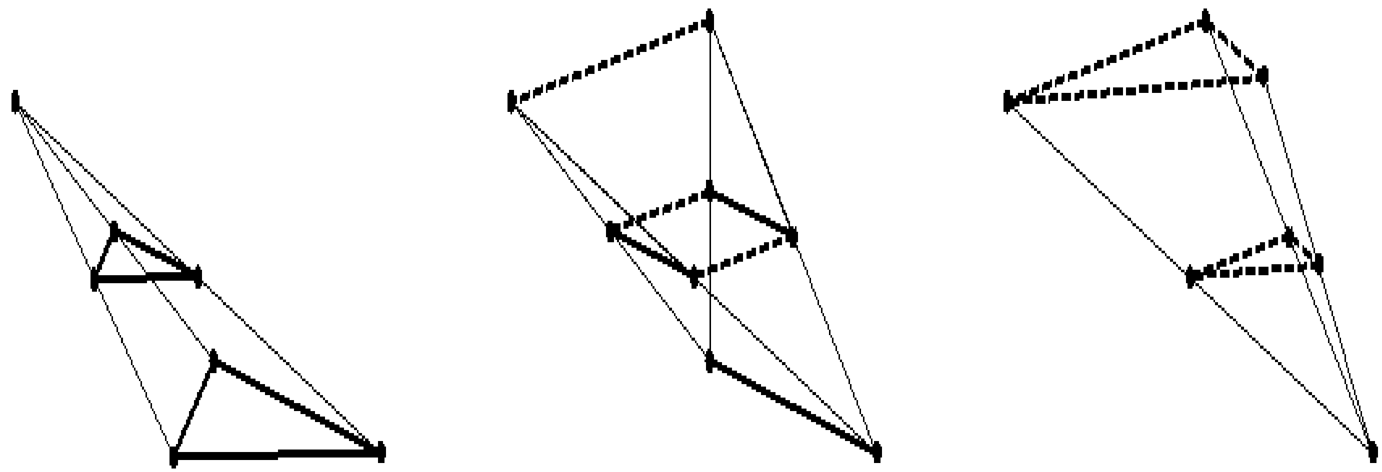
The Newton polytopes are the convex hulls of the supports.

The Cayley polytope

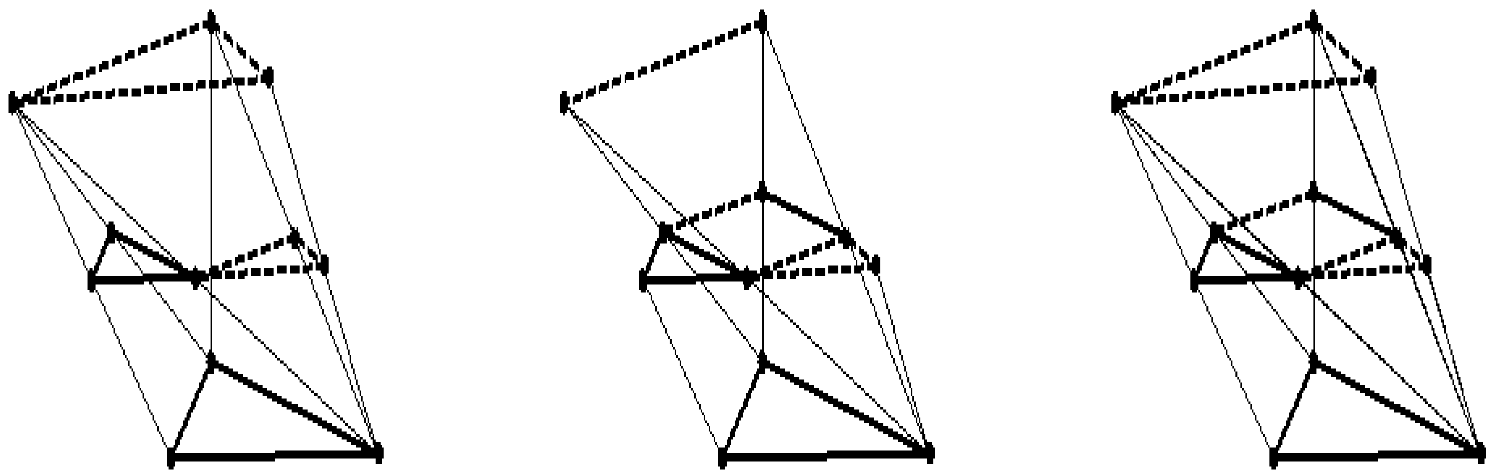


Place one polytope at level 0, the other at level 1.

