Numerical Homotopy Algorithms for Enumerative Geometry

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Schubert Calculus and Schubert Geometry

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Outline of the Talk

- What is Numerical Schubert Calculus?

  numerical analysis to solve problems in enumerative geometry

- Software Demonstration

- Pieri Homotopies compute solutions to intersection problems

  joint with Birk Huber

  parallel implementation with Yusong Wang

- *towards* Littlewood-Richardson homotopies

  with Frank Sottile and Ravi Vakil

Numerical Schubert Calculus

Why?

- Numerical Algebraic Geometry: development of tools in numerical analysis to solve problems in algebraic geometry.
- Provide software to compute (larger) demonstration examples, e.g.: checking instances of Shapiro-Shapiro conjectures.
- Pieri homotopies solve the output pole placement problem in the control of linear systems.

Literature I

H. Schubert: *Kalkül der abzählenden Geometrie.* Teubner, Leipzig, 1879. Reprinted in 1979 by Springer–Verlag with an introduction by S.L. Keiman.

M. Pieri: Formule di coincidenza per le serie algebriche ∞^n di coppie di punti dello spazio a n dimensioni. *Rend. Circ. Mat. Palermo*, 5:252–268, 1891.

J. Rosenthal: On dynamic feedback compensation and compactifications of systems. *SIAM J. Control and Optimization*, 32(1):279–296, 1994.

M.S. Ravi, J. Rosenthal, and X. Wang: Dynamic pole placement assignment and Schubert calculus. *SIAM J. Control and Optimization*, 34(3):813–832, 1996.

F. Sottile: Pieri's formula via explicit rational equivalence. *Can. J. Math.*, 49(6):1281–1298, 1997.

R. Vakil: A geometric Littlewood-Richardson rule. *Annals of Mathematics* 164: 371–421, 2006.

Literature II

B. Huber, F. Sottile, and B. Sturmfels: Numerical Schubert calculus. *J. of Symbolic Computation*, 26(6):767–788, 1998.

J. Verschelde: Numerical evidence for a conjecture in real algebraic geometry. *Experimental Mathematics* 9(2): 183–196, 2000.

B. Huber and J. Verschelde: Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control. *SIAM J. Control Optim.* 38(4):1265–1287, 2000.

F. Sottile and B. Sturmfels: A sagbi basis for the quantum Grassmannian. *J. Pure and Appl. Algebra* 158(2-3): 347–366, 2001.

T.Y. Li, X. Wang, and M. Wu: Numerical Schubert calculus by the Pieri homotopy algorithm. *SIAM J. Numer. Anal.* 20(2):578–600, 2002.

J. Verschelde and Y. Wang: Computing dynamic output feedback laws. *IEEE Trans. Automat. Control.* 49(8):1393–1397, 2004.

Numerical Homotopy Algorithms

to solve polynomial systems

Solve $f(\mathbf{x}) = \mathbf{0}$

locally: Newton's method

$$x^{(k+1)} = \mathbf{x}^{(k)} - \frac{f(\mathbf{x}^{(k)})}{f'(\mathbf{x}^{(k)})}, \quad k = 0, 1, \ldots$$

converges quadratically to a regular solution.

globally: the homotopy

$$h(\mathbf{x}, t) = (1 - t)g(\mathbf{x}) + tf(\mathbf{x}) = 0,$$

defines a deformation from a start system $g(\mathbf{x}) = \mathbf{0}$ to f, as t goes from 0 to 1. The paths $\mathbf{x}(t)$ are computed with predictor-corrector methods.

Conservation of Number Principle

With general complex coefficients we are sure to avoid singularities during the continuation phase.

To solve a specific real instance, we use a cheater's homotopy to go from the generic complex to the specific real instance.

Singularities can only occur at the end of the continuation.

The number of isolated solutions remains the same, except perhaps at the end.