A triangulation of the Cayley polytope



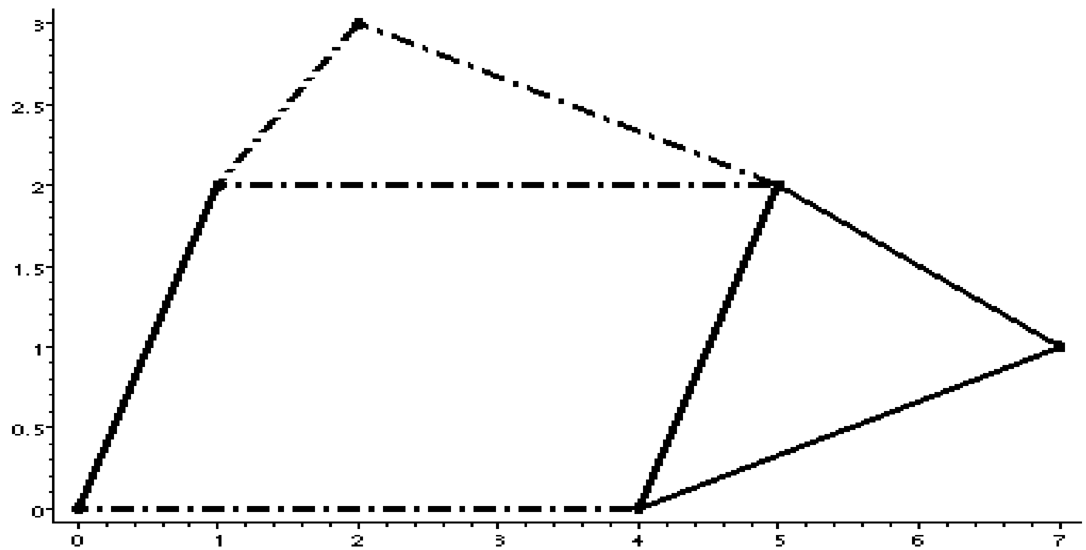
A mixed subdivision induced by a triangulation of the Cayley polytope



Mixed Volumes

Mixed subdivisions visualize Minkowski's theorem:

$$\begin{aligned}\mathrm{area}(\lambda_1 P_1 + \lambda_2 P_2) &= V(P_1, P_1)\lambda_1^2 + 2V(P_1, P_2)\lambda_1\lambda_2 + V(P_2, P_2)\lambda_2^2 \\ &= 5\lambda_1^2 + 2 \times 8\lambda_1\lambda_2 + 4\lambda_2^2\end{aligned}$$



Newton Polytopes and Real Solutions

B. Sturmfels: **On the number of real roots of a sparse polynomial system.** In *Hamiltonian and Gradient Flows: Algorithms and Control*, ed. by A. Bloch, pages 137–143, AMS 1994.

B. Sturmfels: **Viro's theorem for complete intersections.** *Annali della Scuola Normale Superiore di Pisa* 21(3):377–386, 1994.

I. Itenberg and M.-F. Roy: **Multivariate Descartes' rule**. *Beiträge zur Algebra and Geometry* 37(2):337–346, 1996.

T.Y. Li and X. Wang: **On multivariate Descartes' rule – a counterexample**. *Beiträge zur Algebra and Geometry* 39(1):1–5, 1998.

I. Itenberg and E. Shustin: **Viro theorem and topology of real and complex combinatorial hypersurfaces.** *Israel Math. J.* 133: 189–238, 2003. `math.AG/0105198`

Bernshteĭn's second theorem

- Face $\partial_\omega f = (\partial_\omega f_1, \partial_\omega f_2, \ldots, \partial_\omega f_n)$ of system $f = (f_1, f_2, \ldots, f_n)$ with Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ and mixed volume $V(\mathcal{P})$.

$$\partial_\omega f_i(\mathbf{x}) = \sum_{\mathbf{a} \in \partial_\omega A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \qquad \underset{\text{face of Newton polytope}}{\partial_\omega P_i = \mathrm{conv}(\partial_\omega A_i)}$$

<u>Theorem:</u> If $\forall \omega \neq \mathbf{0}$, $\partial_\omega f(\mathbf{x}) = \mathbf{0}$ has no solutions in $(\mathbb{C}^*)^n$,

then $V(\mathcal{P})$ is exact and all solutions are isolated.

Otherwise, for $V(\mathcal{P}) \neq 0$: $V(\mathcal{P}) > \#$isolated solutions.

- Newton polytopes in general position: $V(\mathcal{P})$ is exact for every nonzero choice of the coefficients.

Newton polytopes in general position

Consider $f(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{222}x_1^2x_2^2 + c_{210}x_1 + c_{201}x_2 = 0 \end{cases}$

The Newton polytopes:

$\forall \omega \neq \mathbf{0} : \partial_\omega A_1 + \partial_\omega A_2 \leq 3 \quad \Rightarrow \quad V(P_1, P_2) = 4$ always exact

for all nonzero coefficients

Power Series

Theorem: $\forall \mathbf{x}(t)$, $h(\mathbf{x}(t), t) = (1-t)g(\mathbf{x}(t)) + tf(\mathbf{x}(t)) = \mathbf{0}$,

$\exists s > 0$, $m \in \mathbb{N} \setminus \{0\}$, $\omega \in \mathbb{Z}^n$:

$$\begin{cases} x_i(s) = b_i s^{\omega_i}(1 + O(s)), & i = 1, 2, \ldots, n \\ t(s) = 1 - s^m & \text{for } t \approx 1, s \approx 0 \end{cases}$$

$$\lim_{t \to 1} x_i(t) \in \mathbb{C}^*? \qquad x_i(t) \begin{cases} \to \infty \\ \in \mathbb{C}^* \\ \to 0 \end{cases} \quad \Leftrightarrow \quad \omega_i \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$

m is the *winding number*, i.e. the smallest number so that

$$\mathbf{z}(2\pi m) = \mathbf{z}(0), \quad h(\mathbf{z}(\theta), t(\theta)) = \mathbf{0}, \quad t = 1 + (t_0 - 1)e^{i\theta}, \quad t_0 \approx 1.$$

Face Systems and Power Series

assume $\lim\limits_{t \to 1} x_i(t) \notin \mathbb{C}^*$, thus $\omega_i \neq 0$, a diverging path

- $h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + t f(\mathbf{x}) = \mathbf{0}$ $\quad \left\{ \begin{array}{c} x_i(s) = b_i s^{\omega_i}(1 + O(s)) \\ t(s) = 1 - s^m, s \approx 0 \end{array} \right.$

  substitute power series

$$h(\mathbf{x}(s), t(s)) = \underbrace{f(\mathbf{x}(s))}_{\text{dominant as } s \to 0} + s^m (g(\mathbf{x}(s)) - f(\mathbf{x}(s))) = \mathbf{0}$$

- $f_i(\mathbf{x}) = \displaystyle\sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \mathbf{x}^{\mathbf{a}} \to f_i(\mathbf{x}(s)) = \underbrace{\sum_{\mathbf{a} \in A_i} c_{i\mathbf{a}} \prod_{i=1}^{n} b_i^{a_i} s^{\langle \mathbf{a}, \omega \rangle}}_{\partial_\omega f_i(\mathbf{x}(s)) \text{ dominant}} (1 + O(s))$

  face $\partial_\omega A_i := \{ \ \mathbf{a} \in A_i \mid \langle \mathbf{a}, \omega \rangle = \min\limits_{\mathbf{a}' \in A_i} \langle \mathbf{a}', \omega \rangle \ \}$

  $\Rightarrow \partial_\omega f(\mathbf{b}) = \mathbf{0}, \mathbf{b} \in (\mathbb{C}^*)^n$

key idea in proof of Bernshteǐn's second theorem

Richardson Extrapolation for ω and m

$$\begin{cases} x_i(s) & = & b_i s^{\omega_i}(1+O(s)) \\ t(s) & = & 1-s^m \end{cases}$$