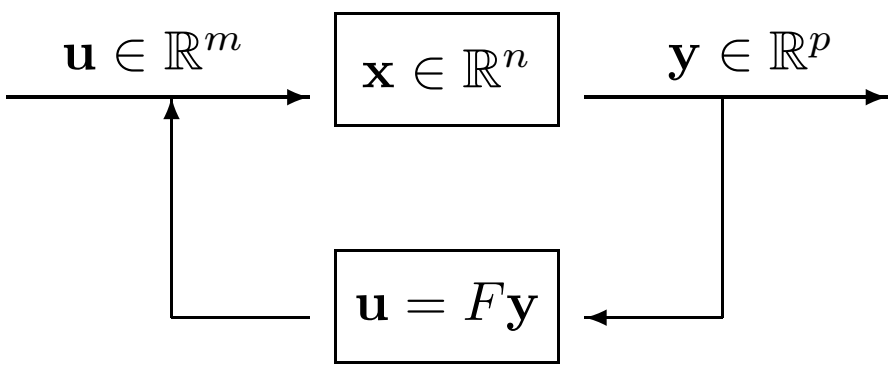
T.Y. Li, T. Sauer, and J.A. Yorke. **The cheater's homotopy: an efficient procedure for solving systems of polynomial equations.** *SIAM J. Numer. Anal.* 26(5):1241–1251, 1989.

A.P. Morgan and A.J. Sommese. **Coefficient-parameter polynomial continuation.** *Appl. Math. Comput.*, 29(2):123–160, 1989. Errata: *Appl. Math. Comput.* 51:207(1992).

Pole Placement by Output Feedback

in linear systems control

Control of an m-input p-output machine by static feedback:



$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} \end{cases}$$

$$\dot{\mathbf{x}} = (A + BFC)\mathbf{x}$$

Problem: Given (A,B,C) and n eigenvalues, find feedback law F.

Equations and Geometry

The feedback laws F, for given (A, B, C) and eigenvalues λ_i, $i = 1, 2, \ldots, n$ must satisfy

$$\det(\lambda_i I_n - (A + BFC)) = 0, \quad i = 1, 2, \ldots, n.$$

or equivalently (using linear algebra):

$$\det \begin{pmatrix} I_p & C(\lambda_i I_n - A)^{-1} B \\ F & I_m \end{pmatrix} = 0, \quad i = 1, 2, \ldots, n.$$

We look for p-planes that intersect n given m-planes nontrivially.

R.W. Brockett and C.I. Byrnes. Multivariate Nyquist criteria, root loci, and pole placement: a geometric viewpoint. *IEEE Trans. Automat. Control*, 26:271–284, 1981.

Software PHCpack

PHC = Polynomial Homotopy Continuation

- Version 1.0 archived as Algorithm 795 by ACM TOMS, 1999.
- Offers implementations of SAGBI and Pieri homotopies to solve problems in enumerative geometry.
- Parallel implementations of the Pieri homotopies developed jointly with Yusong Wang.

  J. Verschelde and Y. Wang: Computing feedback Laws for linear systems with a parallel Pieri homotopy. In Proceedings of HPSEC, pages 222–229, IEEE Computer Society, 2004.
- Current version (in source and executable form) available at `http://www.math.uic.edu/~jan/download.html`.

Hypersurface Pieri Conditions

Input: mp m-planes in $\mathbb{C}^{m+p}$, $L_1, L_2, \ldots, L_{mp}$.

Output: p-planes X : $X \cap L_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, mp$.

$m = 2$, $p = 2$:

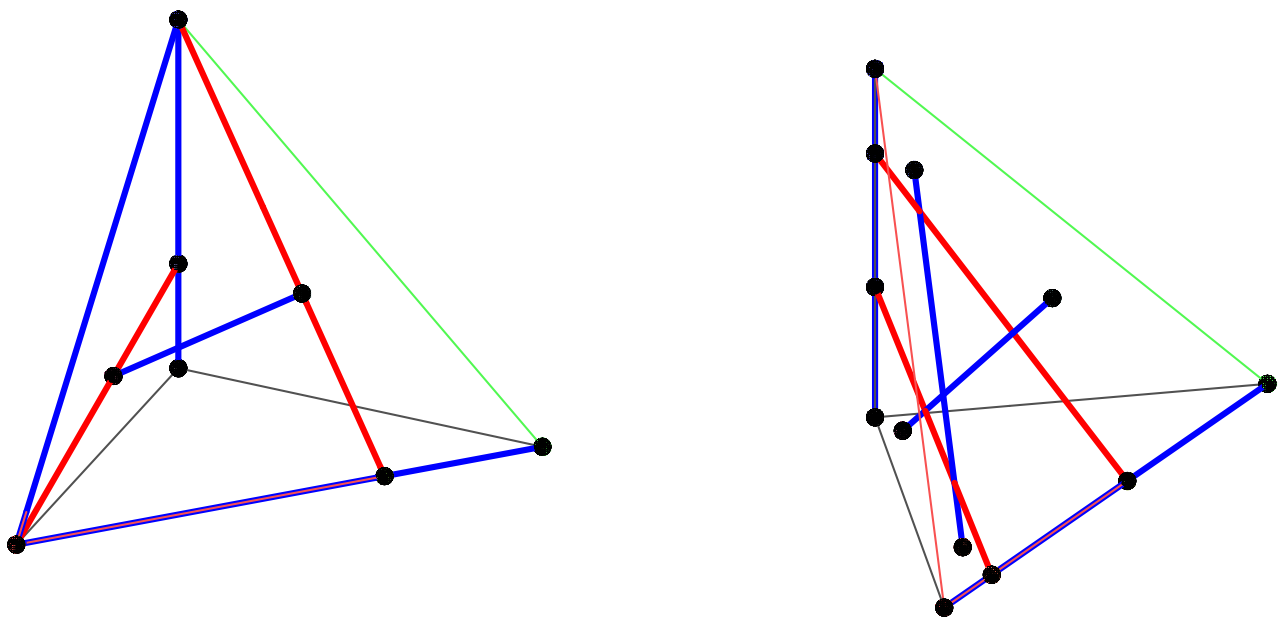
$$\begin{array}{c} \textit{localization} \\ \textit{pattern for } X: \end{array} \quad X = \begin{bmatrix} 1 & 0 \\ \star & 1 \\ \star & \star \\ 0 & \star \end{bmatrix} \star \neq 0, \qquad \det([X|L_i]) = 0, \quad i = 1, 2, 3, 4.$$

Four general lines L_i will determine X uniquely.

We represent X by [3 4]:

the row indices of the bottommost nonzero entries.

Lines Meeting Four Lines in Space



Two **red** lines meet four given **blue** lines in three space.

Localization Patterns and Special Planes

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_4\}$ be the natural standard basis for $\mathbb{C}^4$.

If $L_1 = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$ and $L_2 = \mathrm{span}(\mathbf{e}_3, \mathbf{e}_4)$, then take X as spanned by points on L_1 and L_2.

$$X = \begin{bmatrix} 1 & 0 \\ \star & 0 \\ 0 & 1 \\ 0 & \star \end{bmatrix} \qquad S_{[2\ 4]} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_3) \qquad \textit{special plane}$$

$$\det([X|S_{[2\ 4]}]) = 0 \quad \Leftrightarrow \quad x_{12} = 0 \text{ or } x_{24} = 0$$

Localization Posets and Root Counts

$$\begin{array}{ccccc} & & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} & & \\ & \diagup & & \diagdown & \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & \star \end{bmatrix} & & & & \begin{bmatrix} 1 & 0 \\ \star & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ & \diagdown & & \diagup & \\ & & \begin{bmatrix} 1 & 0 \\ \star & 0 \\ 0 & 1 \\ 0 & \star \end{bmatrix} & & \end{array}$$

$$\begin{array}{ccccc} & & [1\ 3] & & \\ & \diagup & & \diagdown & \\ [1\ 4] & & & & [2\ 3] \\ & \diagdown & & \diagup & \\ & & [2\ 4] & & \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & \diagup & & \diagdown & \\ 1 & & & & 1 \\ & \diagdown & & \diagup & \\ & & 2 & & \end{array}$$

Pieri Homotopies

Two stages to resolve mp intersection conditions:

1. For general inputs L_i, at stage n:

$$h(X,t) = \begin{cases} \det(\ X \ \mid L_i \ ) = 0, i = 1,2,\ldots,n-1 \\ \det(\ X \ \mid (1-t)S_X + tL_n \ ) = 0 \end{cases}$$

   is a Pieri homotopy which moves the nth intersection condition from the special S_X to the general L_n.

2. For specific inputs K_i, $i = 1,2,\ldots,mp$:

$$h(X,t) = \begin{cases} \det(\ X \ \mid (1-t)L_i + tK_i \ ) = 0, \\ \qquad\qquad i = 1,2,\ldots,mp \end{cases}$$

   is a cheater's homotopy, for t from 0 to 1.

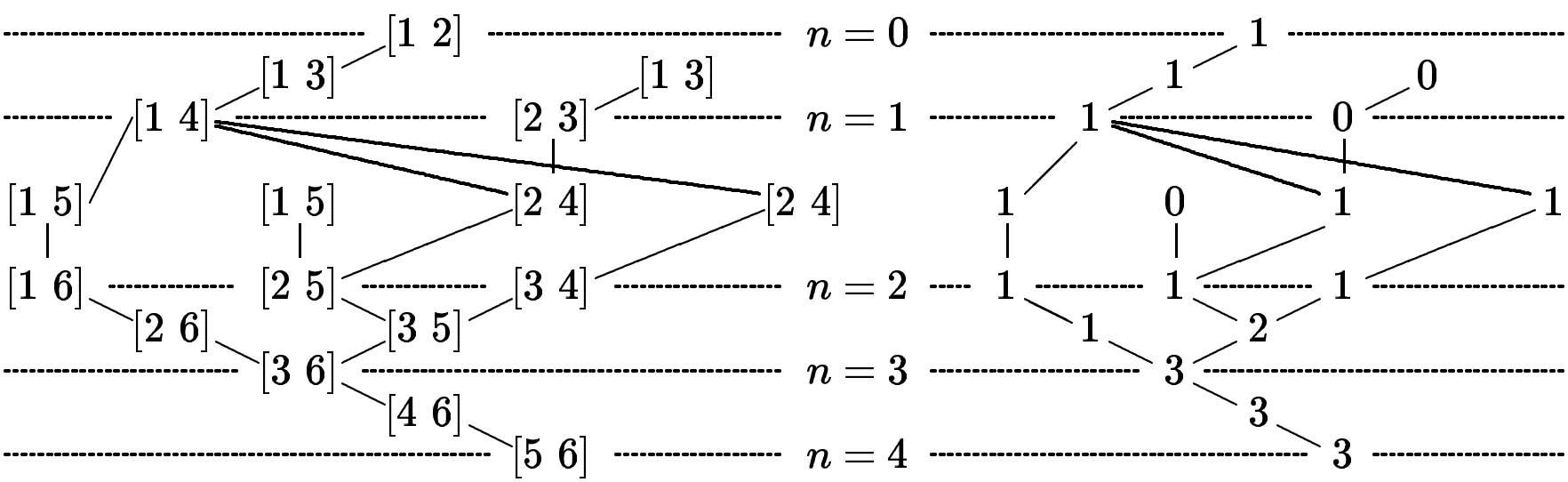
General Intersection Conditions

Input: n general $(m+1-k_i)$-planes in $\mathbb{C}^{m+p}$, $L_1, L_2, \ldots, L_n$;

$k_1 + k_2 + \cdots + k_n = mp$.

Output: p-planes X : $X \cap L_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, n$.

Bottom pivots for $p = 2$, $m = 4$, $k_i = 2$, $i = 1, 2, 3, 4$:



Quantum Schubert Calculus

Input: $n = mp + q(m+p)$ general m-planes in $\mathbb{C}^{m+p}$, $L_1, L_2, \ldots, L_n$, sampled at n interpolation points, $s_1, s_2, \ldots s_n$.

Output: maps $X(s)$ of degree q producing p-planes: $X(s_i) \cap L_i \neq \{\mathbf{0}\}$, $i = 1, 2, \ldots, n$.

For $X(s)$ to meet L_n at s_n, we use the Pieri homotopy:

$$h(X(s,t), s, t) = \begin{cases} \det(\ X(s,t)\ \mid (1-t)S_X + tL_n\ ) = 0 \\ (s-1)(1-t) + (s-s_n)t = 0 \\ \det(\ X(s_i,t_i)\ \mid L_i\ ) = 0 \\ i = 1, 2, \ldots, n-1 \end{cases} \qquad t \in [0,1].$$

As t goes from 0 to 1, the interpolation point goes from $(1,0)$ to $(s_n, 1)$, from ∞ to s_n.

General Schubert Conditions

Input: k general matrices, $F_1, F_2, \ldots, F_k \in \mathbb{C}^{n \times n}$;

k brackets $\mathbf{a}_i = [a_{i1}\ a_{i2}\ \cdots\ a_{ip}]$, $a_{i1} < a_{i2} < \cdots < a_{ip}$, $i = 1, 2, \ldots, k$

Output: all p-planes X: $\mathrm{Rank}([\ X \mid F_{1:a_{ij}}\ ]) = p + a_{ij} - 1$,

$i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$.