$$x_i(s_k) = b_i h^{k\omega_i/m} s_0(1+O(h^{k/m}s_0))$$

Geometric sampling $0 < h < 1$

$$1-t_k = h(1-t_k) = \cdots = h^k(1-t_0)$$

$$s_k = h^{1/m}s_{k-1} = \cdots = h^{k/m}s_0$$

- $\log|x_i(s_k)| = \log|b_i| + \frac{k\omega_i}{m}\log(h) + \omega_i \log(s_0)$

  $+\log(1+\sum_{j=0}^{\infty} b'_j (h^{k/m}s_0)^j)$

  $v_{kk+1} := \log|x_i(s_k+1)| - \log|x_i(s_k)|$

Extrapolation on samples

$$v_{k..l} = v_{k..l-1} + \frac{v_{k+1..l} - v_{k..l-1}}{1-h}$$

$$\omega_i = m\frac{v_{0..r}}{\log(h)} + O(s_0^r)$$

- $e_i^{(k)} = (\log|x_i(s_k)| - \log|x_i(s_{k+1})|)$

  $-(\log|x_i(s_{k+1})| - \log|x_i(s_{k+2})|)$

  $= c_1 h^{k/m} s_0(1+0(h^{k/m}))$

  $e_i^{(kk+1)} := \log(e_i^{(k+1)}) - \log(e_i^{(k)})$

Extrapolation on errors

$$e_i^{(k..l)} = e_i^{(k+1..l)} + \frac{e_i^{(k..l-1)} - e_i^{(k+1..l)}}{1-h_{k..l}}$$

$$h_{k..l} = h^{(l-k-1)/m_{k..l}}$$

$$m_{k..l} = \frac{\log(h)}{e_i^{(k..l)}} + O(h^{(l-k)k/m})$$

the system of Cassou-Noguès

$$f(b,c,d,e) = \left\{ \begin{array}{rcl} 15b^4cd^2 + 6b^4c^3 + 21b^4c^2d - 144b^2c - 8b^2c^2e & & \\ -28b^2cde - 648b^2d + 36b^2d^2e + 9b^4d^3 - 120 & = & 0 \\ 30c^3b^4d - 32de^2c - 720db^2c - 24c^3b^2e - 432c^2b^2 + 576ec & & \\ -576de + 16cb^2d^2e + 16d^2e^2 + 16e^2c^2 + 9c^4b^4 + 5184 & & \\ +39d^2b^4c^2 + 18d^3b^4c - 432d^2b^2 + 24d^3b^2e - 16c^2b^2de - 240c & = & 0 \\ 216db^2c - 162d^2b^2 - 81c^2b^2 + 5184 + 1008ec - 1008de & & \\ +15c^2b^2de - 15c^3b^2e - 80de^2c + 40d^2e^2 + 40e^2c^2 & = & 0 \\ 261 + 4db^2c - 3d^2b^2 - 4c^2b^2 + 22ec - 22de & = & 0 \end{array} \right.$$

Root counts: $D = 1344$, $B = 312$, $V(\mathcal{P}) = 24 > 16$ finite roots.

$$\begin{array}{l} \partial_{(0,0,0,-1)} f(b,c,d,e) = \\ m = 2 \end{array} \left\{ \begin{array}{r} -8b^2c^2e - 28b^2cde + 36b^2d^2e = 0 \\ -32de^2c + 16d^2e^2 + 16e^2c^2 = 0 \\ -80de^2c + 40d^2e^2 + 40e^2c^2 = 0 \\ 22ec - 22de = 0 \end{array} \right.$$

Some further recommended reading

B. Huber and J. Verschelde: **Polyhedral end games for polynomial continuation.**
Numerical Algorithms 18(1):91–108, 1998.

J. Verschelde: **Toric Newton Method for Polynomial Homotopies.**
J. Symbolic Computation 29(4-5): 777–793, 2000.

Sparsity and Unimodular Transformations

$$f(\mathbf{x}) = \begin{cases} x_1^3 x_2^{-1} + c_1 = 0 \\ x_1 x_2^2 + c_2 = 0 \end{cases} \qquad f(\mathbf{x} = \mathbf{y}^U) = \begin{cases} y_1 + c_1 = 0 \\ y_1^{-2} y_2^7 + c_2 = 0 \end{cases}$$

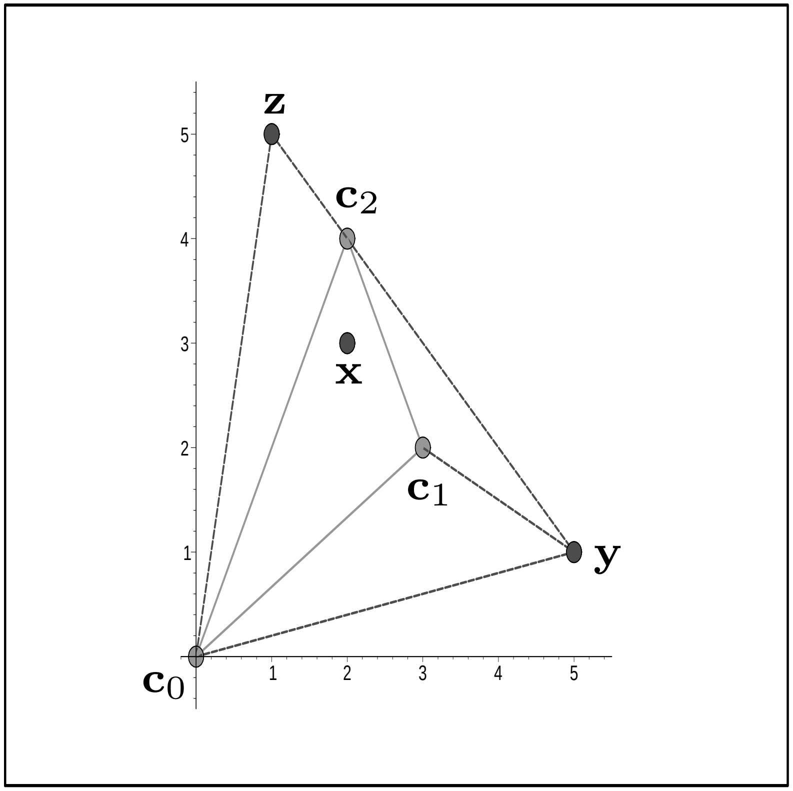
The substitution $\mathbf{x}^V = (\mathbf{y}^U)^V = \mathbf{y}^{VU} = \mathbf{y}^L$ is elaborated as

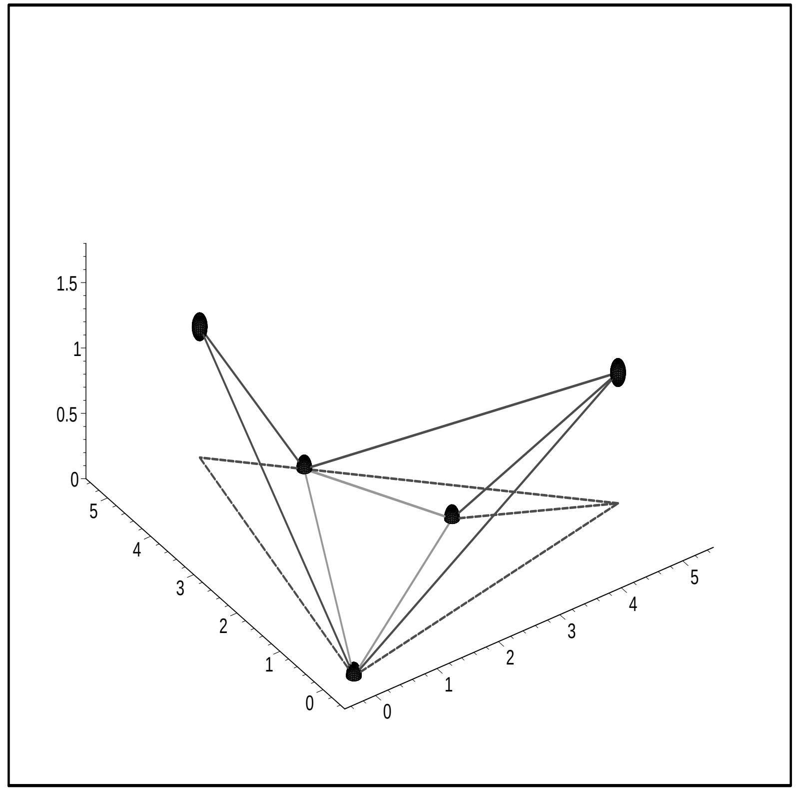
$$\begin{aligned} \begin{pmatrix} x_1^3 \cdot x_2^{-1} \\ x_1^1 \cdot x_2^2 \end{pmatrix} &= \begin{pmatrix} (y_1^0 y_2^1)^3 \cdot (y_1^{-1} y_2^3)^{-1} \\ (y_1^0 y_2^1)^1 \cdot (y_1^{-1} y_2^3)^2 \end{pmatrix} \\ &= \begin{pmatrix} y_1^{3\cdot 0 - 1\cdot(-1)} \cdot y_2^{3\cdot 1 - 1\cdot 3} \\ y_1^{1\cdot 0 + 2\cdot(-1)} \cdot y_2^{1\cdot 1 + 2\cdot 3} \end{pmatrix} = \begin{pmatrix} y_1^1 \cdot y_2^0 \\ y_1^{-2} \cdot y_2^7 \end{pmatrix}. \end{aligned}$$

factorization $VU = L$:
(U unimodular, det(U)=1)