Example:

$$\text{in } \mathbb{C}^6: \quad \Omega_{[2\ 4\ 6]}(F_1) \cap \Omega_{[2\ 4\ 6]}(F_2) \cap \Omega_{[2\ 4\ 6]}(F_3) = 2[1\ 2\ 3].$$

A general 3-plane X in $\mathbb{C}^6$:

$$X = \begin{bmatrix} 1 & \star & \star & \star & 0 & 0 \\ 0 & 1 & \star & \star & \star & 0 \\ 0 & 0 & 1 & \star & \star & \star \end{bmatrix}$$

$\Omega_{[2\ 4\ 6]}(F)$ *is described by*

13 *determinantal equations on* X

A Geometric Littlewood-Richardson Rule

following Ravi Vakil

We resolve

$$\text{in } \mathbb{C}^6: \quad \Omega_{[2\ 4\ 6]}(F_1) \cap \Omega_{[2\ 4\ 6]}(F_2) \cap \Omega_{[2\ 4\ 6]}(F_3)$$

formally as

$$\begin{aligned} &[2\ 4\ 6][2\ 4\ 6][2\ 4\ 6] \\ &= (1[2\ 3\ 5] + 2[1\ 3\ 5] + 1[1\ 2\ 6])\,[2\ 4\ 6] = 2[1\ 2\ 3] \end{aligned}$$

via the repeated application of Ravi's checker game:

1. resolving one intersection condition occurs in a poset of $n(n-1)/2$ levels (bubble sort on $1\ 2\ \cdots\ n$);
2. for k conditions we have a poset of posets, of $k-1$ levels.

1	1 2 3 4 5 6	1([2 4 6],[2 4 6])
2	1 2 3 4 6 5	1([2 4 6],[2 4 6])
3	1 2 3 5 6 4	1([2 4 6],[2 4 6]) + 1([2 6 4],[1 4 6])
4	1 2 3 6 5 4	1([2 4 5],[2 4 6]) + 1([2 6 4],[1 4 6])
5	1 2 4 6 5 3	1([2 4 5],[2 4 6]) + 1([2 6 3],[1 4 6])
6	1 2 5 6 4 3	1([2 5 4],[2 4 6]) + 1([2 6 3],[1 4 6])
7	1 2 6 5 4 3	1([2 5 4],[2 3 6]) + 1([2 6 3],[1 4 6])
8	1 3 6 5 4 2	1([2 5 4],[2 3 6]) + 1([4 5 2],[1 3 6]) + 1([3 6 2],[1 4 6])
9	1 4 6 5 3 2	1([2 5 3],[2 3 6]) + 1([4 5 2],[1 3 6]) + 1([3 6 2],[1 4 6]) + 1([6 3 2],[1 2 6])
10	1 5 6 4 3 2	1([2 4 3],[2 3 6]) + 1([5 4 2],[1 3 6]) + 1([3 6 2],[1 4 6]) + 1([6 3 2],[1 2 6])
11	1 6 5 4 3 2	1([2 4 3],[2 3 6]) + 1([5 4 2],[1 3 6]) + 1([3 5 2],[1 4 6]) + 1([6 3 2],[1 2 6])
12	2 6 5 4 3 1	1([2 4 3],[2 3 6]) + 1([5 4 1],[1 3 6]) + 1([3 5 1],[1 4 6]) + 1([6 3 1],[1 2 6])
13	3 6 5 4 2 1	1([3 4 2],[2 3 6]) + 1([5 4 1],[1 3 6]) + 1([3 5 1],[1 4 6]) + 1([6 2 1],[1 2 6])
14	4 6 5 3 2 1	1([4 3 2],[2 3 6]) + 2([5 3 1],[1 3 6]) + 1([6 2 1],[1 2 6])
15	5 6 4 3 2 1	1([4 3 2],[2 3 4]) + 2([5 3 1],[1 3 6]) + 1([6 2 1],[1 2 6])
16	6 5 4 3 2 1	1([4 3 2],[2 3 4]) + 2([5 3 1],[1 3 5]) + 1([6 2 1],[1 2 6])

Towards Littlewood-Richardson Homotopies

- A checker game defines the undegeneration of one moving flag with respect to a fixed flag.
- After each undegeneration, a cheater's homotopy is applied to move a flag to a given flag.

work in progress ...