$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 7 \end{bmatrix}$$

Pivoting to update a triangulation





$\mathbf{c}_0 = (0,0)$ $\quad$ $\mathbf{x} = (2,3)$ $\quad$ $\mathbf{x} = +\frac{1}{8}\mathbf{c}_0 + \frac{1}{4}\mathbf{c}_1 + \frac{5}{8}\mathbf{c}_2$

$\mathbf{c}_1 = (3,2)$ $\quad$ $\mathbf{y} = (5,1)$ $\quad$ $\mathbf{y} = -\frac{1}{3}\mathbf{c}_0 + \frac{9}{4}\mathbf{c}_1 - \frac{7}{8}\mathbf{c}_2$

$\mathbf{c}_2 = (2,4)$ $\quad$ $\mathbf{z} = (1,5)$ $\quad$ $\mathbf{z} = +\frac{1}{8}\mathbf{c}_0 - \frac{3}{4}\mathbf{c}_1 + \frac{13}{8}\mathbf{c}_2$

barycentric coordinates

Incremental Polyhedral Continuation

$$g(\mathbf{x}) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2 + c_{100} = 0 \\ c_{211}x_1x_2 + c_{210}x_1 + c_{201}x_2 + c_{200} = 0 \end{cases}$$

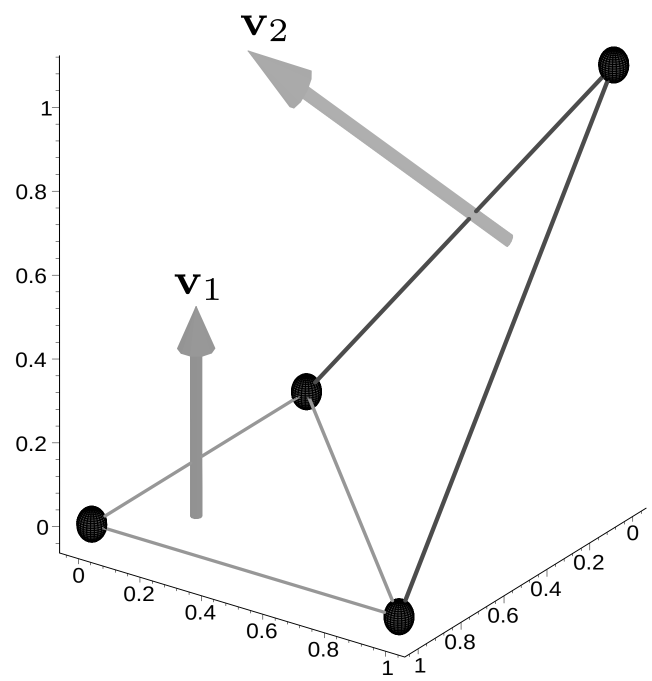
$$\mathbf{v}_1 = (0,0,1)$$

$$g_1(\mathbf{x},t) = \begin{cases} c_{111}x_1x_2 + c_{110}x_1 + c_{101}x_2t + c_{100} = 0 \\ c_{211}x_1x_2 + c_{210}x_1 + c_{201}x_2t + c_{200} = 0 \end{cases}$$

$$\boxed{x_1 = \widetilde{x}_1 t \quad x_2 = \widetilde{x}_2 t^{-1}}$$

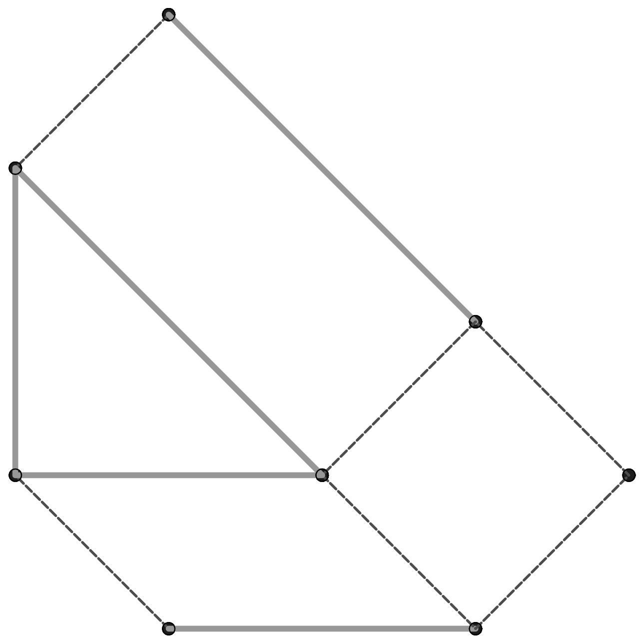
$$\mathbf{v}_2 = (1,-1,1)$$

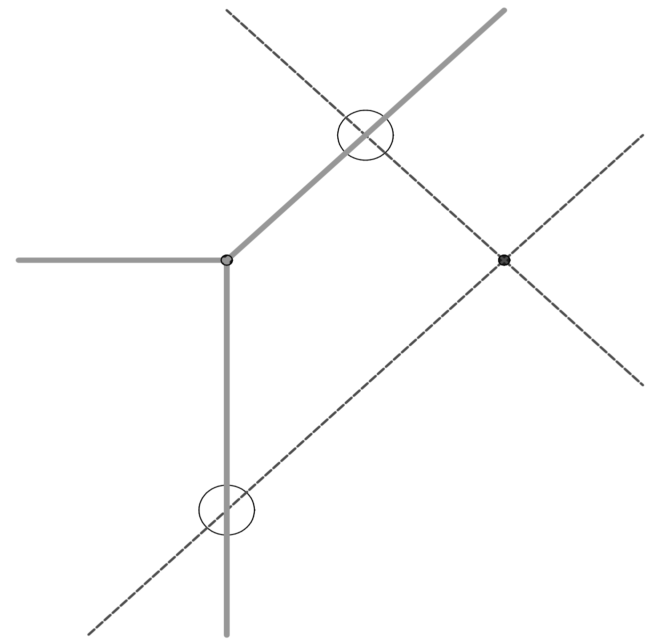
$$g_2(\widetilde{\mathbf{x}},t) = \begin{cases} c_{111}\widetilde{x}_1\widetilde{x}_2 + c_{110}\widetilde{x}_1t + c_{101}\widetilde{x}_2 + c_{100} = 0 \\ c_{211}\widetilde{x}_1\widetilde{x}_2 + c_{210}\widetilde{x}_1t + c_{201}\widetilde{x}_2 + c_{200} = 0 \end{cases}$$



Mixed Cell Configurations and Normal Fans

normals to mixed cells are in the intersections of normal cones to the edges





$$\text{Find all } \mathbf{v} \text{ satisfying } \begin{cases} \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle = \langle \widehat{\mathbf{b}}, \mathbf{v} \rangle & \forall \widehat{\mathbf{a}}, \widehat{\mathbf{b}} \in \partial_{\mathbf{v}} \widehat{A}_i \\ \langle \widehat{\mathbf{a}}, \mathbf{v} \rangle \geq \langle \widehat{\mathbf{b}}, \mathbf{v} \rangle & \forall \widehat{\mathbf{a}} \in \widehat{A}_i \setminus \partial_{\mathbf{v}} \widehat{A}_i, \forall \widehat{\mathbf{b}} \in \partial_{\mathbf{v}} \widehat{A}_i \\ \quad v_{n+1} = 1 & \qquad\qquad i=1,2,\ldots,n \end{cases}$$

Polyhedral Homotopies

Let $g(\mathbf{x}) = \mathbf{0}$ have the same Newton polytopes $\mathcal{P}$ as $f(\mathbf{x}) = \mathbf{0}$, but with randomly choosen complex coefficients.

I. Compute $V_n(\mathcal{P})$:			II. Solve $g(\mathbf{x}) = \mathbf{0}$:	
I.1	lift polytopes		II.1	introduce parameter t
I.2	mixed cells	$\Leftrightarrow$	II.2	start systems
I.3	volume of mixed cell		II.3	path following

III. Solve the specific system $f(\mathbf{x}) = \mathbf{0}$:

$$h(\mathbf{x}, t) = (1-t)g(\mathbf{x}) + tf(\mathbf{x}) = \mathbf{0}, \quad \text{for } t \text{ from 0 to 1.}$$

coefficient-parameter continuation

Some references on mixed volumes and polynomial systems

B. Huber and B. Sturmfels: **A polyhedral method for solving sparse polynomial systems.** *Math. Comp.* 64(212):1541–1555, 1995.

I.Z. Emiris and J.F. Canny: **Efficient incremental algorithms for the sparse resultant and the mixed volume.** *J. Symbolic Computation* 20(2):117–149, 1995.

I.Z. Emiris: **Sparse Elimination and Applications in Kinematics**. *PhD thesis*, UC Berkeley, 1994.

J. Verschelde: **Homotopy Continuation Methods for Solving Polynomial Systems**. *PhD thesis*, KU Leuven, 1996.

B. Sturmfels: **Polynomial equations and convex polytopes.** *Amer. Math. Monthly* 105(10):907–922, 1998.

Recent computational advances

more efficient use of linear programming:

T.Y. Li and X. Li: **Finding mixed cells in the mixed volume computation.** *Found. Comput. Math.* 1(2): 161–181, 2001. Software available at `http://www.math.msu.edu/~li`.

T. Gao and T.Y. Li: **Mixed volume computation for semi-mixed systems.** *Discrete Comput. Geom.* 29(2):257-277, 2003.

and parallel mixed-volume computations:

A. Takeda, M. Kojima and K. Fujisawa: **Enumeration of all solutions of a combinatorial linear inequality system arising from the polyhedral homotopy continuation Method.** *Journal of the Operations Research Society of Japan* 45(1): 64–82, 2002.

Y. Dai, S. Kim and M. Kojima: **Computing all nonsingular solutions of cyclic-n polynomial using polyhedral homotopy continuation methods.** *J. Comput. Appl. Math.* 152(1-2): 83–97, 2003.

T. Gunji, S. Kim, M. Kojima, A. Takeda, K. Fujisawa, and T. Mizutani: **PHoM – a polyhedral homotopy continuation method for polynomial systems.** `http://www.is.titech.ac.jp/~kojima/sdp.html`.

The software PHCpack

J. Verschelde: **Algorithm 795: PHCpack: A general-purpose solver for polynomial systems by homotopy continuation.** *ACM Transactions on Mathematical Software* 25(2): 251-276, 1999.

Available via `http://www.math.uic.edu/~jan/download.html`.

Modes of operation:

1. As a blackbox: `phc -b input output`.
2. In toolbox mode (call `phc` with other options).
3. The library PHCpack, in Ada with C interface.

PHCpack is menu-driven and file oriented

```
Welcome to PHC (Polynomial Homotopy Continuation) Version 2.1(beta).

Running in full mode.  Note also the following options:
  phc -s : Equation and variable Scaling on system and solutions
  phc -d : Linear and nonlinear Reduction w.r.t. the total degree
  phc -r : Root counting and Construction of start systems
  phc -m : Mixed-Volume Computation by four lifting strategies
  phc -p : Polynomial Continuation by a homotopy in one parameter
  phc -v : Validation, refinement and purification of solutions
  phc -e : SAGBI/Pieri homotopies to intersect linear subspaces
  phc -c : Irreducible decomposition for solution components
  phc -f : Factor pure dimensional solution set into irreducibles
  phc -b : Batch or black-box processing
  phc -z : strip phc output solution lists into Maple format

Is the system on a file ? (y/n/i=info)
```

Papers documenting the usefulness of PHCpack

C.W. Wampler: **Isotropic coordinates, circularity and Bezout numbers: planar kinematics from a new perspective**. Proceedings of the 1996 ASME Design Engineering Technical Conference. Irvine, CA, Aug 18–22, 1996. (CD-ROM).

F. Sottile: **Real Schubert Calculus: Polynomial systems and a conjecture of Shapiro and Shapiro**. Experimental Mathematics 9(2): 161-182, 2000.

B. Haas: **A Simple Counterexample to Kouchnirenko's Conjecture**. Beitraege zur Algebra und Geometrie/Contributions to Algebra and Geometry, 43(1):1-8, 2002.

E. Lee, C. Mavroidis, and J. Morman: **Geometric Design of Spatial 3R Manipulators**. Proceedings of the 2002 NSF Design, Service, and Manufactoring Grantees and Research Conference, San Juan, Puerto Rico, January 7-10, 2002.

E. Lee and C. Mavroidis: **Solving the Geometric Design Problem of Spatial 3R Robot Manipulators Using Polynomial Continuation**. Journal of Mechanical Design, Transactions of the ASME, 2002 (in press).

F. Xie, G. Reid, and S. Valluri: **A numerical method for the one dimensional action functional for FBG structures**. Can J. Phys. 76: 1-21, 2002.